RATIONAL *and*

APPLIED MECHANICS

by

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THIS BOOK IS DEDICATED TO
HIS FORMER STUDENTS
WHOSE CONSTANT INTEREST AND ZEAL
WERE FOR NEAR HALF-A-CENTURY
THE INSPIRATION OF THE TEACHER
AND WHOSE WARM FRIENDSHIP
IS STILL THE INSPIRATION OF
THE AUTHOR

ERRATA.

Pages 252—253. In joint paragraph change $\sqrt{\frac{2}{3}gr}$ to $\sqrt{gr}$.

PAGE

338. Line 19 change G to O.
341. Line 1 change (11) to (12); and 365 to 366.
341. Line 15 change 365 to 366.
344. Line 6 from bottom change $\frac{dz}{dx}$ to $\frac{dz}{dx_1}$.
344. Line 3 "given" not "giiven".
345. Line 3 change $-$ to $+$.
347. Last line change all R's to V's.
348. Line 6 change "vlaue" to "value".
350. Line 1 change IX to X.
352. Line 4 in numerator change $=$ to $-$.
353. Line 3 change l to l_1.
353. Lines 9 and 13 change VIII to VI.
354. Equation (2) near bottom change Wa to W_1a.
359. Line 19 change V_1 to V_2.
360. Line 7 change $C''C'''$ to $C''D''$.
362. Line 7 from bottom in value of y change l_2 to x, making $y = -W_1x - \frac{wx_2}{2}$.
362. Last line change lz to l_2.
363. Line 3 change lz to l_2.
365. Middle of page change $\frac{dx}{dz}$ to $\frac{dz}{dx}$.
373. Line 4 insert P before *due*.
376. Line 3 insert A_1 between $=$ and $+$.
376. Line 19 change $=$ to $-$.
377. Fig. 391 put B' at right-hand support.
377. Line 18 change M_2 to M in the formula.
377. Line 18 change A_1 to A.
380. Line 19 change 332 to 392.
384. In foot-note correct spelling of "stress".
385. Line 17 change "in (9)" to "above".
386. In foot-note correct spelling of "vertical".
386. Line 18 change "joice" to "joist".
390. Line 2 from bottom invert the fraction containing E.
391. Line 20 insert l before $\sqrt{2}$.
392. Line 15 change 14 to 42.
396. Line 4 cut off the bar over second fraction.
397. Fig. 409 change θ to β.
399. Line 3 change $\div$ to $\times$.
402. Line 2 change Fig. 2 to Fig. 414.
408. Line 20 spell "Theorem" correctly.
409. Line 7 change 24 to 12 in the fraction $-\frac{w_1l_1^3}{24}$.
411. Line 4 from bottom insert $\div$ under bar, thus: $\sqrt{2an \div (pn)^2}$.
412. Line 9 change m to $\frac{1}{n}$.
416. Line 2 change Δ to Δ_1.
417. Line 5 spell "bending".
418. Line 12 change $\frac{2EI}{l^3}$ to $\frac{l^3}{2EI}$.
454. Line 3 raise the expression $\epsilon^{\frac{wz}{p_1}}$ into the line.

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PREFACE.

This book is written primarily for students entering upon their second collegiate year. Their knowledge of mechanics is limited to what was gained during their study of physics and from practice in laboratory and shop; hence no apology is needed for making matters plain and easy in the first chapters of a work which later on leads up to the Theory of Structures, Internal Stress, Motion under Complicated Conditions, Forms of Energy, and higher Graphics.

It is assumed that at every stage of progress the student can by reasonable effort read the book understandingly *by himself* (always with pencil and paper at hand), so as to be ready to grasp the conditions of concrete or ideal problems brought up in class, and follow statements made in the language of mathematics and mechanics. In other words, he must here learn to translate the laws of the world of matter, motion, and force into mathematical terms.

There is here no attempt to embrace the special work of any one branch of Engineering or Architecture; but the aim has been to make an opening into every field and to make it possible for the student in any technical branch to intelligently read and use carefully prepared professional Papers and Manuals, and to solve new problems as they arise. Some readers may think that I have ventured into too many fields, but not far enough in some. I shall not have wholly failed, if the student says as he finishes a Chapter,—"I wish the Author had gone further in this interesting subject." The Author has been a *Teacher,* and his effort now, as always, has been to guide, and help his young companions to *climb alone;* and when he thinks he has helped them sufficiently, he has turned to another group.

Accordingly, Chapter XI is devoted to Elementary Graphical Statics, and incidentally, some of the laws which obtain in Framed Structures are presented; while the subject of Redundant Members in Frames is deferred to Chapter XXIII, and even there, the treatment is purely elementary, inasmuch as a thoro discussion of Indeterminate Problems belongs in an advanced professional course. In the same way Hydraulics and Thermodynamics are introduced in Chapters XXVI and XXVII.

It was a most important step in the development of Applied Mechanics, when Internal Stress and the theory of Elasticity were introduced into the study of solids which had previously been regarded as rigid. The Theory of Structures is impossible without the Moduli of Elasticity and the Distribution of Stress. The behavior of a loaded beam with a *fixed end* is quite beyond the scope of many books on Mechanics: Hence these matters are introduced as strictly appropriate and even essential.

The Author hastens to acknowledge his great obligation to Professor Rankine, the eminent Scotch engineer and author, the perfect master of Mechanical Analysis. His profound insight into the conditions of a problem, and his ready command of the methods of Mathematics made difficult things seem so easy to him, that he failed to see the pits into which his students fell. His reasoning was almost sub-conscious, and he was given to writing down the second equation first, and then deriving the first from the second. Hence it was always a great achievement, when a student could truthfully say that he could "read Rankine."

Newton explained his superiority to other men as due to his ability to hold his attention steadily upon the details of a problem without a waver for a longer time than other men. Rankine seemed to see as by a flash of lightning.

In the book which follows, the reader will find no lightning flashes; but it is hoped that he will find some evidence of a steady light, and a series of logical steps which a strict attention to business will enable him to surmount. In the first few chapters the steps are short and easy, and the exposition full. In the later chapters the steps are longer and higher, as the climber is supposed to have greater reach, and stronger mathematical legs.

Thruout the book the aim has been to be rational, to make every step reasonable, and every demonstration intelligible. The secret of a lucid analysis is, like that of untangling a snarl of yarn, viz: to *get hold of the right end of the thread.*

It is often necessary to repeat what has been said before, and to call special attention to a new application of a principle which may have been forgotten. The Theory of Couples will be found as serviceable as the Parallelogram of Forces.

No attempt is made in this book to deal with commercial matters. Prices like fashions change, and markets rise and fall, but the fundamental laws of mechanics are eternal; and once they are fairly mastered, the engineer should never be at loss under new conditions. He must at times partly make, and wholly utilize, a new environment.

The Author has not cared to multiply problems. To go beyond clear illustrations of general principles and useful methods of analysis, is to waste time and opportunity. To ring endless changes upon bodies falling in a vacuum, and projectiles flying thru empty space, is to kill time and "keep students busy." No problems are more interesting and useful than those one finds in the shops, yards and mills of his own neighborhood, if he has eyes to see and lives near a busy community. A *personally conducted tour* of a few hours, should fill a note book with problems.

In occasional numerical problems the Author has steadily refused to split hairs while dealing with gross weights and constants given in round numbers. In these days of slide-rules, and Moduli given to the nearest million, it is ridiculous to insist upon six-place logarithms and to thousandths of a foot or a fraction of an ounce. Accordingly, a Four-Place Table of Logarithms is given in the Appendix, and quite

generally the value of π is given as $\frac{22}{7}$ (which is more accurate than 3.14, and, considering both latitude and elevation, the value 32 for g is used as sufficiently accurate.

The Author is under many obligations to his former student, Samuel A. Burgess, Esq., for having read with care the greater part of the proof sheets. Errors still remaining in the book will be corrected in the second edition, if readers will kindly point them out.

A WORD TO TEACHERS: The order in which subjects are taken up was adopted to meet the needs of engineering students who early in their professional study take up frames and structures. The subjects of motion and deformation usually come later.

If a class is greatly interested in Stability, Chapters XXIV and XXV may well follow Chapter XI.

If a class wishes to study Motion, and to postpone Strength and Elasticity, it may skip for a time Chapters IX, X, XI, and XII.

A class not ready to use the Integral Calculus, should for a while pass over Chapters VII, VIII, IX and X.

The Author hopes that no student will be allowed to wholly omit Chapter XI, or to make use of the Moment of Inertia of Surfaces without first mastering Chapters IX and X.

Chapters XXVI and XXVII may well follow Chapter XVIII.

Finally, it is hoped that no teacher will attempt to rush a class of Undergraduates thru the book in forty weeks.

CALVIN M. WOODWARD.

St. Louis, Jan. 1, 1913

APPLIED MECHANICS.

CHAPTER I.

INTRODUCTORY.

1. Force is an action between two bodies. It is not a tendency, it is a real thing as experience readily shows. *A* pushes or pulls *B*, and at the same time *B* pushes or pulls *A*; there are two bodies and one action or force. There can be no force unless there be at least two bodies. The bodies may be at rest or they may move. There may be an action, a push or pressure, between two bricks in a wall which does not move. There may be a pull or tension between two cars which are in motion.

1. In Applied Mechanics the general surface of the earth is assumed to be at rest. In Celestial Mechanics full account is taken of the motions of the earth and of other heavenly bodies.

All forces are more or less distributed, either over surfaces or thru volumes. The pressure between one's foot and the floor, or between steam and a piston, is distributed over the surface of contact; in one case unevenly, in the other uniformly; in one case without motion, in the other with motion. The earth, in a most mysterious way, pulls upon every particle of matter:—an apple hanging on a tree, a brick in a wall, and an iron ball flying thru the air; and of necessity the apple, the brick, and the ball pull the earth. This is called the "force of gravitation." Of all the forces with which mechanics deals, gravitation is the most common, and the most inexplicable. We know, however, that nothing can escape it, and that it is not sensibly affected by such distances as are generally considered in this book, or by intervening objects.

2. It is highly important that the student regards the force of gravitation as a pull, distributed thruout the whole volume of a body. Moreover the action (or pull) of gravitation upon a body must not be confused with the action between the body and the platform or foundation upon which the body may rest. When a body like a block of stone rests upon the ground, there are two actions between the stone and the earth, viz: a pull down and a push up, which exactly balance, one action being distributed thru volumes, the other being distributed over the surface of contact.

3. The plural word "volumes" is used advisedly. One volume is that of the stone which is made up of an infinite number of particles;

the other volume is that of the earth which also consists of an infinite number of particles; and there is an action between every particle in the stone and every particle in the earth. What we call the *weight* of a body is the *resultant* of all the separate pulls, and its direction is towards and away from the earth's center. The stone pulls the earth *just as much* as the earth pulls the stone. See "ATTRACTION" in a later Chapter.

2. When Sir Isaac Newton, the master mind in Mechanics, stated as *a fundamental law that action and reaction were equal and opposite,* he meant only this: That when there is an action between A and B, the measure of A's action on B is exactly equal to the measure of B's action on A, and in the opposite direction. This sounds as though there were two forces for one action, but such is not the case. There is only one force, but its action may be viewed from the standpoint of A, or from the standpoint of B. That is, if we are considering the condition or behavior of A as regards rest or motion, we only take into account the action of B upon it, and we give that action its proper direction; if, however, we are considering the status of B, we take account only of A's action upon it.

1. This law of Newton does not explain the stability of the block of stone which rests on the ground in a former illustration. In that case there were two actions between the earth and the stone, a pull and a push, that is, an attraction and a pressure; and the pull and push balanced.

It is true in the case considered, that if there were no pull, there would be no push, because we have assumed that the stone was at rest, and hence the two actions must balance. If there was but one action, the stone would not be at rest; this must be made clear.

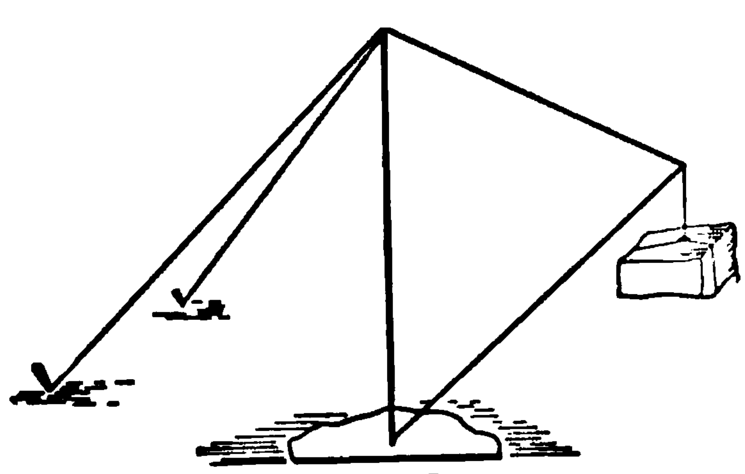


Fig. 1

2. Suppose a stone to be suspended by a derrick, and to be at rest. There is now an action between the stone and the earth; the earth is pulling *down* on the stone and the stone is pulling *up* on the earth; that is one case of action. In the next place there is an action between the stone and the derrick; the stone is pulling down on the derrick, and the derrick is pulling up on the stone—this is a second case of action. Thus we see that the stone is acted upon by two bodies: the earth which pulls down and the derrick which pulls up, and as the stone is at rest we know that these two pulls are equal in magnitude and exactly opposite.

Now suppose the rope or chain thru which the derrick acts, is cut or breaks. The action between the stone and the derrick ceases, and only the action between the stone and the earth remains. Hence it cannot be at rest; and the stone and the earth begin at once to approach each other. However, the earth's motion is so immeasurably small that we leave it wholly out of account, and merely say: "the stone *falls*."

3. Stress. Thus far we have thought of A and B as two separate bodies, but it is evident that they may be separate parts of the same body like adjacent strata in the earth, adjacent leaves in a book, adjacent links in a chain, adjacent particles in a steel rod or a concrete post. When A and B are adjacent parts of a continuous body, their mutual action, no matter what its character may be, is called *Stress*. If it be a pull it is called *tensile stress*, and the continuous body, rope, wire, rod or bar is said to be in *tension*. If the adjacent layers press against or upon each other in a post, strut, block or wall, the post, strut, block or wall is in compression, and the stress is called *compressive stress.*

The link of a chain illustrates fairly well adjacent parts of a continuous body. Suppose in Fig. (1) that the stone is suspended to the derrick boom by means of a chain. The chain is in tension and every link is acted upon by two forces (independently of the earth's pull or attraction upon the material in the link) the downward pull of the link below it, and the upward pull of the link above it. Furthermore, it is evident that the upward pull of the topmost link must equal in magnitude the downward pull of the lowest link, plus the pull, or attraction, of the earth upon all the material in the intermediate links.

4. Magnitude of forces. Units. We are immediately conscious that forces or actions have magnitude; one pull or attraction is greater or less than another; one push, pressure or repulsion is greater or less than another. In order to express magnitude with precision we must have a well known unit of force with which all other forces may be compared numerically. A certain pull or push shall be called *one*, and like other units in common use, it shall have a name. The unit of time in mechanics the world over is a *second* (or a multiple of a second); the unit of length or distance most commonly used in the Anglo-Saxon countries is a *foot* (or a multiple thereof), though the meter, a French unit, is in common use in physical text-books and laboratories. The unit of value in North America is a *dollar*. In all these cases we know by observation, experience and frequent use just what these units are, and what a given number of seconds, feet, meters or dollars means.

5. Units of force. When, however, we come to the unit of force and its name, we find great diversity in the text books, lecture rooms, and laboratories of every engineering school in the land, in spite of all efforts of the users to bring about the adoption and general use of a common unit.

1. The first and most common is the *avoirdupois pound* by which we measure the weights of all sorts of things, building materials, foods, crops, ores, manufactured articles, etc. (*i. e.*, the pull of the earth upon them), the tensions in ropes, cords, chains, drawbars, rods and eye-bars; and the pressures of gases (air, steam, etc.), and of liquids (water, oils, etc.).

2. The second unit of force is the *poundal.* This unit of force is about one-thirty-second of an avoirdupois pound, or half an ounce. Its use is somewhat limited, but is not unfrequently found in works on applied electricity, where it measures the electrical forces of attraction and repulsion.

3. The third unit of force is a *kilogram,* or its one-thousandth part called the *gram.* This unit of force is of French origin, but its use, like that of the meter, is world-wide in physics, chemistry and engineering. In magnitude the Kilogram is about 2.2 pounds.

6. The meter (which was intended to be one ten-millionth of a quadrant of the earth's meridian, and which is approximately so), is really the distance between two engraved marks on a platinum-iridium bar, carefully preserved at Paris. An accurate copy of this bar is in the archives at Washington. The meter bar was received by the President of the United States in 1890; the entire metric system is now legalized in all the territory under our flag. The *foot,* consisting of 12 inches, is defined as a certain part of a *meter,* shown by the relation that

$$1 \text{ meter} = 39.37 \text{ inches}.$$

1. At the same time in 1890 a standard block of metal (platinum-iridium) was received and deposited at Washington, which, at the level of the sea, and at 45° north latitude was declared to weigh one *kilogram* (kg.). Another block, under the same conditions, is declared to weigh one *pound* (lb.). The relation is very exactly

$$\text{One kilogram} = 2.20462 \text{ lbs.}$$

2. The relation of the kilogram to the meter is shown by this: A kilogram is the weight, at the level of the sea, latitude 45°, of one one-thousandth part of a cubic meter of distilled water at a temperature of 0° centegrade, or 32° Fahrenheit. Accordingly, a cubic meter of water weighs 1,000 kgs. under standard conditions.

3. It will be well for the student to fix these standard relations well in his mind as he must think readily in terms of each.

1 meter = 39.37 inches.
1 foot = 0.3048 meter. 2·54 centimeters = 1 inch.
1,000 meters = 1 kilo-meter = $\frac{5}{8}$ mile, nearly = 3281 feet.
1 kilogram = about 2.2 lbs. 453.6 grams = 1 lb.
1 ton = 2,000 lbs. = 907.2 kgs.

For greater precision (rarely necessary) see an encyclopedia.

7. Units of mass and time. There are in every system of units a unit of *time* and a unit of *mass*, which will be fully explained when we come to expound the laws of motion and moving bodies.

Meanwhile, the student must not be confused or disturbed by the use in text books and scientific journals of such expressions as "a mass of so many pounds," or "a mass of so many kilograms"; the meaning merely is:—a quantity of material upon which the earth's attraction is so many pounds weight, or so many kilograms. The simplest way to find the magnitude of the earth's pull or attraction upon a given quantity or mass of material is to weigh it by a spring balance, a process which need not be explained. The essential thing is to bear constantly in mind that the earth's pull upon a mass is always a *force*, and that that force should never be confused with the material pulled or acted upon.

8. Mechanical problems. 1. Problems in mechanics are always given under conditions which are more or less ideal, *i. e.*, not real. For instance, in the case of the stone hanging upon the boom of a derrick, it was assumed that the action of the air upon the stone was either nothing at all, or that its action was self-balanced, and yet we know that such could not be the fact. We know that the pressure of the air upon every square foot of the surface of the stone is about a ton. We also know that the upward pressure of the air upon the bottom or lower surface is a little greater than the downward pressure upon the top or upper surface; consequently the total up-and-down action of the air is not self-balanced. The difference or buoyancy is, however, so small compared with the other forces acting (the pull of the earth and the pull of the rope or chain), that it was neglected, and the magnitude of the earth's pull was assumed to be equal to the pull of the rope.

2. Again, there are always air currents, which are ignored, unless those currents are very strong, as in the case of high winds. In like manner we shall often assume that solids and gases are of uniform density tho we know that they always vary more or less. We shall at times suppose that the surfaces of solid bodies are *smooth*,

tho we know none that are perfectly smooth; and that the earth's pull on a body when a few feet above the sea level, or a few feet below it, is just the same as it is at sea level; and that change of latitude has no effect upon the earth's attraction, tho it is well known that strictly speaking such is not the case. Sometimes we shall assume that bodies are perfectly rigid tho we are positive that no bodies are strictly rigid.

8. Such false assumptions do not render the problems and their solutions useless. The solutions are just as accurate as are the assumptions. If the latter were sufficiently accurate, or "near enough," then the solution is near enough. Both the student and the teacher, however, are warned against the danger of ignoring conditions which are so far from *real* conditions, that the results of solutions are very misleading and mischievous. It is rarely wise to ignore friction, internal stresses and elasticity. Even with the best assumed conditions, problems are still somewhat ideal. In the problems which are solved or given for solution in this book, care will be taken to have the assumed conditions approximate closely real conditions, so that the conclusions reached may have value, and serve as guides for future conduct and use.

9. The recognition of forces. Fig. 2 should represent a heavy block resting on a table. This block supports a smaller block and a man who holds a loaded basket on his head. Partial support is given to the large block by a "flexible and weightless" rope passing round the large block at the lower end and after going over a "smooth" peg, sustains a weight of 40 lbs. at the upper end. The student is to give the *number, character, "sense"* and *magnitude* of the forces or actions upon each individual body: *A, B, C, D, E and P. Character* shows whether it is a push or a pull, and how it is distributed. "Sense" tells the direction of the action of a force on the body just then under consideration.

For example, the Man is acted upon by three forces:

P Fig. 2 P

1. The *downward pull* of the earth's attraction of 150 lbs., which is *distributed* thru the *volume* of his entire person.

2. A *downward pressure* of 20 lbs. from the loaded basket *distributed* over the upper *surface* of his head.

3. An *upward pressure* of 170 lbs. from the large block *distributed* under the soles of his shoes.

The student should write out similar statements for the other bodies in the problem, paying special attention to his language as well as to his thought.

10. Ideal conditions. Several matters will be noticed which require mention. We have assumed several conditions which are more or less erroneous.

1. We have left the air entirely out of consideration whatever its action may be.

2. We have assumed that the peg over which the rope passes is perfectly smooth.

3. That the rope itself, tho strong, is perfectly flexible and imponderable, so that the tension in it is the same (viz.: 40 lbs.) from end to end.

In so-called practical problems, it is generally said that such trifles are unimportant, but it is easy to see that they might not be trifling or unimportant.

4. Finally, we must *infer* some things which the statement of the problem does not give. This inference is the result of a process of reasoning of which one is hardly conscious. How did we know that the large block pressed or lifted up against the man's shoes with a force of 170 lbs.? Had both man and block been either descending or ascending faster and faster, the action between shoes and block would not have been 170 lbs., but since both were *standing still,* we know that the forces acting on the man *must balance.* Hence the only upward force must alone be equal to the sum of the two downward forces.

5. Such reasoning we call "Common Sense," but it is strictly in accordance with the axioms or laws of mechanics. It will be seen later on that the range of common sense can be greatly increased as the laws and conditions which obtain in the interaction of bodies, at rest and in motion, are known and understood.

11. The transmission of forces. In the case of a chain under tension, it is easy to see how a pull upon the link at one end is transmitted, or *passed along,* thru all the links to the other end.

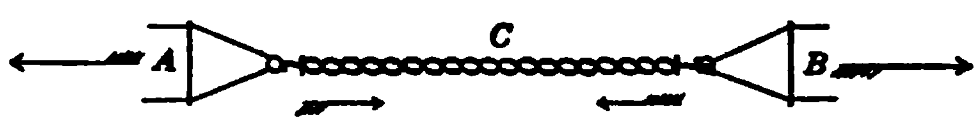


Fig. 3

Accordingly, we say that there is *an action between A and B by means of the chain C.* Still more simply, we may say, when we are considering the status of *B,* "*A* is pulling *B.*" It is evident that the length of the chain is not

of any account so long as the chain, like the rope in the problem, is without weight, and therefore straight.

In like manner a body *A* may act upon or push a body *B*, by means of a strut, post, brace or leg. We shall see later that a force can be transferred or transmitted sideways (laterally) thru solid bodies.

12. Forces acting at point. In the statement and solutions of ideal problems in mechanics, perhaps the most striking assumption is, that finite forces act at points. Now all forces in reality act at surfaces or thru volumes, yet a point has neither volume nor surface. Nevertheless the notion of *concentrating* a distributed action or force is most natural and very convenient. Common experience, and therefore common sense, convinces us that a trap door, a platform, or a warehouse floor, if sufficiently rigid, could be supported by a single prop or post, *provided it be put in the proper place.*

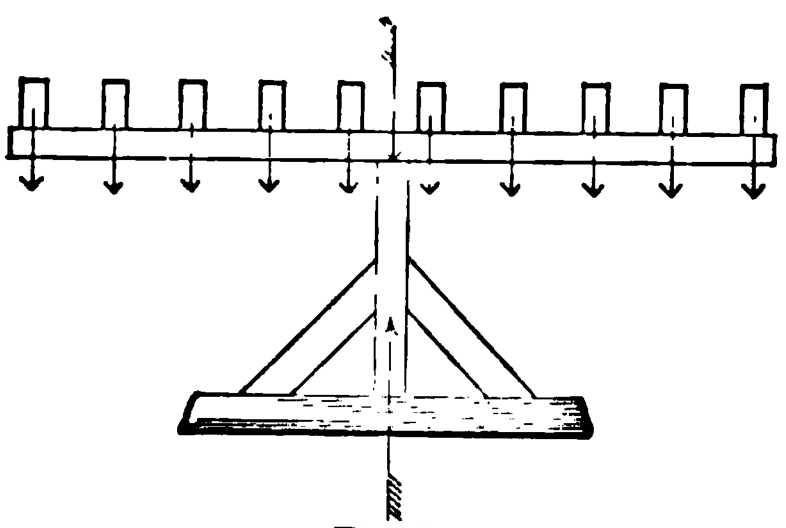


Fig. 4

Imagine a stiff plank, loaded with bricks, balanced upon the top of a small square post. This is easily done in fact, and still more easily imagined. Now the earth's attraction upon plank and bricks must be exactly balanced by the lift of the post; hence the total downward action must somehow be concentrated, or transmitted, to a point or small area just above top of the post. Similarly, the upward action of the post may be centered within a small area immediately in contact with the surface of action of the plank. These two areas may ideally be made as small as we please, *i. e.*, each may be called a "*physical point*" and the two total actions may be thought of as acting at a common point.

13. The graphical representation of a force. It will now be easy to represent an action upon a body (from any source whatever) by a straight line or arrow, the direction in which the arrow points showing the "sense" or direction of the action; the *length* of the line drawn to scale (so many pounds or tons to the foot, or so many kilograms to the meter), showing the magnitude of the force or action; and the *position* of the line showing the "line of action," any point of which, in or on the body acted upon, may be taken as the physical "point of action" of the acting body. In figure 4, the upper arrow shows the direction, magnitude and position of the earth's total attraction on the plank and bricks; and the lower arrow shows the direction, magnitude and position of the upward action of the post upon the plank.

14. Abstract forces. Altho no force can exist, except as an action between bodies, it will be convenient, when we study the concentration (composition or combination) of separate or distributed forces, to leave the description of the bodies whose actions we are considering, wholly out of our thought, and treat the forces as independent things; that is, we withdraw (or abstract) the forces from the bodies creating them, and reason about them abstractly, just as one might add six dollars to eight dollars, thereby producing fourteen dollars without having any money at all. So we may say: "Suppose we have a force fully represented by the arrow line AB, and, in the same plane with AB, another force fully represented by the arrow line CD. Our problem is now to find a single force XY whose action shall in all essentials represent, or be equal to, the combined action of the two given forces."

In the solution of this problem, we take no thought of the bodies producing the given actions. We care not what they are, just where they are, nor why they act, except that we must know their relative positions. Of course, we assume that they act upon the same rigid body, but we do not care *in this problem* what that body is. In other words our problem is purely abstract.

1. Thruout this book we shall mingle abstract problems freely with those which approximate more or less closely real ones. With abstract problems, we shall aim to illustrate general methods, establish general laws, and derive general formulas, and then apply them to the solutions of problems derived from all sorts of sources and conditions, of a practical character. The number of possible problems is infinite; many are fanciful and useless; many are too difficult for our range of mathematics; many are intensely practical and admit of easy solutions. In our analysis we shall draw freely upon the student's knowledge of Algebra, Geometry, plane and solid, and Trigonometry; and later on we shall assume Analytic Geometry and Calculus. The more thoroughly these have been mastered the better, but the more difficult operations will be clearly explained.

PART I.

STATICS.

CHAPTER II.

The Combination and Resolution of Forces.

STATICS is that department of Mechanics in which all the bodies, acting and acted upon, are at rest or stationary. A body at rest under the action of two or more other bodies is said to be in Equilibrium; and the forces acting upon it are said to Balance.

15. Forces having a common line of action. The simplest possible example is that of two forces which exactly balance or neutralize each other. It is evident without any attempt at proof that they must not only act along the same line, but they must have equal magnitudes and opposite directions. Under no other conditions can two forces balance.

1. In so far as the state of rest or motion of a rigid body is concerned, two balancing forces may be left out of account, while we are considering the actions of other forces upon the same body. This cancellation of two balancing forces, is analagous to the striking off of equal terms in the members of an algebraic equation.

2. **Conversely,** we may at any time, to facilitate a solution, introduce two balancing forces, representing the actions of two imaginary bodies upon the body we are considering, without in any way affecting its state of rest or motion.

3. When several forces, of different magnitudes, have a common line of action, while some act in one direction and some in the other, they are readily reduced to two by adding those which have a common direction, and then reducing the two to one by allowing the smaller to cancel or neutralize an equally large part of the other; or the cancellation and addition may be carried on simultaneously. All this is readily expressed in the language of algebra.

Let F be the numerical measure of the magnitude of a force, and let one direction along the common line of action be considered as positive, and the other direction negative.

ΣF represents a *process* of algebraic addition of as many numbers (of units of force) as there were of assumed forces, acting upon a (real or imaginary) body. Finally, if we represent the magnitude

of the single force which *results* from the addition by the letter R, we have our first equation in mechanics.

$$\Sigma F = R \tag{1}$$

If the forces under consideration actually balance, the value of R is zero, and the equation becomes

$$\Sigma F = 0 \tag{2}$$

16. Conversely. Since forces having a common line of action may be combined to form a single force, so a force may be resolved or separated into any number of component forces having the same line of action and preserving the algebraic sum.

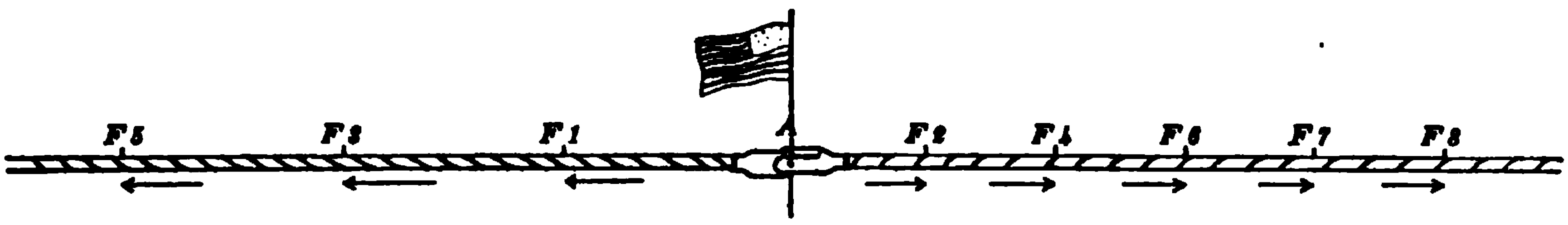


Fig. 5

The student should not fail to see the full meaning of these equations. An illustration will surely be of value. Fig. 5 is to represent a "tug of war" between a team of three men and a team of five boys. The three men have hold of one rope and are pulling to the left. The boys have another rope and they pull to the right. Both ropes are attached to a pin carrying a flag. Ropes, pin and flag are "imponderable," *i. e.*, their weight is not taken into account. Of course the effort of each team will be to pull the pin and flag its way. As soon as the signal is given to "Pull!" eight bodies (boys and men) act (thru the mediation of the ropes) upon the body (pin) A. Let us assume that forces acting towards the right are positive and those to the left are negative, and that

$$F_8 = +200 \text{ lb} \quad F_7 = +180 \text{ lb} \quad F_4 = +175 \text{ lb} \quad F_1 = -306 \text{ lb}$$
$$F_3 = -284 \text{ lb} \quad F_6 = +140 \text{ lb} \quad F_2 = +150 \text{ lb} \quad F_5 = -255 \text{ lb}$$

Now in this case ΣF means

$$+180 + 140 + 200 - 284 + 175 + 150 - 306 - 255$$

and since the algebraic sum is zero,

$$\Sigma F = 0.$$

Hence the forces balance, and that the pin A stands still.

If on a second pull $\Sigma F < 0$, the men win; if $\Sigma F > 0$, the boys win.

17. An action in Statics does not produce motion; on the contrary an action prevents motion. A brick in the wall is acted upon by the

brick above it, and the brick below it, and by the earth which attracts it, but it does not move, because the forces balance each other, that is: the action of the lowest brick is equal and opposite to the combined action of "gravitation" and the uppermost brick.

1. When forces do not balance, and motion is caused or modified, we have a problem in Kinetics, which will be discussed in later chapters.

18. Couples. When a body is acted upon by two other bodies in such a manner that the two forces have equal magnitudes, different but parallel lines of action, and opposite directions, they cannot balance, neither can their joint actions have a tendency to move the body up or down, to the right or to the left. Their sole tendency is to make the body *turn*, or, if it be already turning, to modify that turning.

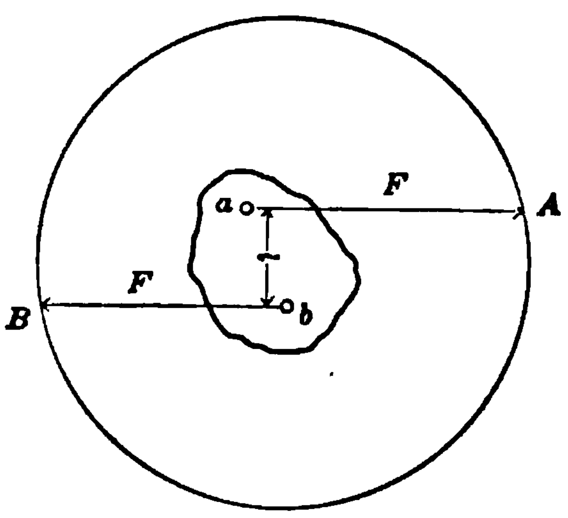


Fig. 6

1. Suppose a piece of cork to be floating upon a tub of water. The pull of the earth upon it is exactly balanced by the upward pressure of the water under it so that it is at rest, and we may ignore these two balanced forces. Now suppose that two needles are prest into the upper surface of the cork at a and b, Fig. 6, and that two parallel threads are stretched to the edge of the tub in a plane parallel to the surface of the water, as shown. Suppose that at A and B gentle but *equal* pulls upon the threads are given simultaneously. Instantly the cork begins to turn "right handed." These two new forces form a "right-hand couple," whose tendency is obvious. The perpendicular distance between the two lines of action is called the *arm* of the couple.

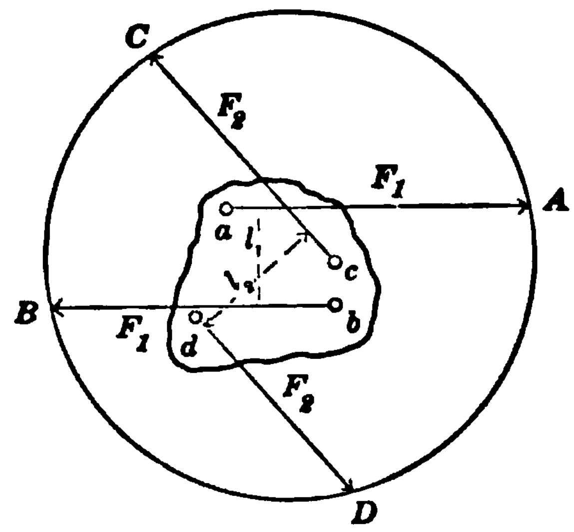


Fig. 7

2. Next suppose the cork at rest again and that two more needles are put in at c and d, Fig. 7, and that threads are carried across and *fastened* at C and D, so that they are parallel to each other and in a plane parallel to the surface of the water. If now the equal pulls at A and B are again applied, the cork stands still, and it will be found that the threads cC and dD are in tension. Moreover, since the cork does not move either towards C or towards D, the tension in cC and in dD are equal; they thus form a "left-hand couple." We consequently have two couples which balance each other. It is evident that the turning

effect or tendency of a couple must depend partly upon the *common magnitude* of the two forces, and partly upon the *distance between the two lines of action.* In short, *the magnitude of a couple's influence is measured by the product of one of its forces by its arm.* This product is called a "*moment,*" and it will be represented by the capital letter M, so that

$$M = Fl$$

This will be proved when we come to resolve a couple into Fl unit couples; **26.**

3. The student must bear in mind that F in this equation is the *number* of units of force in one of the given forces, and l is the *number* of units of length in the given distance (between the parallel lines of action), and that therefore M is the *number* of units of moment.

When we speak of "the *force* of a couple," we shall mean the common magnitude F.

19. The unit moment is not a fundamental unit, but it is derived from two which are fundamental: the foot and the pound; hence a moment is read as so many "foot-pounds, foot-tons," etc.

The word "*foot-pound*" as here used must not be mistaken for a later use of the same word as the *unit of work.* The moment unit expresses and measures the turning action of a body, but its action does not require actual motion. An *unbalanced* moment would turn a body; but in all problems in Statics, moments are balanced and no motion results.

20. The Freedom of a Balancing Couple. Referring again to Fig. 7, it is evident that the needles c and d could have been placed anywhere on the cork, and that the threads drawn straight from them could have been made fast anywhere on the circumference of the tub, provided only that the threads were separate and parallel, and that they were in a horizontal plane, so as to be prepared to act as a left-hand couple. It is important that the student see that these conditions are sufficient as well as necessary.

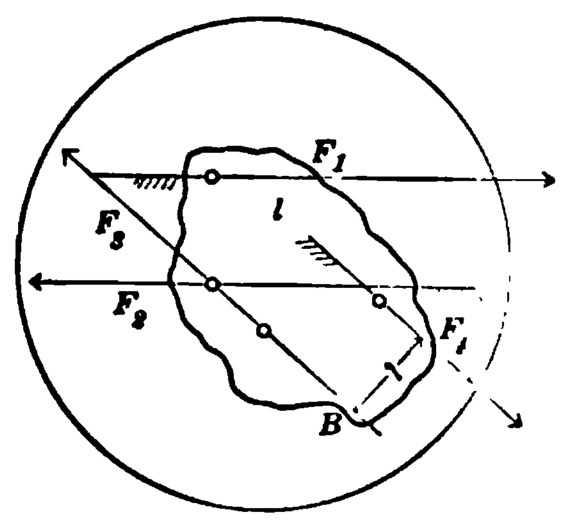


Fig. 8

1. Suppose the body B, Fig. 8, to be acted upon by the two forces F_1 and F_2 forming a right-hand couple with the arm l. It is evident that the "point of application" of F can be anywhere on the line of action, in each case.

2. The direction of the two lines of force can be any direction in the horizontal plane of the couple. For suppose new points of attachment are taken and a new direction to the parallel lines be chosen

as shown by F_3 and F_4, so that both force and arm remain numerically the same (leaving the couple still right-handed) the moment must still be $M = Fl$.

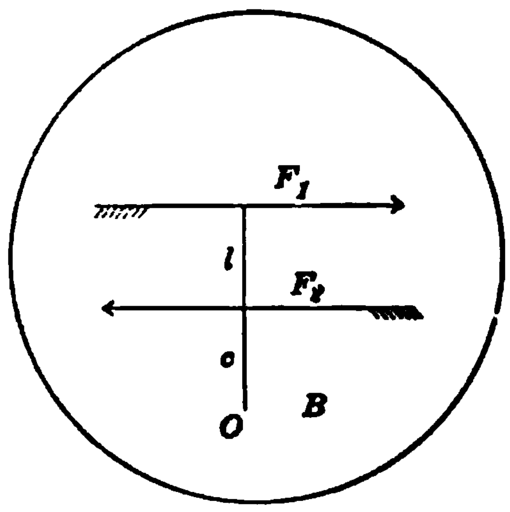


Fig. 9

3. The turning effect with reference to *any point* in the plane of the couple taken as an axis is always the same. Let a vertical at O Fig. 9 be taken as an axis about which the couple M (*i. e.*, the two forces with the arm l) has a tendency to turn the body B. The tendency of F_1 taken alone is evidently measured by $+F(c+l)$, and the tendency of F_2 is obviously $-Fc$. The tendency of the couple is then the sum of the tendencies of its parts, viz.:

$$F(c+l) - Fc = Fl.$$

The arbitrary distance c has disappeared from the equation and there is nothing in the result to indicate where O was taken. This means that whether c be long or short, positive or negative, the result is the same.

21. Hence we conclude, that *a couple acting upon a rigid body can be shifted at will in its own plane to new points and directions without affecting its turning tendency provided only that its moment and sign be unchanged.*

Since a given couple can be balanced by one of any number of left-hand couples constituted as above described, these left-hand couples must have the same moment and be equivalent to each other.

22. We therefore conclude in general that *two couples are equivalent which have the same moment, which lie in the same plane and have the same sign* (that is they tend to turn a body in the same direction). It follows that a couple F_1l_1 may be replaced by a couple F_2l_2 provided $F_1l_1 = F_2l_2$, as: (10 ℔)×(6 feet) = (4 ℔)×(15 feet); = (20 ℔)×(3 feet).

23. The moment of a force. If the axis be taken in the line of action of one of the forces of a couple, the moment of the couple is identical with the product of the other force by its distance from the axis.

Definition. The *product of a force, by its distance from a given point (or axis) is called the Moment of that force with reference to that point or axis.*

24. The turning effect or tendency of a couple is not changed by a shifting of it to a parallel plane.

Proof. Suppose we have two couples M_1 and M_2 equal in all respects, but acting in parallel planes H and K. Suppose in the plane H, a

third couple M_3 equal to M_1 in all respects except the direction of its turning tendency; it may be so shifted as to completely neutralize M_1 and hence balance it. It will now be shown that it can equally well balance M_2 in the parallel plane K. Let M_3 be so shifted that the arms of M_3 and M_2 are parallel, when M_3 becomes M_3' as shown in Fig. 10. It is now easily seen that the four equal forces F_2, F_3, F_2', F_3' balance, for, by symmetry, the concentration of F_2 and F_3 is a parallel force of $2\,F$ thru O, which is midway between them. In like manner the concentration of F_2' and F_3' gives a parallel force of $2\,F'$ thru O, in the direction opposite to $2\,F$, so that the resultant is zero. Hence M_2 and M_3 balance. Now since M_1 and M_2 can separately be balanced by the same couple, they must be equivalent.

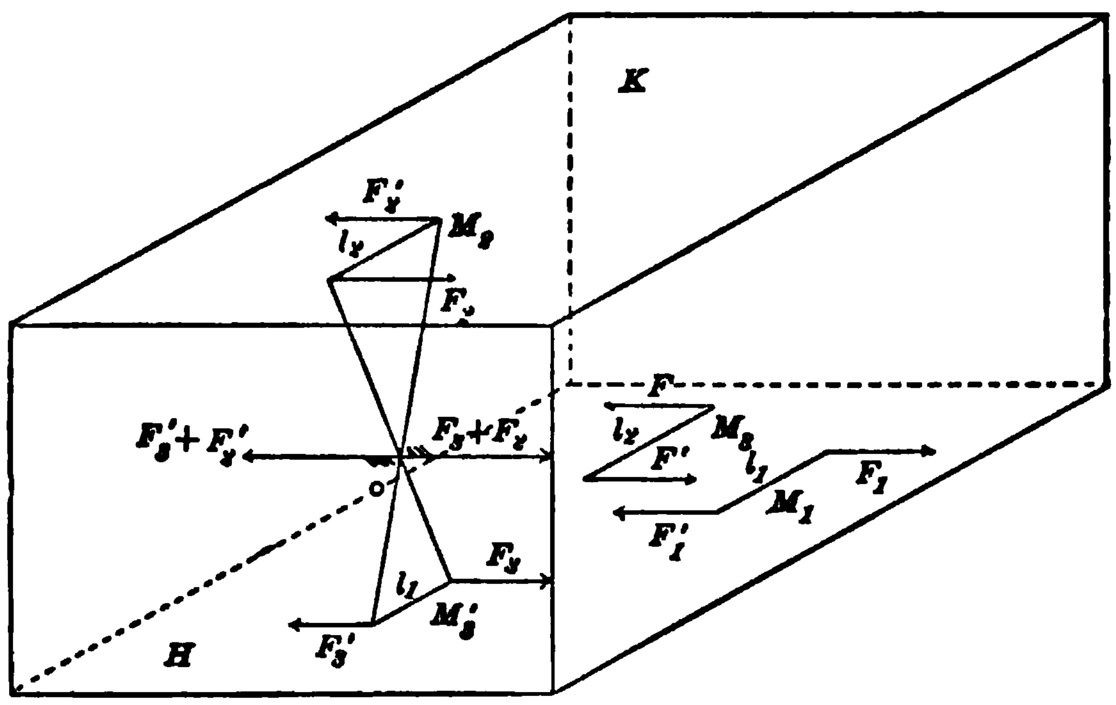


Fig. 10

25. Definitions of the "axis of a couple" and of positive and negative couples.

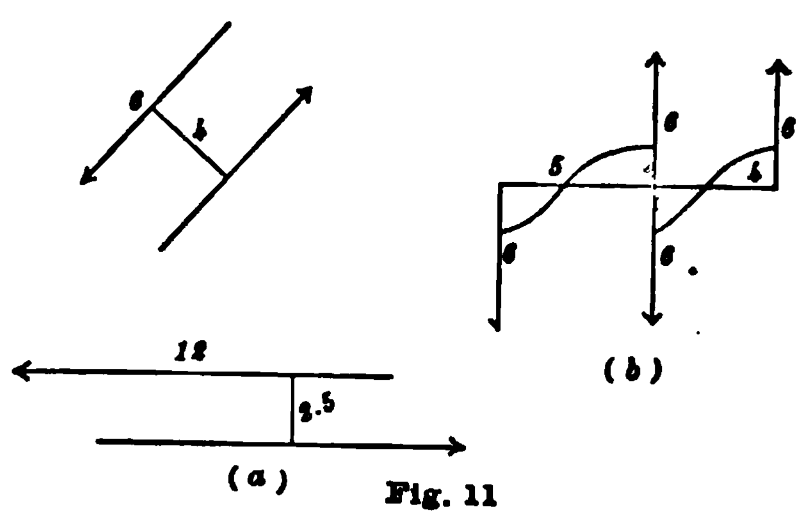
It has been seen that the point O (see Fig. 9), assumed for the purpose of computing the resulting tendency of a couple, may be anywhere in the plane of the couple. Now through O imagine a line drawn perpendicular to the plane of the *couple*; it will be perpendicular to all the parallel planes to which the couple may be shifted. Hence O may be any point in space, and the perpendicular line will be called the "*axis of the couple.*"

1. This axis is a purely ideal matter; it has no dynamic function. In Statics since the forces which act on a body balance, whether they form couples or not, the *axis* of a couple is not a restraining device, for it does not react upon the body in any way. It is merely an aid in our analysis.

2. Now as to the sign of a couple. When we stand in front of the dial of a tower clock, the hands appear to us to move "right-handed," or "clock-wise." To a person *in the tower*, however, the motion of the same hands appears to be left-handed. It is therefore necessary to determine in every case what "right-handed" means. To do this, take a point O as an origin on the assumed axis, and let one end of the axis, $+X$, be called positive, and the other, $-X$, be negative; and let

$+X$ be so taken that the direction *from* $+X$ *to* O shall be ***the absolute direction in which one must look*** in order that the couple may appear to him to be right-handed or positive. Left-handed couples are negative. In the case of the Y-axis, look from $+Y$ towards O.

26. The addition of couples having the same axis. Suppose two couples in parallel planes act on the same body. First, let one or both be shifted till they act in the same plane. Second, while preserving the *product* value of *force* by *arm*, let one or both factors be so modified that the forces in the two couples shall have a common magnitude. Third, let their positions be so changed that a force of one couple shall directly balance and neutralize a force of the other couple. Fig. 11*a* shows the two couples in the same plane, and Fig. 11*b* shows them so shifted and modified that two forces neutralize each other, and only one couple remains which has an arm equal to the sum of the two arms, and a moment equal to the sum of the given moments.



Example—Let M_1 have a force of 4 (units) and an arm of 5 (units); and M_2 have a force of 6 (units) and an arm of 3 (units).

Let them be shifted and modified while still equivalent, and then combined into one couple $M_3=38$.

1. It is evident that any number of couples with a common axis could be added in the same way. A negative or left-handed couple would have the effect of shortening the arm of a positive couple. Show this.

2. An equally simple method of adding co-axial couples would be to reduce them all to a common arm, so that when superposed the result would be one arm with equal and opposite sets of co-linear forces, making a single couple.

Example.—Given two co-axial couples:

$$\left.\begin{matrix} F=7 \\ l=8 \end{matrix}\right\} \textit{positive, and } \left.\begin{matrix} F=12 \\ l=\ 2 \end{matrix}\right\} \textit{negative.}$$

Reduce to a common arm $l=4$, and combine by superposition.

26¹. The resolution of couples. The converse of the last section shows at once that a couple may be resolved (or separated) into any number of couples by inserting at different points on the arm equal and directly opposite forces, each of the given magnitude F, and each

parallel to the given forces. The forces introduced balance each other in every pair, and therefore they do not affect the condition of the body acted upon. The given couple $(F)\times(AB)$, is resolved into five component couples, having the same axis, forces of a common magnitude, and different arms.

Still further, every one of the ten forces can be resolved into any number of components (the same for each of the two forces of a couple). Thus each of the above five couples may be resolved into several couples with equal arms, as is the one shown in (*b*).

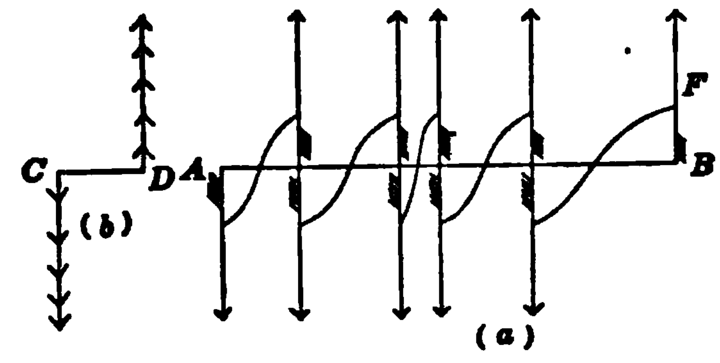


Fig. 13

It thus appears that the original couple may be resolved into any number of component couples.

When these two methods of resolving a couple into components are combined, it is evident that two couples, acting on the same body, having the same axis, and equal moments, may be shown to be equivalent, because they can be resolved into the *same number of equal component couples.*

It follows that a couple which has an arm 7 units long and a force equal to 10 units has a turning tendency 70 times as great as the moment of a couple which has a unit arm and a unit force. Hence the turning tendency or moment is measured by the *product of its force, by its arm, i. e.*, the product of the two numerical measures. This is the proof which was promised in **18.**

The student must never confuse his thought by saying that we can *really* multiply a force by an arm; that would be absurd. What we *really* do is to multiply one *number* by *another number*, and the product is still a number.

What are the "*conditions*" under which it is proper to say that a length of three feet and a force of seven lbs, give us a moment of 21 foot-lbs.?

From all this it follows that the essential things about any couple are: the *direction* of its *axis*, its *moment*, and its *sign.*

27. Couples as vectors. It is now evident that the *essential* things of a couple may be represented by a single straight line or vector: It must have *length* to represent the *moment* of the couple; it must have *direction* to show the axis, and therefore the *plane* of the couple; it must show the "sense" or tendency by **pointing** in the **direction in which the couple must be viewed** to appear right-handed. Couple vectors will be used in discussions which follow.

28. The combination of couples which do not have the same axis, *i. e.*, which are *not* in the *same* or *parallel planes.*

1. Let there be two couples in planes perpendicular to each other, and acting on the same body. They can be combined as follows: Reduce the couples to equivalent couples so that the four forces will have a common magnitude. Next, shift and if necessary *turn* the *couples,* each in its own plane, till there are two equal and directly opposite forces *acting along* the *line of intersection* of the two planes. These two co-linear forces balance each other, and there remain two equal forces which form a couple in a *third plane* which is parallel to the intersection of the first two. The arm of this *resultant couple* is evidently the hypotenuse of a triangle whose legs are the reduced arms of the given couples.

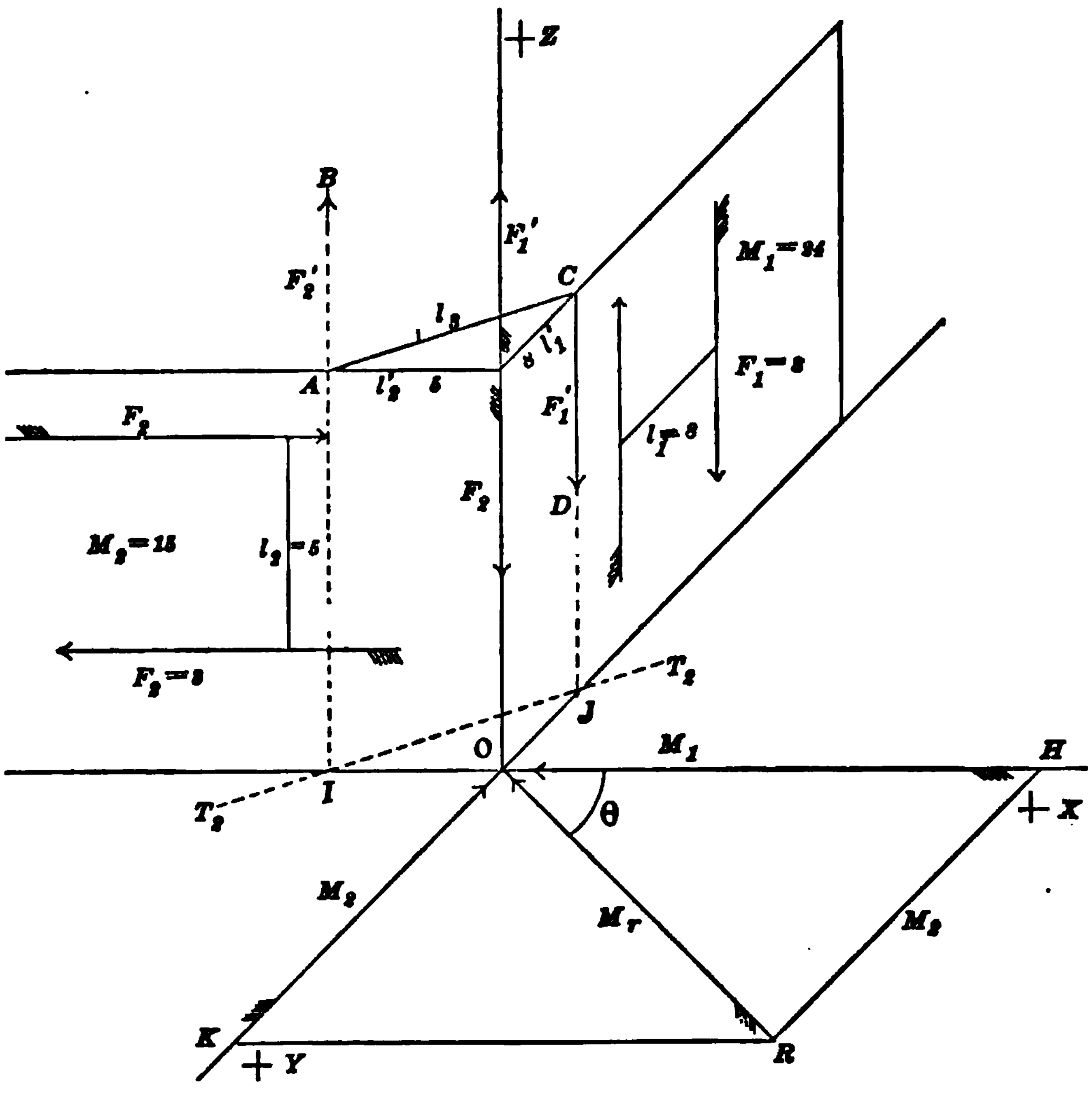


Fig. 13

Illustration. Let OX, OY, OZ be rectangular axes. Fig. 13. Let $M_1=24$ be a couple in the plane YZ, right handed to one *looking* from $+X$ towards O. Let $M_2=15$ be a couple in the ZX plane, right handed to one looking from $+Y$ towards O. Reduce each couple to an equivalent couple with a force 3, and a proper arm; so that $M_1=3\times8$, and $M_2=3\times5$. When the reduced couples are shifted to the intersection OZ, so that two of the forces neutralize each other, there remains only a couple having one force in the plane YZ acting down, and the other in the plane ZX acting up, with an arm which equals $\sqrt{(64+25)}$; so that the *resultant couple* is found to be:

$$3\sqrt{(64+25)}=\sqrt{[(24)^2+(15)^2]}=M=\sqrt{(M_1^2+M_2^2)}$$

The axis of M is evidently a line in XY (or parallel to that plane) perpendicular to the trace T_2T_2 of the third plane $ABCD$, on XY.

Fig. 13 also contains a graphical solution of the problem by the method of vectors.

The *direction* of M is determined by the equations:

$$\sin\theta=\frac{M_2}{M},\qquad \cos\theta=\frac{M_1}{M},\qquad \tan\theta=\frac{M_2}{M_1}$$

The moment M_1 is laid off to scale on $+X=HO=24$. In like manner M_2 is represented fully by $KO=15$, and then M by the diagonal of the rectangle.

3. The student should prove by geometry that the triangles RHO and JIO are similar and that consequently RO is perpendicular to T_2T_2.

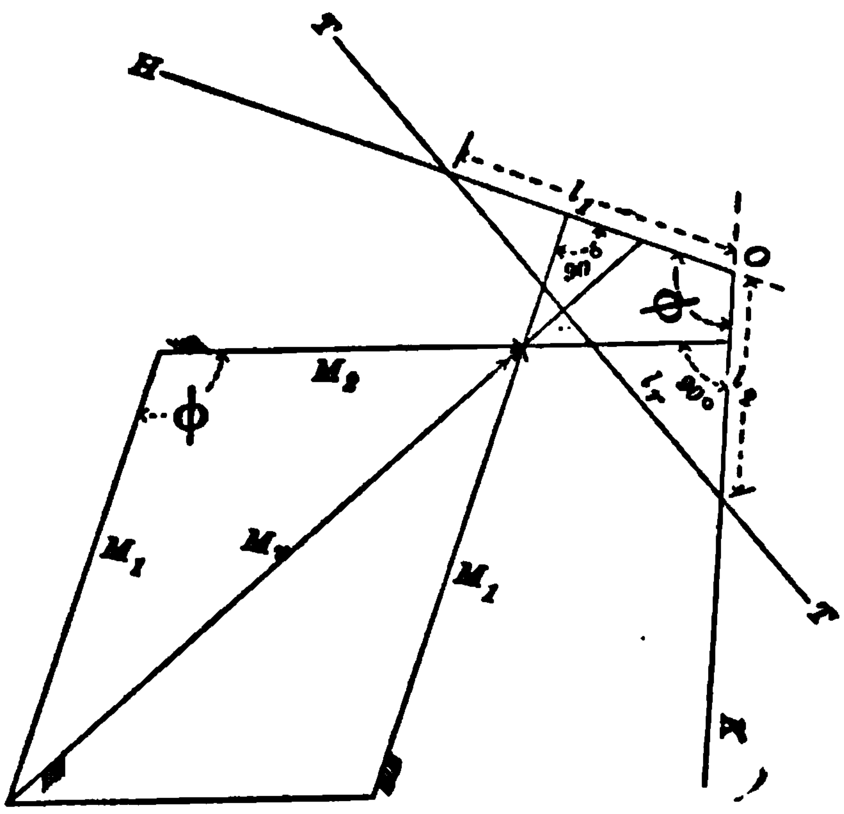


Fig. 14

29. When the planes are oblique to each other the analysis is the same, but the triangle of arms is oblique. 1. The algebraic solution is as follows:—Let the given couple be reduced to equivalent couples, the magnitude of F being common. Then

$$M_1=Fl_1,\text{ and } M_2=Fl_2$$
$$\text{and } M=F\,(l_1^2+l_2^2-2l_1l_2\cos\phi)^{\frac{1}{2}}$$

in which ϕ is the angle HOK (Fig. 14.)

2. In like manner the Graphical solution gives M in magnitude and shows the direction of the axis which is perpendicular to the plane of the new couple. $M=\sqrt{(M_1^2+M_2^2-2M_1M_2\cos\phi)}$.

Let both planes be perpendicular to the paper, and let their traces intersect at O. M_1 fully represents the couple in OH, and M_2 fully represents the couple in OK; and consequently M fully represents the resultant couple in the plane TT.

30. It is readily seen that two couples in non-parallel planes cannot possibly balance each other; and it is as easily seen that two couples in non-parallel planes may be balanced by a third couple, and that the moment of the balancing couple must be equal to M, the resultant of the other two, and the arrow line of its axis must be just the opposite of that of M.*

*A resultant couple has been defined as the couple which balances the couple which balances the given set of couples.

31. Three couples balance. It follows from the last two problems that when the representative vectors of three couples, are of such lengths and directions that they may form a plane triangle, when they are drawn *in order, as they point,* they balance, and the body on which they act is in equilibrium.

Thus the triangle ABC represents fully three couples acting on the same body, each with a definite tendency to turn it in a definite direction, and yet they balance and (so far as these couples are concerned) the body is at rest. The triangle ABC is called the *moment triangle of equilibrium.*

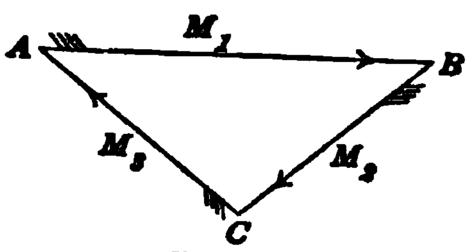


Fig. 15

32. The resultant of three or more couples. Having found the resultant of two couples as shown above, a third couple may be combined with the resultant of the two, and a resultant of three be found. This last may then be combined with a fourth, and so on for any number of couples.

1. When the planes of three couples are the rectangular co-ordinate planes, the problem is simple. The resultant of M_1 and M_2 is $M' = \sqrt{(M_1^2+M_2^2)}$; and the resultant of M' and M_3 is

$$M = \sqrt{(M_1^2+M_2^2+M_3^2)};$$

its vector is the diagonal of a rectangular solid constructed on M_1, M_2, and M_3 as confluent edges.

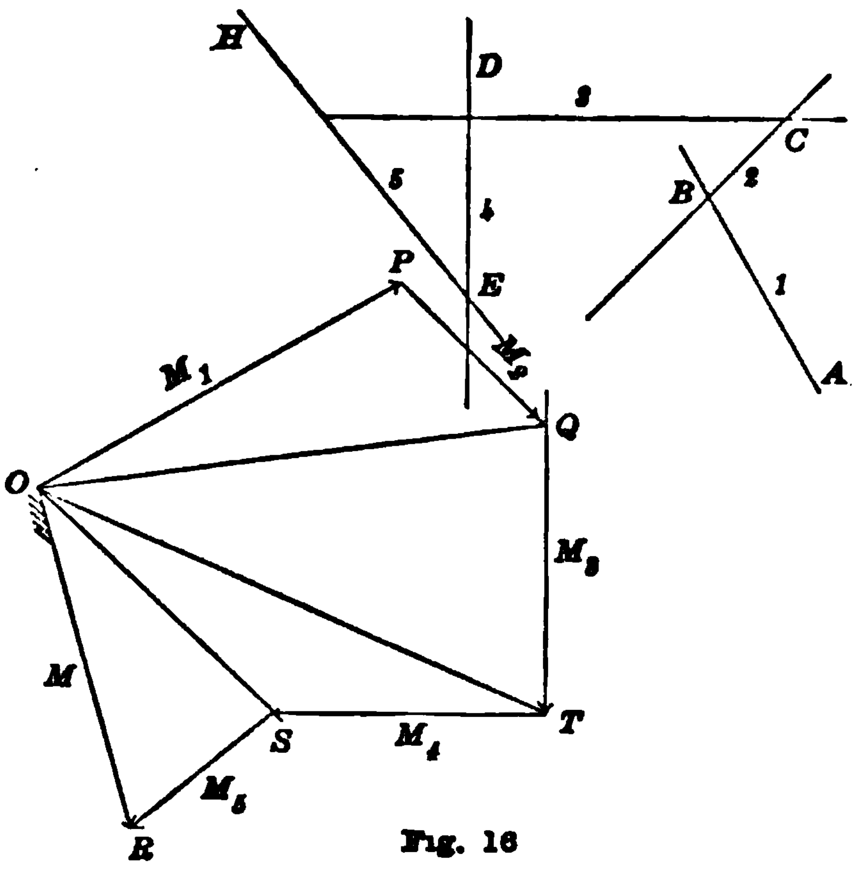


Fig. 16

2. If the planes of several couples are perpendicular to a common plane, the graphical solution is most simple. In Fig. 16 let the plane of the paper be the common perpendicular plane. Let the planes of the given couples have the traces AB, BC, CD, DE, EH, etc. From any point O (Fig. 16) draw M_1 fully representing the couple in AB; from the end of the vector M_1, draw M_2 fully representing the couple in BC; OQ, then, fully represents the resultant of M_1 and M_2. From the extremity of the vector M_2, draw M_3, then M_4, and so on. OT is the resultant of M_1, M_2 and M_3; and $OR = M$ is the resultant of them all. The *balancing couple* is represented by the vector RO, pointing from R towards O.

If the planes of the given couples have no common perpendicular

plane, the polygon $OPQSTR$ is not a plane figure; it is a polygon in space, but OR is still the vector of the resultant couple.

33. The combination of a couple and a single force acting in a plane parallel to the plane of the couple.

Suppose that among the forces which act upon a body at rest there are two which form a couple and a third force whose line of action is in, or parallel to, the plane of the couple. Let the irregular outline (Fig. 17) represent the stationary body; in this case, let the curved arrow and the numerical value of M represent the moment of the couple; and let the force F be represented by OA. It will now be shown that the resultant of M and F is the single force F_r.

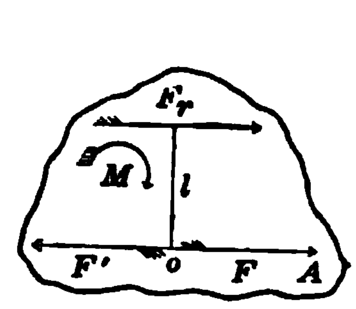


Fig. 17

(*a*) If M and F are not in the same plane, shift M to a parallel plane containing F.

(*b*) In place of the original force and arm of the couple, let $M = Fl$, in which F has the magnitude of the given force F, and $l = M/F$.

(*c*) Shift and turn the couple till one of its forces F', has the line of action of the given force F, but acting *opposite* to it, thereby canceling it, so that nothing is left but the *force* F_r *acting parallel* to the *given* F, and of the same magnitude. *The distance moved to the left* is $l = M/F$; that is, if an observer standing at O sees the couple right-handed as he looks *down*, the *Resultant* of M and F, will be F_r on his *left*, when he wheels about and looks in the direction OA.

Problem. The student should show that a given *left*-handed couple would have caused the resultant force to appear on his *right.*

34. The converse of the above proposition is evidently true, viz.: A given force F may be resolved into a couple and an equal force F acting at any point in space, distant l from the line of action of F, the moment of the couple being Fl and its axis being perpendicular to the plane of F and l.

35. The resolution of couples. A given couple may be resolved into two or more couples graphically, if the vector of the given couple is the line AB (Fig. 18 *b*), by drawing two component vectors AH and HB; or by three vectors which may be the edges of a rectangular solid. (Fig. 18 (*a*)).

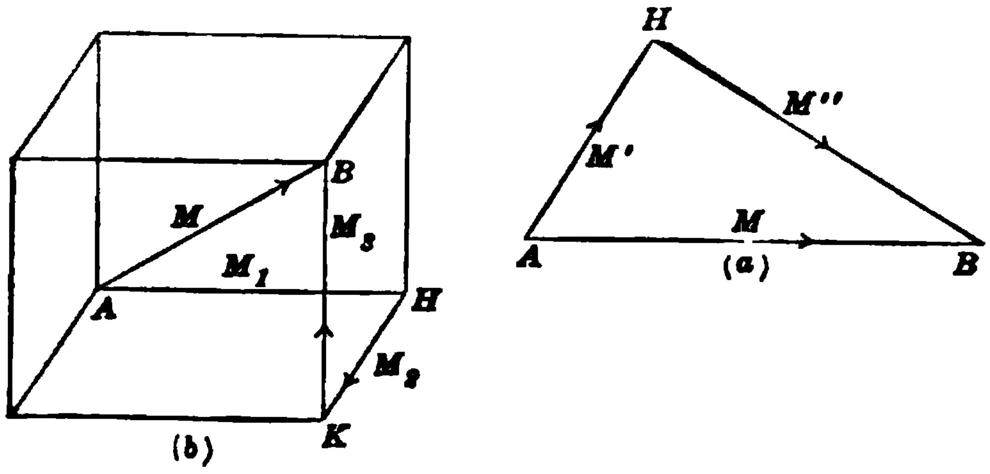


Fig. 18

36. Resolution of a force into a force and two couples. A single force acting on a body, may be resolved into two couples and an equal single force acting at any point on the given body that may be chosen.

Let the point A (Fig. 19) be the point of action of a given force F. It may be resolved by the introduction of two balancing forces $+F$ and $-F$, at B, into an equal parallel force at B, and a couple Fy, right-handed as seen from $+X$, as shown in the figure. Still more, the force $+F$ at B may be itself resolved into an equal parallel force $+F$ at O, and a couple $(-Fx)$ (which is left-handed seen from $+Y$). We have thus resolved a force, acting at the point (x, y) and parallel to OZ, into an equal force acting at O, and two couples; one having a moment $+Fy$ about the X-axis, and one having a moment $-Fx$ about the Y-axis.

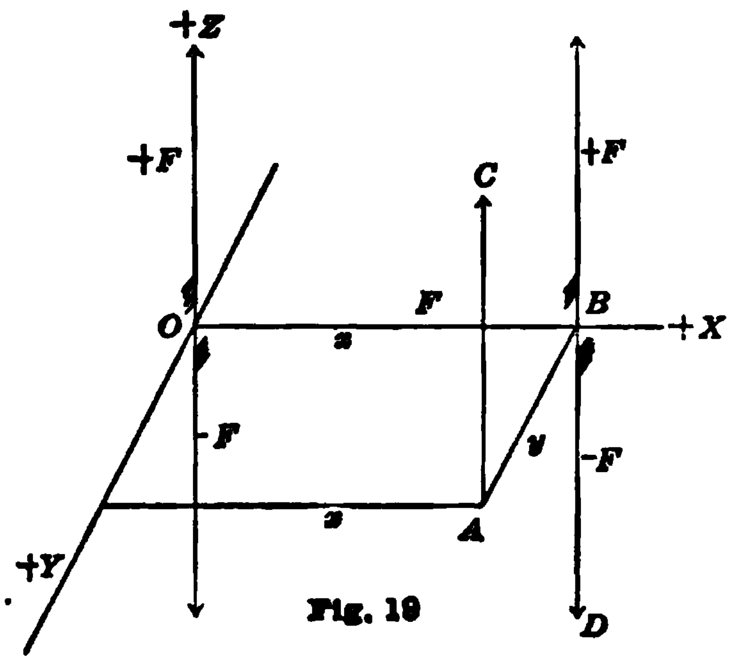


Fig. 19

The use of this method of resolution will be illustrated by several examples in Chapter III, and will be freely employed in subsequent chapters.

CHAPTER III.

Parallel Forces Which Balance.

37. Three parallel forces balance. Since they could not balance, if all acted in the same direction, the largest must act in a direction opposite to that of the other two; hence we must have

$$F_1+F_2-F_3=0$$

or, numerically,
$$F_3=F_1+F_2. \tag{1}$$

If we resolve F_3 into two forces respectively equal to F_1 and F_2, we have two couples which must balance each other. Since they could not balance if they were not co-axial, and since the couples cannot be co-axial unless the forces are co-planar, we reach one im-

portant conclusion: that **three parallel forces cannot balance unless they act in the same plane.**

Since the couples balance, their moments have equal magnitudes and opposite signs.

Ex. Let the three forces be represented by a scale drawing, Fig. 20. As we look down upon the drawing, F_1b is right-handed, and F_2a is left-handed. Hence $F_1b - F_2a = 0$,

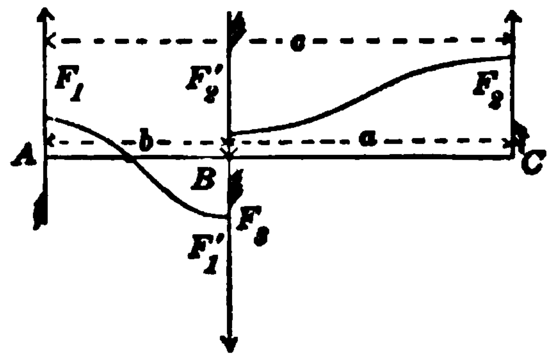


Fig. 20

$$\text{or } \frac{F_1}{F_2} = \frac{a}{b} \tag{2}$$

or the *forces* (of the balancing couples) are inversely as their *arms.*

If we let the distance between the forces having the same direction be known as c, we have in this case

$$c = a + b$$

Now erect an axis at A, and take moments. We have at once

$$F_3b = F_2c, \text{ hence } \frac{F_2}{F_3} = \frac{b}{c} \tag{3}$$

If we take C as our axis, and take moments, we have

$$F_1c = F_3a \text{ hence } \frac{F_1}{F_3} = \frac{a}{c} \tag{4}$$

Combining (2), (3), and (4) we have

$$F_1 : F_2 : F_3 = a : b : c; \tag{5}$$

a very useful *triple proportion.*

38. Deductions. From the above discussion several important conclusions follow. When three parallel forces acting on the same rigid body balance, we know:

1. The three lines of action lie in the same plane.
2. Their algebraic sum is zero.
3. The line of action of the largest force lies between the lines of action of the other two.
4. The magnitude of each force is proportional to the distance between the other two.
5. In the language of algebra

$$\begin{gathered} F_1 + F_2 + F_3 = 0 \\ F_1 : F_2 : F_3 = a : b : c \\ a + b = c \end{gathered} \tag{6}$$

When a body acted upon by three parallel forces is at rest, we know that the above equations must be true.

Problems.

1. Given two parallel forces 18 lb and 40 lb, 8 feet apart. Find the balancing force, and the distances between it and them.

2. Given the parallel forces $P=+80$ lb, $Q=-64$ lb, and the distance between them, 12 feet. Find the balancing force, and its position.

3. Three parallel co-planar forces act on the same straight bar at given points A, B, and C, and balance. The force at $A=35$ lb. Find the forces at C and B.

Fig. 21

Use the proportions in equation (6) for magnitudes.

4. Suppose the force A makes an angle of 30° with the bar. How does the obliquity affect the result?

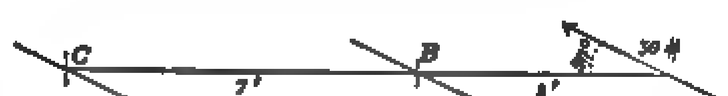


Fig. 22

39. The resultant of two parallel forces. Since F_1, F_2, and F_3 balance, it is evident that one must balance the combined action of the other two. Now the single force which is equivalent to the combined action of two or more forces is called the "Resultant" of those forces. In the present case the resultant of two must be equal and directly opposite to the balancing force*. Hence if we know all about the third or balancing force, we know all about the resultant.

Thus, if P, Q and S, acting on the same body, balance, the resultant of P and Q is directly opposite to S; and the resultant of P and S is directly opposite to Q.

The student should adapt the equation (6) to this case; and write deductions as to the *resultant* as was done for the balancing force in the last section.

40. Graphical solution for the resultant of two parallel forces. The following elegant solution is taken from Rankine's Applied Mechanics.

Fig. 23

Ex. Given, in Fig. 24, two parallel forces, F_1 and F_2, fully represented in magnitude, relative position, and direction by AP and BQ. To find their resultant graphically.

Solution. 1. Draw thru P and B lines parallel to the (imaginary) line AQ. 2. Draw thru A and Q lines parallel to the (imaginary)

* This justifies Rankine's definition: "The resultant of a system of forces is the force which balances the single force which balances the given forces."

line PB. The intersection of the new lines thru P and Q will give the arrow head of the resultant; and the two lines thru A and B will intersect at the tail end of the resultant, which is therefore CR.

In (*a*) Fig. 24, the given forces act in the same direction: in (*b*), they act in opposite directions.

The geometrical proof of the correctness of these solutions is left to the student. He should prove in each case that CR fulfills all requirements of *magnitude, direction and position.*

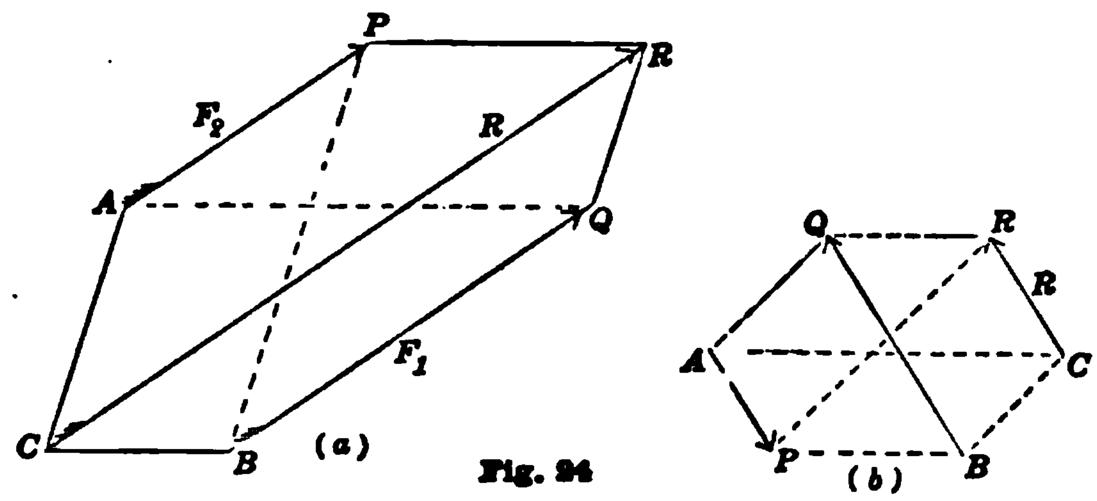


Fig. 24

41. The lever. Case I. Suppose a rigid beam carrying a hook at each end is balanced on a knife edge, *i. e.*, the sum of the pulls of the earth on the beam and hooks is equivalent to a single pull directly over the knife edge. This pull (or weight) is balanced by the upward lift of the knife edge, or "*Fulcrum*," so that when we come to consider the action of other forces on the beam, we can completely neglect the weight of the beam itself.

Let us represent, Fig. 25, the distance from hook to hook by l, and the length of the two arms by a and b respectively.

Suppose F_1 and F_2 are applied to the hooks and that they balance. We have at once

$$R=F_1+F_2$$

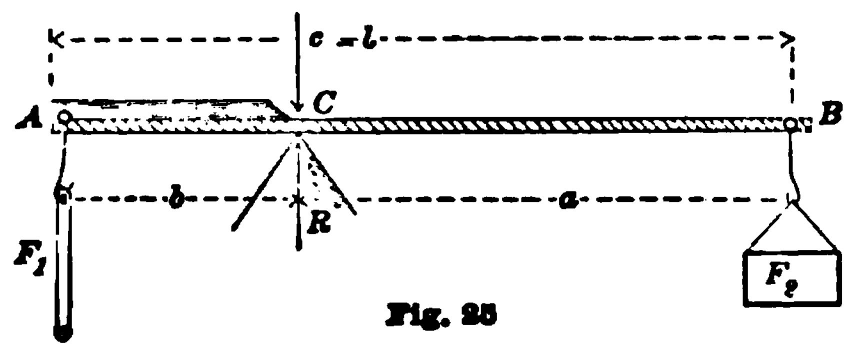


Fig. 25

which must be the additional support given by the fulcrum —to balance F_1 and F_2.

From the proportion equation derived from (6)

$$a : b : l = F_1 : F_2 : R$$

$$a = \frac{F_1 l}{R} = \frac{F_1 l}{F_1+F_2}, \text{ and } b = \frac{F_2 l}{R} = \frac{F_2 l}{F_1+F_2}.$$

Again, if F_1, F_2, and a are given, we have as before, $R=F_1+F_2$, and from the proportion

$$\frac{F_1}{F_2} = \frac{a}{b}, \quad b = \frac{F_2}{F_1} a$$

and $l=a+b$.

If F_1 and the two lengths, a and b, are known

$$\text{then } F_2 = \frac{b}{a} F_1$$

$$\text{and } R = \frac{a+b}{a} F_1.$$

This case was formerly spoken of as a "lever of the **first class.**"

42. Case 2. Let the lever AB and its hooks be balanced on C by means of a counter weight D, so that the weight of lever and counter-weight may again be neglected. See Fig. 26. Let F_2 be a known force acting down as shown at a distance a from C. An unknown force acts up at B whose distance from F_2 is c. We are to find the force F_1 and the additional load put upon the support at C.

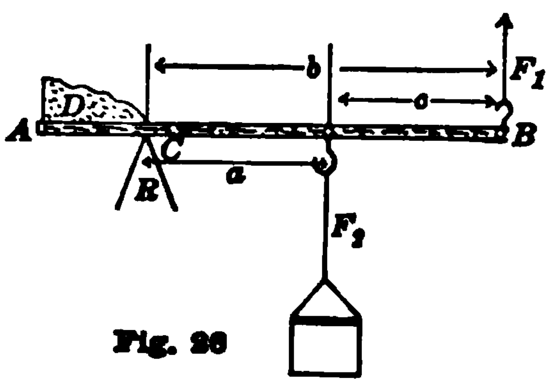


Fig. 26

It is evident at a glance that the turning moment of F_1 about a horizontal axis at C is negative (left-handed), and equal to $F_1(a+c)$. It is also evident that the turning moment of F_2 about the same axis is right-handed, and equal to F_2a.

These two moments must balance, hence

$$F_1(a+c) = F_2a$$

so that

$$F_1 = \frac{a}{a+c} F_2.$$

But $a+c=b$ as defined for the lever, so that we have

$$\frac{F_1}{F_2} = \frac{a}{b} \text{ as in formula (6).}$$

The resultant of F_1 and F_2 must be equal to their difference and it must act opposite to the support of the fulcrum, as the latter is the balancing force. Hence the fulcrum action due to F_1 and F_2 is

$$R = F_2 - F_1$$

Fig. 26 illustrates what was once known as a "lever of the **second class.**"

43. Case III. **A lever of the third class** is illustrated by Fig. 27.

Ex. 1. A man, with a pitchfork six feet long, supports at the far end a mass of hay weighing 24 lb. One hand is at A, acting down, and the other hand at B, 18 inches away, is acting up. What are the

forces at A and B if the handle is horizontal and all actions are vertical?

The hand at A may be thought of as the fulcrum.

Ex. 2. In the above example, for pitchfork substitute fish pole; for 6 feet, put 20 feet, and for hay weighing 24 lb, put a fish weighing 5 lb. Find the forces at A and B, capable of just supporting the weight of the fish.

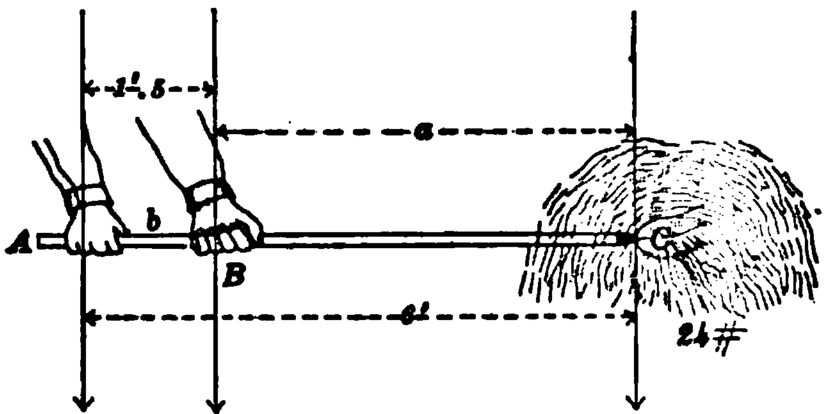


Fig. 27

44. A Word about Friction. Whenever the surface of one body is in contact with the surface of another, there is always a more or less effective interlocking of the minute particles of one body with the particles of the other, so that they are able to resist separation by sliding. This ability to resist sliding (or relative motion) is variously called *adhesion*, *cohesion*, *sticking*, or *friction*. The term *Friction* is used in Mechanics as the name of the force with which the bodies in contact resist a mutual sliding.

There is, of course, a limit to the ability of material in contact to resist a sliding motion. When the external forces tending to produce sliding are so great that the limit of frictional resistance is passed, then sliding takes place.

Fig. 28 represents a "rough" heavy body resting upon a horizontal plane which is not smooth. The body is stationary in spite of the tension T in a rope which *tends* to draw it sideways. As the body A stands still, the slender blocks (b) stand on end in equilibrium. The friction between A and B is sufficient to balance T. Between A and B there are two different actions both distributed over the surface of contact, one *normal* or vertical, one *tangential* or horizontal. Each of these forces is represented by two arrows. The two arrows on A show how A is acting on B; the two arrows on B show how B is acting on A. It is evident that B's horizontal action on A is equal to the horizontal action of the rope T, and in the opposite direction; hence they balance.

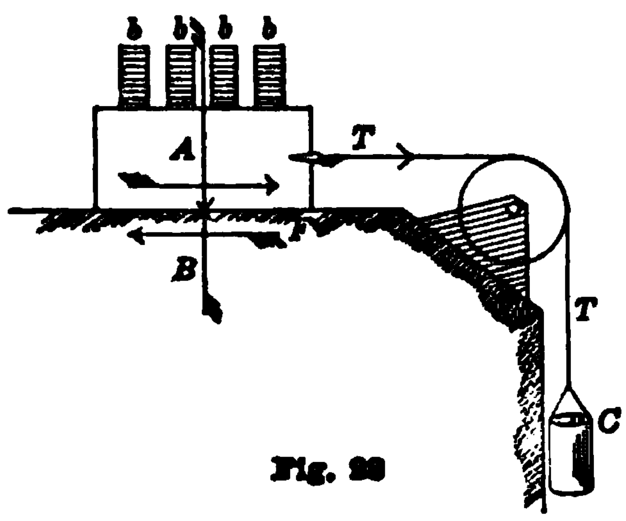


Fig. 28

Let F denote the friction shown by the horizontal arrow on A or B. Since A stands still we have

$$F = T$$

If more shot or pebbles are put into C, T will be increased, and of course F will be increased also, *as long as A stands still.* When T increases to a certain point, A starts and slides. F has reached its limiting value F_1, and the *difference* between T and F_1 makes A move and perhaps causes the slender blocks to fall over.

In statical problems the limit of frictional ability is not reached. The laws which govern friction will be discussed later when motion is studied, but it is well, at this point, to remark that the frictional ability of surfaces in contact depends upon several things: the normal action (pressure) between the two; the character of the surfaces (rough, polished, or lubricated), the temperature, etc. In Mechanics, the word "smooth" is used only in cases of ideal bodies where the friction is zero.

45. Applications. CASE I. Show how the action of steam on the piston of a locomotive engine has a tendency to move the loco-

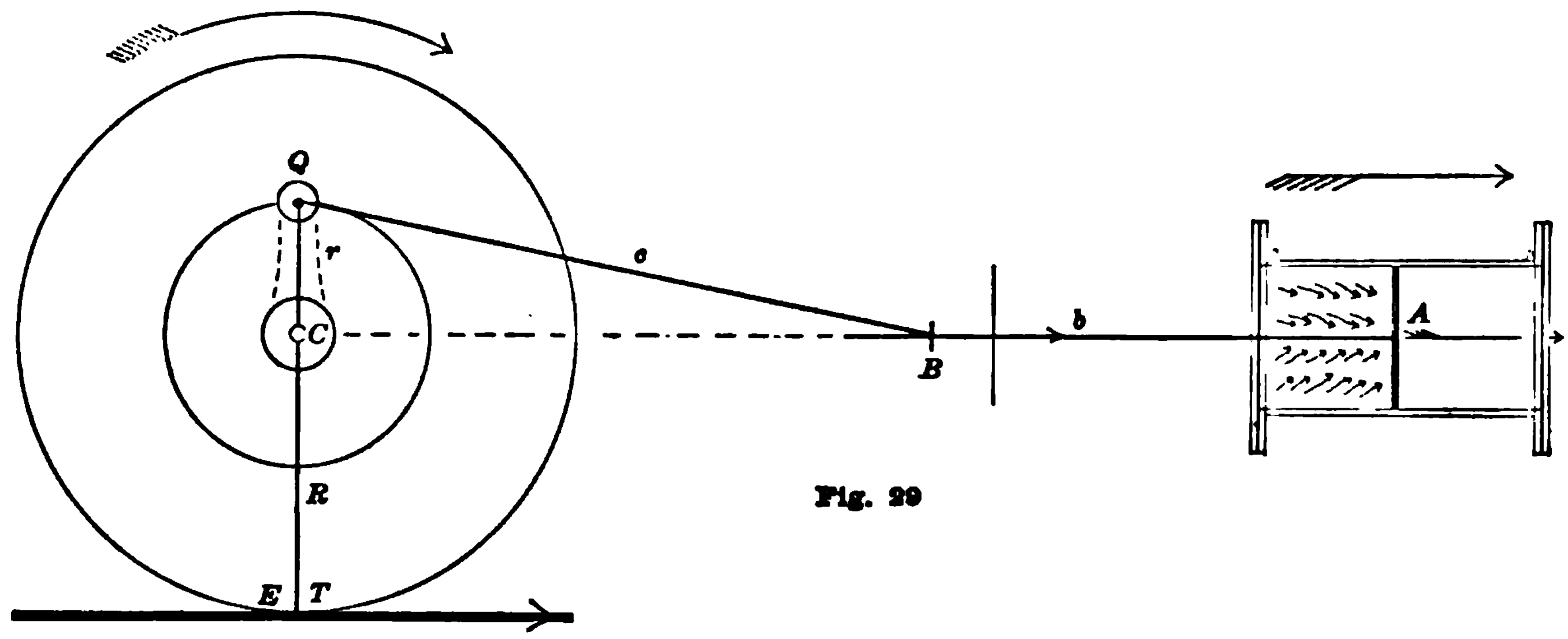


Fig. 29

motive; *first, when the crank pin is above the axle* of the driving wheel (Fig. 29).

Solution. 1. Suppose the steam at high pressure is as shown in the cylinder. It presses on the piston towards the right and equally on the cylinder head towards the left. The cylinder is bolted to the frame, so the action of the steam on cylinder head is to push or move the frame of the engine *to the left.* If the area of the piston face is A square inches, and the steam pressure per square inch is p lbs., in excess of the "*back pressure*" on the other face, the force tending to *move the frame to the left,* or backwards, is Ap.

The steam pressure, Ap, upon the piston is transmitted thru the piston rod b to the crosshead B, thence thru the connecting rod c to the crank-pin Q. Omitting for the present the small amount of frictional resistance offered:—by the cylinder upon the circumference

of the piston, by the packing in the cylinder-head upon the piston-rod; and by an upper guide D upon the cross-head,—it is clear that the horizontal force acting *to the right* upon the crank-pin at Q is Ap.

The crank and wheel form a *rigid lever* which (omitting the pull of the earth which is balanced by the vertical action of the track) is acted upon by *three* forces which balance. (See Fig. 30.)

First, The force Ap on the crank pin;

Second, The *frictional force* of the track upon the circumference of the wheel at E, which force we will call T. This force prevents slipping.

Third, The *reaction* of the frame thru the journal box at C upon the axle, which reaction must equal the sum $Ap+T$. Thus we see that the *direct action* of the *axle upon the frame* is the *resultant* of Ap and T, which is $Ap+T$ *acting towards the right.* We found in the first paragraph of this solution that the action of the steam upon the cylinder-head tended to move the engine frame of the locomotive to the *left*, by a force of Ap pounds. We have in the last sentence found that the action of the axle of the driving wheel tended to move the frame to the *right* by a force of $Ap+T$ pounds. The algebraic sum of these horizontal forces is

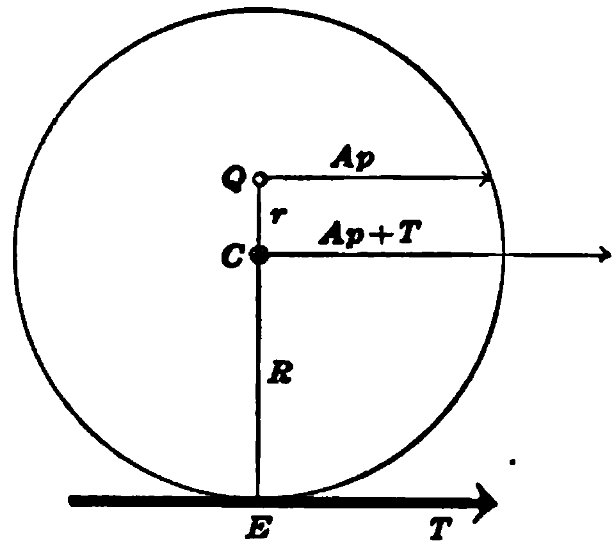


Fig. 30

$$Ap+T-Ap=T$$

which is exactly the horizontal action or *friction* between the track and the wheel.

We shall find the numerical value of T by solving the problem of the lever consisting of the radius R and the crank r. By the law of the lever

$$\frac{Ap}{T}=\frac{R}{r} \text{ or } T=\frac{rAp}{R}$$

which is the measure of the resultant forward action of the steam in our cylinder upon the locomotive; in other words, this is the force which tends to move the locomotive, and which *will* move it unless some force not considered above holds it. The student must not fail to see clearly that the action T of the *track upon the wheel* is *forward*, and hence the locomotive moves forward.

46. Second, when the crank is below the axle. CASE II. See Fig. 31.

1. In this case the action of the steam upon the *cylinder head* and thru it upon the frame of the locomotive, is Ap *to the right.*

2. The action of the steam upon the *piston* puts compression or thrust into the piston-rod and into the connecting-rod, and has a *tendency* to move the *crank-pin towards the left.*

3. As before, the rigid body of wheel and crank forms a lever,

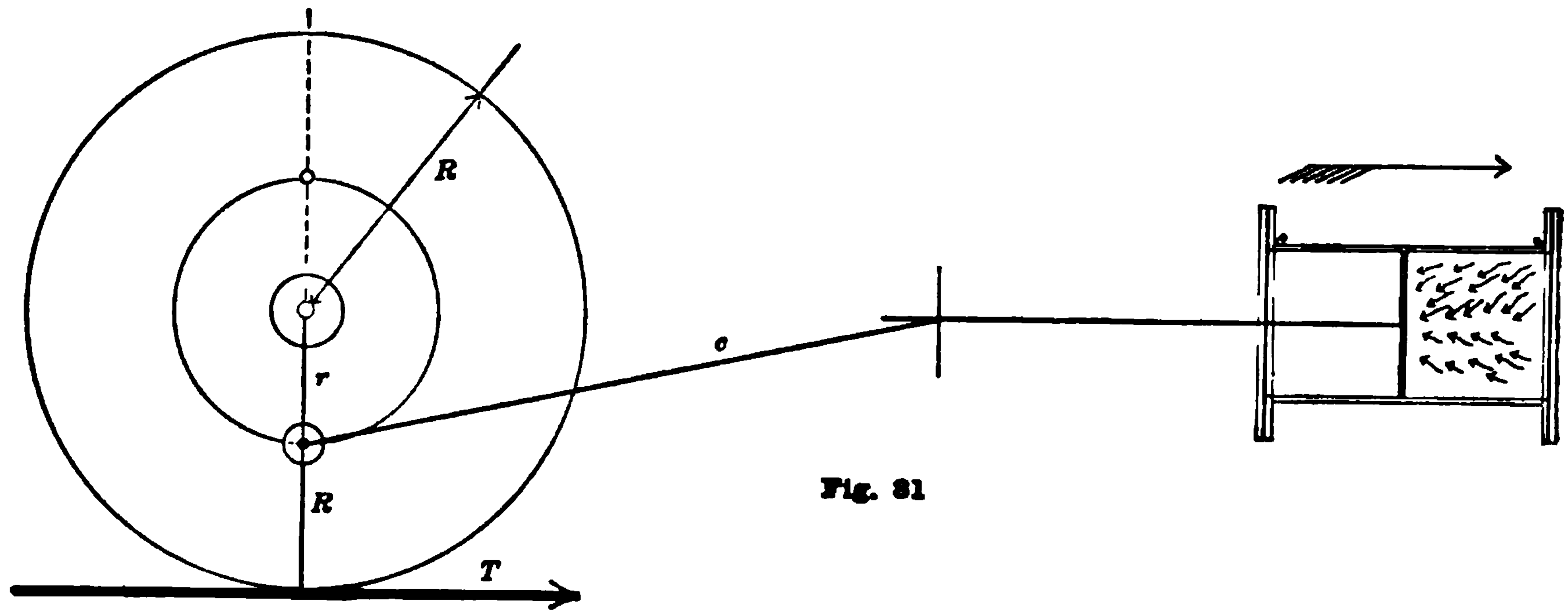


Fig. 31

with two parallel forces, Ap and T *acting in opposite directions.* Fig. 32. Their resultant acts *towards the left* thru the axle upon the frame, namely, $Ap-T$. (See the lever of the third class), **42.**

4. The sum of the actions upon the frame is Ap on the cylinder-head, which is *positive* to *the right,* and $Ap-T$ which is *negative* to the *left.* Their resultant Action is:

$$Ap-(Ap-T)=T,$$

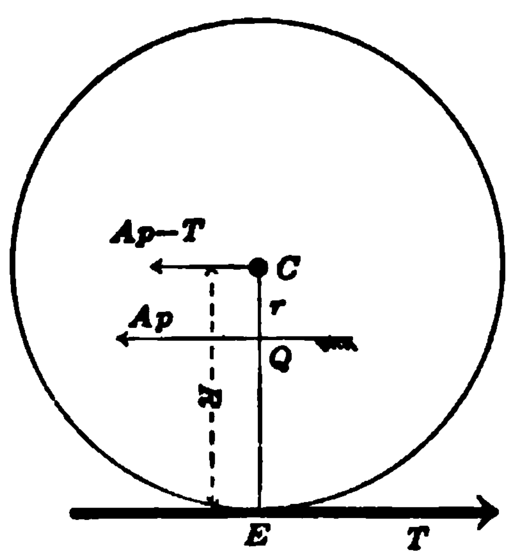


Fig. 32

and T is *just the same as before.* This shows that the steam is urging the locomotive forward *just as much* when the piston rod is pushing back, as when it is pulling forward.

In the foregoing discussion it is very important to keep in mind the vast difference between the action of the *Frame upon* the driving *axle* (thereby making a balancing force with T and Ap), and the action of the *axle upon the frame* thereby being the resultant of T and Ap. It is this *resultant* which counts in the discussion.

47. Remarks. From the value of $T=\dfrac{rAp}{R}$ in each case, it appears:—

1. That the higher the steam pressure p, the greater the traction.
2. The larger the cylinder cross-section, the greater the traction.
3. The longer the crank-arm, the greater the traction.
4. The smaller the radius of the wheel, the greater the traction.
5. Since T is limited by the frictional ability of the surfaces in contact (wheel and rail) which ability depends, as everybody knows,

upon the load on the wheel, and the character of the surfaces (clean or sanded, wet or dry, cold or hot, greasy or icy, etc.) there is in every case a greatest possible value of T, which we will call T_1.

Since the value of T_1 is made greater by increasing the roughness or "biting" ability of the rail, the engineer, when he needs greater traction, operates a device which sprinkles sand upon the rail. But even then there is a limit to the value of T_1.

Hence the greatest value of the steam pressure is

$$p_1 = \frac{RT_1}{Ar}.$$

If the engineer attempts to increase T_1 by increasing p_1, the wheel will *slip*, and as is well known the traction at once falls below T_1. However, the traction can be increased by the addition of one or two driving wheels, each a duplicate of the first and each carrying, approximately, the same load. The crank or cranks must be similarly placed, and the new connecting-rod, or rods, from wheel to wheel, must always be horizontal. The action of the main connecting rod is then distributed between the wheels, thus permitting a greater steam pressure without causing the wheels to slip. T is then ΣT.

Problem.

Let the student make skeleton drawings of a horizontal engine driving a paddle-wheel steamboat, and then write out analyses similar to the above.*

The problem in the text is introduced to show that the principles of the lever enable one to explain how steam drives a locomotive. The formula found holds for a *single* cylinder and *only when the crank pin is directly above* or *below the axle.*

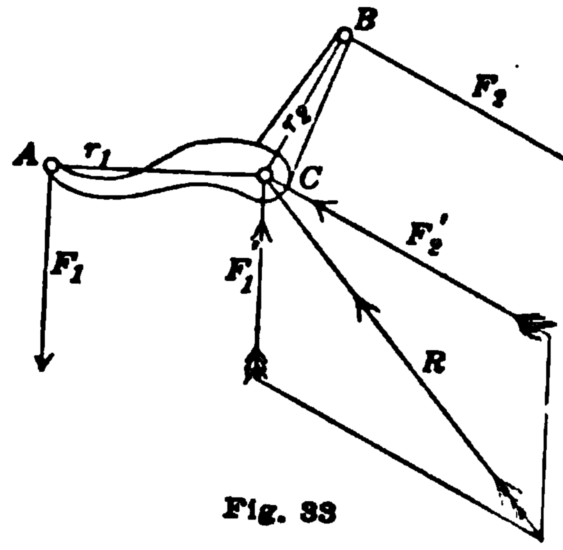


Fig. 33

46. Bent Levers. Bent levers are analyzed as are straight levers by the use of moments. A lever (without weight) acted upon by forces F_1 and F_2, with lever arms r_1 and r_2 respectively, is in equilibrium if the moments F_1r_1 and F_2r_2 are equal, *provided* the axle at C is properly supported. The angle ACB may have any value.

* The captain of a Mississippi steamboat once tried to convince the author that his engine propelled the boat only when it was pulling forward with the crank pin above the shaft. When the crank was below the axle and the steam was pushing back, he thought the engine was a hindrance and not a help. He knew so little of the principles of Applied Mechanics that no reasoning could convince him that he was wrong; but his boat went ahead with *every stroke* just the same.

Hence $\frac{F_1}{F_2} = \frac{r_2}{r_1}$ or the forces are inversely as the arms, as in the case of a straight lever.

The Support reactions. The reactions upon the fixed axis of a bent lever must not be overlooked. In the last figure the force F_1 calls into being the equal acting force $F_1' = F_1$; and the force F_2 in like manner brings into play the action $F_2' = F_2$. (See **34.**) The resultant of F_1' and F_2' (see Chapter V.) is the force

$$R = \sqrt{(F_1^2 + F_2^2 - 2F_1F_2\cos ACB)},$$

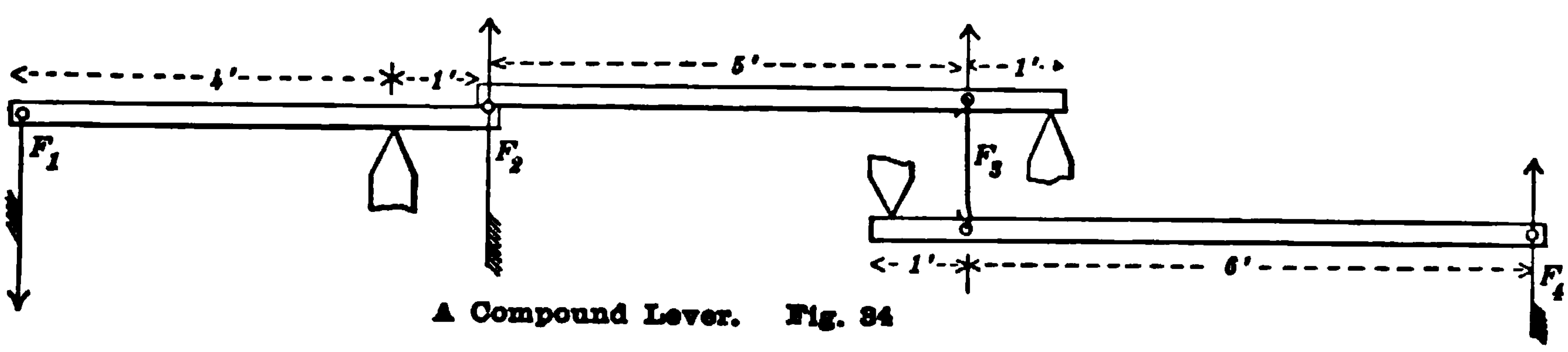
which *must be balanced* by the *reaction* of the journal box of the axle. The magnitude and direction of this supporting force must never be ignored.

47. Compound levers. The chief use of levers is to transmit and modify force. A force is applied to a lever at A, and the result is a very different force applied to a body at B. A very small force at A may produce a very large force at B, given a strong lever and a sufficient support as a fulcrum, and there is no limit to the ratio of the force at B to that at A. Hence, the first great Engineer of history, Archimedes, said:—"Give me a spot whereon to rest my lever [that is, a fulcrum] and I will move the world."

1. It is often necessary or at least profitable to separate a ratio into the product of two or more ratios, which means that one lever may be made to operate a second lever, and, if needed, the second may operate a third. Thus: Fig. 34.

It is evident that

$$F_2 = 4F_1; \; F_3 = 6F_2 = 24F_1; \; F_4 = \frac{F_3}{7} = \frac{24}{7} F_1.$$



A Compound Lever. Fig. 34

The first lever is of the "First Class"; the next of the "Second Class," and the last is of the "Third Class." In every analysis the fulcrum should be taken as the moment axis. A compound lever may be designed to make the ratio F_4 large or small. In Fig. 34 the levers must be so proportioned as to balance when F_1 is zero, so that their weight may be neglected without error. The same will be true of those examples which follow unless otherwise accounted for.

2. Fig. 35 is equivalent to a compound lever consisting of three simple levers. The student will readily see that, assuming no friction and no stiffness in any rope (a very violent assumption):

$$\frac{W}{F_1} = \frac{R_1}{r_1} \cdot \frac{R_2}{r_2} \cdot \frac{R_3}{r_3}$$

That is, the action of W is exactly balanced by the action of F_1, thru the intervening cords and pulleys. If there be friction at the axles, or stiffness in the cords, then the weight W will remain in equilibrium in spite of some departure from the value given by the formula:

$$F_1 = \frac{r_1\ r_2\ r_3}{R_1 R_2 R_3} W.$$

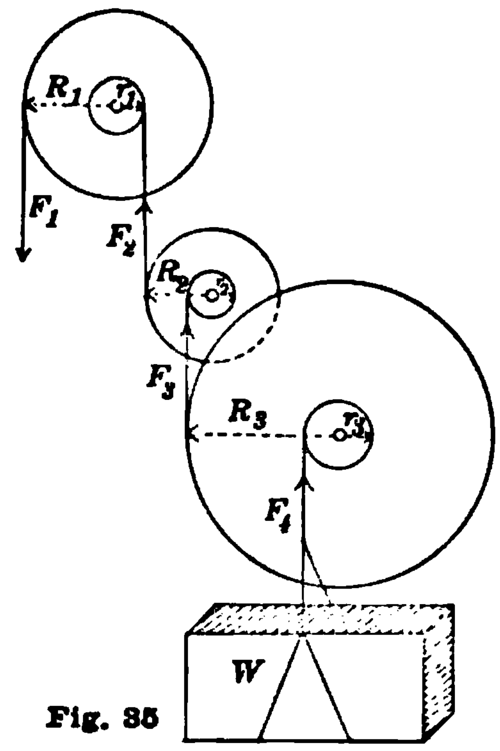


Fig. 35

In Fig. 35, all pulleys and drums are mounted on fixed supports. In the same way gear wheels with their axles act as levers in a train of wheel work. Combinations of pulleys with fixed and with movable blocks will be discussed later in the book.

Problem.

48. Pullman Car Truck. In an ordinary Pullman car, one-fourth of the weight of the car body and contents is brought to each set of three wheels on one side of a truck. It is then by a system of levers distributed in such a way that each wheel carries the same load, viz.: one-twelfth of the entire weight of the car.* The student should inspect a truck, and by means of a drawing explain how this is done.

Testing Machine of St. Louis Bridge.

49. A fine example of compound levers is found in the testing machine designed by Col. Henry Flad and used by Capt. James B. Eads in testing the strength and elasticity of the materials which were put into the great Eads Bridge at St. Louis. The arrangement of the levers is shown in the "figured" drawing, Fig. 36.

This machine was capable of exerting a tensile or compressive force of 100 tons, by means of fluid pressure in a thick iron cylinder, and a system of compound levers.

* A car "truck" is the low steel carriage with six wheels upon which one end of the car-body rests.

When a specimen of steel was to be tested in tension it was coupled in between the two pins shown in the horizontal section near the figure (6″) in Fig. 3. The draw-bar, or piston-rod, is of steel 3 inches in diameter; it passes into the cylinder and is screwed into the plunger or piston. Liquid, glycerine and water, was forced into the cylinder by a force pump operated by hand. The plunger was 7¼ inches in diameter, and a liquid pressure of 6,000 lbs. per square inch gave a total of 100 tons. The cylinder itself was held in place by large lugs between blocks on the frame of the machine.

When a specimen was to be tested in *compression* it was inserted between the parallel steel faces on either side of the letter E. When the plunger was forced out it pushed the specimen against the outer cross-head (on the left) which brought the four long tension rods into action; they transferred their joint action to the short arm of the first lever, as was done directly in the former case.

The extreme delicacy of the apparatus is shown by the fact, reported by Capt. Eads, that when a specimen was under a stress of nearly 100 tons, a common lead pencil dropt into the can at K would cause the long arm of the last lever to fall; and upon the removal of the pencil the arm would rise again. Every lever turns on a steel "knife-edge" which acts as a fulcrum.

The first lever is a bent lever of the first class. Its short arm is 10¼ inches; its long arm is 9 feet 4¾ inches; so that the ratio is 1 to 11.

The second lever is of the third class. The short arm is 12 inches; and the long arm is 13 feet. Hence the ratio is 1 to 13.

The third lever is a simple straight lever of the first class. Its short arm is 7 inches, and the long arm is 8 feet 1.86 inches, so that the ratio is 1 to 13.98.

All the levers were balanced so that the third lever, which should oscillate between very narrow limits, stood horizontally when there was neither tension nor compression in the specimen. A suspended pan shown at the end of the third lever was in great part the balancing force for all levers.

When the liquid used to drive the piston or plunger in the thick cylinder was applied, the action began upon the specimen, which had been inserted into the machine. The balance of the levers was maintained by pouring fine shot into the pan, and the weight W, of the shot, having been carefully weighed in an accurate balance (such as is used in a chemical laboratory) furnished the means for measuring the tensile and compressive stress applied to the specimen. The formula for such a calculation was:

If P denotes the required total stress

$$P = 11 \times 13 \times 13.98 \times w = 1999.2\ w = 2000\ w, \text{ very nearly.}$$

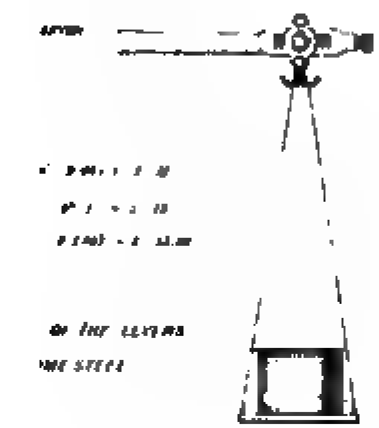


FIG. 3 HORIZONTAL SECTION

If w was 100 pounds, P was 100 tons, which was regarded as all that it was prudent to apply to the machine.*

Suppose a steel cylinder 8/10 of an inch in diameter is under tension, and that the shot dropped into the can, to produce a balance, weighs 8 lbs. and 3½ ounces. What is the tension *per square inch* of the cross-section in the specimen?

47. Another warning. The student who has seen, in what has been shown, how a small force at one point can by means of levers produce or balance a large force at another point, must not jump to the conclusion (as so many self-styled inventors have done) that the increase in the force thus produced is all clear gain. Levers are of practical use, as abundant experience shows, but they do not enable one to get something out of nothing.

Levers are generally used to move things, to lift heavy loads, to do work. We shall discuss *motion* later on, but it is timely to say now, that when motion results, there is full compensation for any apparent gain in force by a loss of motion; and a balancing loss in force for any gain in distance moved.

Ex. For example, if a force of 20 lbs. (see Fig. 37) can balance a weight of 1,000 lbs., which is 50 times as great, the body pulling down on the rope at P must pull the rope down 50 feet in order to raise W one foot.

See chapter XXI on Work and Energy.

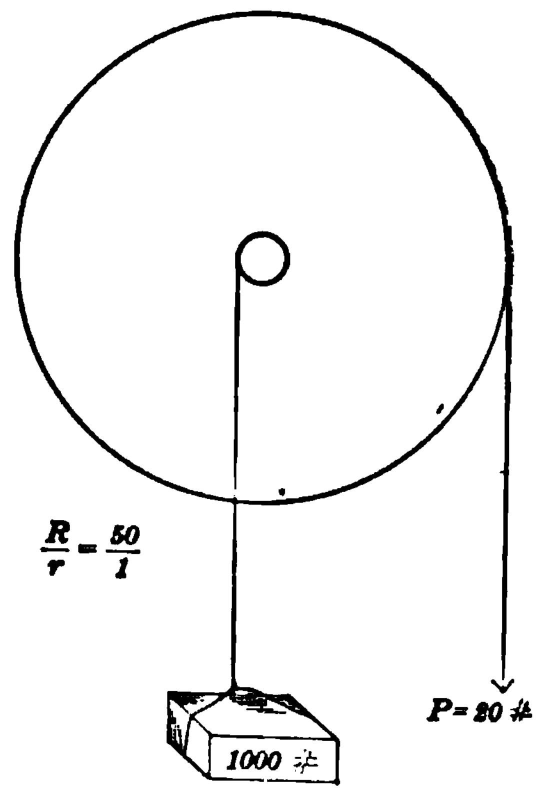


Fig. 37

48. The Relation between any number of Parallel Co-Planar Forces acting upon a body in equilibrium. Since the forces balance, there can be no *resultant* tendency to move the body in any direction or to turn it around. Hence, the algebraic sum of the forces must be zero; and the sum of their moments with reference to an axis erected at *any point* in their plane must also be zero. Hence, the *Equations of Equilibrium.*

$$\left.\begin{array}{l}\Sigma F = 0\\ \Sigma M = 0\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

* This elegant machine is still in use in a St. Louis Manufacturing establishment. Figure 36 is reduced from a lithographic plate in the author's "History of the St. Louis Bridge."

For further description of the machine, and the elegant method employed for determining elongation, modulus of Elasticity, and Elastic limit, devised by Chancellor Chauvenet of Washington University, the reader is referred to the History, published in 1881.

In point of fact the balancing forces can always be resolved into a number of separate couples which balance.

1. Example. Let the forces be $F_1 \ldots\ldots\ldots F_4$ in Fig. 38. The force $F_3=16$ can be resolved into $8+4+4$, which, taken with $F_1=8$, $F_2=4$, and $F_4=4$, form three couples whose algebraic sum is evidently zero, just as $\Sigma M=0$ for any axis as O.

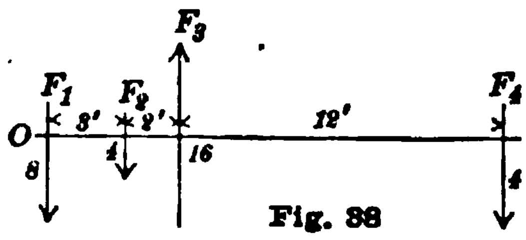


Fig. 38

2. Example. Six parallel co-planar forces act upon the bar AB, as shown in Fig. 39. What and where is the single force which will balance them? Take the line AB as the axis of X, and any point on it as O, as an origin. Let the unknown force be P and its distance *to the right of* O be x. Substitute in the Eq's of Equilibrium, and solve. Let forces acting down be positive so that $(+F)\ (+x)=+Fx$ a right-hand moment.

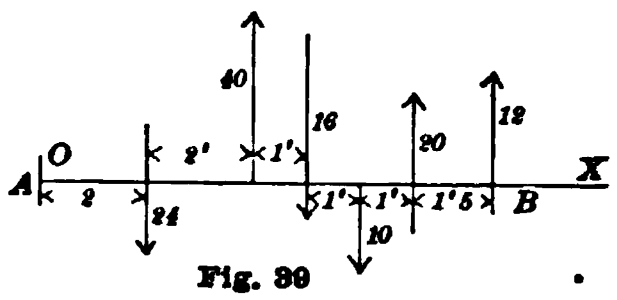


Fig. 39

3. Ex. Suppose we have a rigid frame $ABCD$, lying on a smooth horizontal plane (or floating on the horizontal surface of water), so that its weight is balanced and left out of account. Suppose further that the frame is acted upon by a set of parallel forces all in a horizontal plane, and that they balance. To make the situation seem concrete, let every arrow in Fig. 40 represent the tension in a wire which is fastened to a pin in the frame. Every wire is connected with a spring balance, which does not appear in the drawing. All dimensions are known, and all the forces are given with the exception of two: P and Q. These two are to be found.

Solution. Since the forces balance, there can be no resultant force in any direction, and no resultant moment. Hence we must have $\Sigma F=0$ and $\Sigma M=0$; in which summations both P and Q are included.

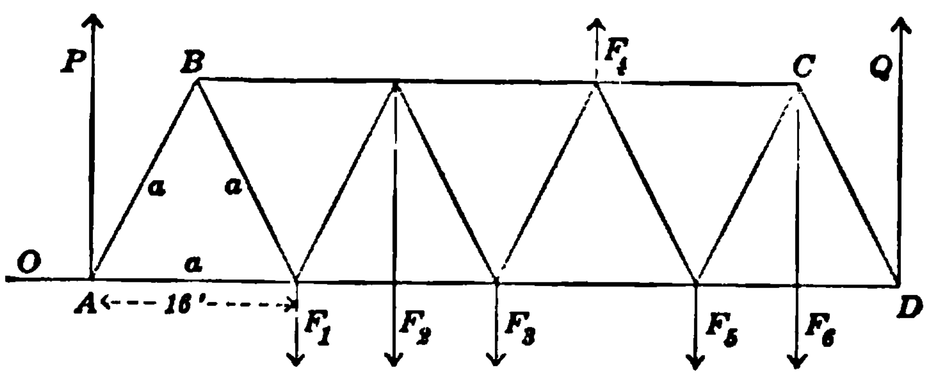


Fig. 40

If the moment axis is taken in the line of action of one of the unknown forces, that force will not appear in the moment equation, since a force cannot have a moment about an axis which it intersects. Hence, at first take the axis at A.

Let the triangles of the frames be equilateral with a side 16 feet long.

Let	$F_1=350$ lb	$F_3=100$ lb	$F_5=200$ lb
	$F_2=600$ lb	$F_4=-740$ lb	$F_6=400$ lb

The equations of equilibrium then become

$$350+600+100-740+400-P-Q=0$$
$$\text{or } P+Q=910; \text{ and}$$

$$16\times350+24\times600+32\times100-40\times740+48\times200+56\times400-64Q=0.$$

It is at once evident that this last equation is clumsy; a "half-panel" length could have been used as a unit of length instead of the foot, so that the "arms" would have been 2, 3, 4, 6, etc. Thus:

$$2\times350+3\times600+4\times100-5\times740+6\times200+7\times400-8Q=0$$
$$\text{or } Q=400\text{lb}$$

and therefore $P=510$lb

Let the student take his axis at D (thereby eliminating Q) and recalculate the value of P.

The student must not suppose that the equation $\Sigma(F)=0$, and *two* moment equations, arising from the use of two different axes, furnish three independent equations sufficient for finding three unknown forces. In fact, the moment equations are not independent.

49. Again, suppose the frame is a bridge truss (Fig. 41) in a vertical plane carrying half of a loaded roadway. Let it be assumed that all forces are vertical and coplanar, and that the loads are placed at the joints. Assume any convenient dimension for the sides of the triangles which are equilateral. Find the supports at A and B. It will be well to let every bar in the frame have the length 2. It will be seen that the *absolute* length has no effect upon the values of V_1 and V_2.

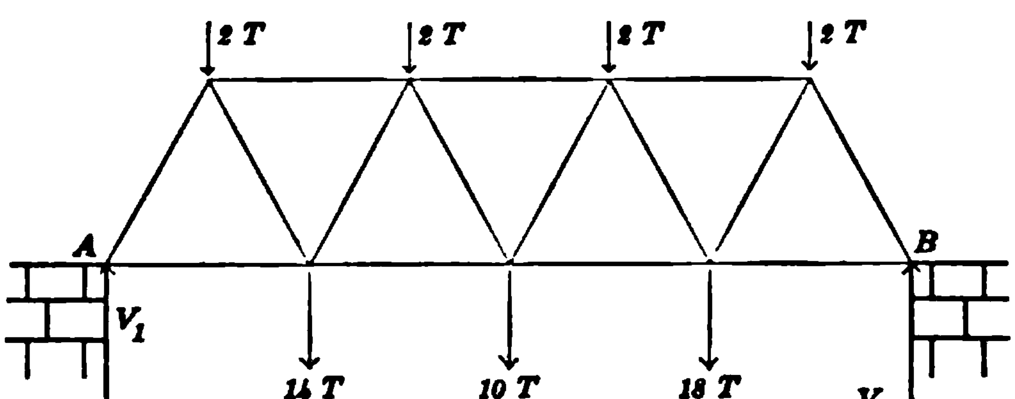


Fig. 41

Ex. 1. Figure ABC (Fig. 42) is a cantilever truss, designed to support a heavy load at C; and smaller loads at the upper pins. It is assumed to be rigid in a vertical plane.

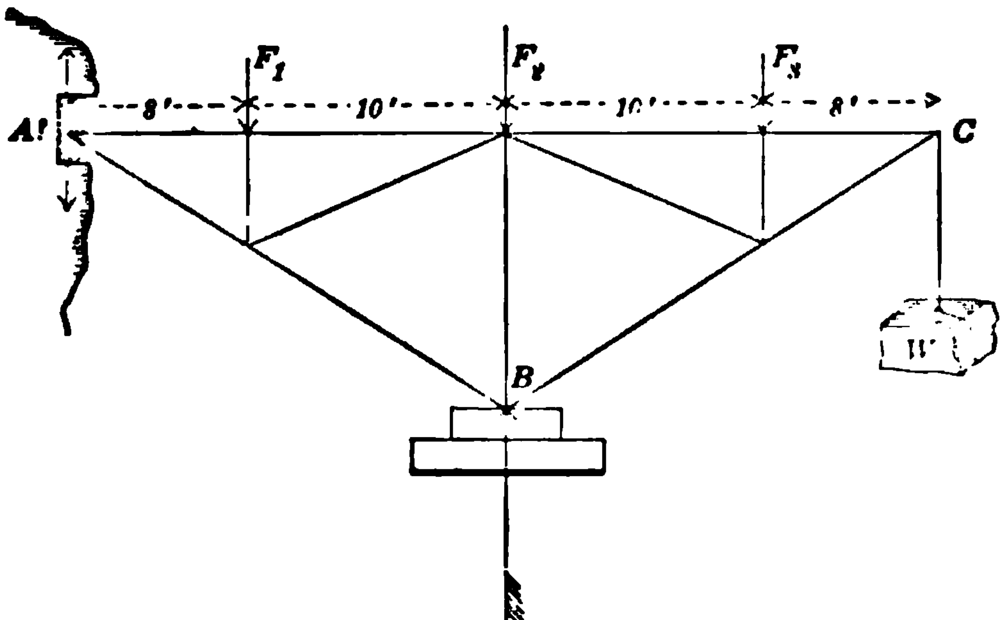


Fig. 42

Let $F_1 = 800$lb $F_3 = 500$lb $F_2 = 1,000$lb $W = 1,600$lb

Find the supporting forces at A and B.

Ex. 2. Let $F_1 = 1,200$lb $F_3 = 200$lb
$F_2 = 1,000$lb $W = 60$lb

Find the supporting forces at A and B.

50. To find the resultant of a set of parallel co-planar forces which do not balance. The *magnitude* of R is evidently equal to the algebraic sum of the given forces. Hence $R = \Sigma F$. Its *position* is found from the necessary fact that the moment of R about any axis is equal to the algebraic sum of the moments of its components about the same axis; that is: $Rx_r = \Sigma(Fx)$.

$$\text{or } x_r = \frac{\Sigma Fx}{R} = \frac{M}{R}$$

This is really the same as was shown in **18**, when we combined a force and a couple. Every moment comes from a couple, and in all statical problems that couple is always easily found.

Ex. 1. Suppose a floating platform is to be acted upon by a system of parallel forces in a horizontal plane which do not balance, and that we are to find the *balancing* force.

In order to find the required balancing force, or balancing couple, find the resultant; and then balance it. Let the platform, Fig. 43, be rectangular, and let the given forces all act parallel to the edge OY, and let their distances from OY be known The data and calculation may be tabulated as shown in the table. Each force is resolved into a force along OY and a couple, as explained in **34**.

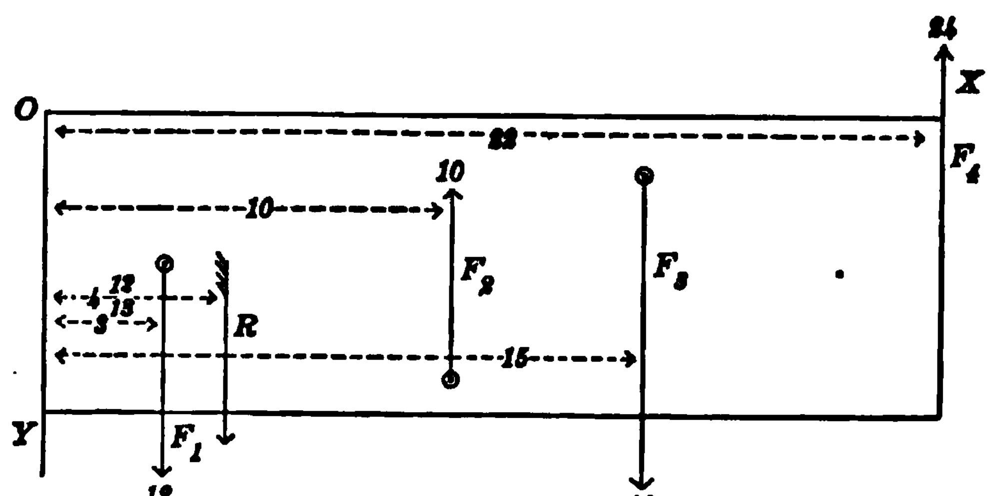


Fig. 43

The forces at O are then added, and the moments about a vertical axis OZ, are also added.

So that $R = \Sigma F$
and $M = \Sigma(Fx)$

The resultant force along OY is $+26$, and the resultant moment is $+128$. The resultant force and the resultant couple combined (see **33**) give a single force $+26$ at a distance from O equal to $M/R=4\frac{12}{13}$. Hence the single resultant of the given forces is a force of 26 lbs. acting in a positive direction and at a distance $x_r=4\frac{12}{13}$. It follows that the *balancing* force is 26 lbs, and that it acts in the *negative direction* along the line of R. If this force, F_5, were added to the given forces, the table would show complete equilibrium, since we should have $R=0$ and $M=0$.

F	x	Fx
$+12$	3	$+36$
-10	10	-100
$+48$	15	$+720$
-24	22	-528
$R=26$		$M=128$

Ex. 2. Find the resultant of the following system of forces applied to a similar body with opportunity for negative values of x. It will be found that the *resultant* is a couple, and that only a couple can balance the given forces. See Table.

F	x	M
$+40$	$+4$	
-16	$+6$	
-36	-2	
$+12$	-5	

51. Parallel forces in space. It is easy to conceive a rigid body acted upon by a variety of other bodies along parallel lines, with concentrated forces and lines of action so adjusted and proportioned that the body is at rest. In fact, every car that stands on the track, every house that stands on posts or columns, every table that stands on legs, every loaded wharf that stands on piles, furnishes an illustration of this proposition.

Since the forces balance, there must be no resultant tendency to move the body in any direction. This gives us one equation between the magnitudes of the forces:

$$\Sigma F=F_1+F_2+\ .\ .\ .\ F_n=0$$

In like manner, since there can be no resultant tendency to turn the body about any axis in space, we have another general equation involving both magnitudes and directions: $\Sigma M=0$, the axis of M being any line in space; more especially a line perpendicular to the given forces.

It was shown in **35** that a moment about any axis in space could be resolved into three component moments about three rectangular axes; that is

$$M=\sqrt{(M_1^2+M_2^2+M_3^2)}$$

if the axes referred to are OX, OY and OZ. These axes may be taken

at will. Let OZ be parallel to the common direction of the lines of action of all the forces. It follows that

$$M_3=0$$

inasmuch as a force can have no direct tendency to turn a body about a line parallel to its own line of action. There remains therefore $M=\sqrt{(M_1^2+M_2^2.)}$ Now, M cannot be zero, unless both M_1 and M_2 are zero. Hence the "Equations of Equilibrium" are

$$\left.\begin{array}{r}\Sigma(F)=0\\ M_1=0\\ M_2=0\end{array}\right\}$$

52. The choice of co-ordinate axes. In accord with what appears to be the best usage, in all space problems in this book, the axis OZ will generally be taken *vertical*; the axes OX and OY are so taken that positive rotation about OX looking from $+X$ towards O carries $+Y$ into $+Z$; and a positive rotation around OY carries $+Z$ at once into $+X$. It is well to note the sequence of letters and numbers: X, Y, Z; 1, 2, 3. Rectangular axes will accordingly stand thus: (Fig. 44). All the co-ordinates of P are shown positive. Arrows on the axes show how one *must look* at a couple, or moment.

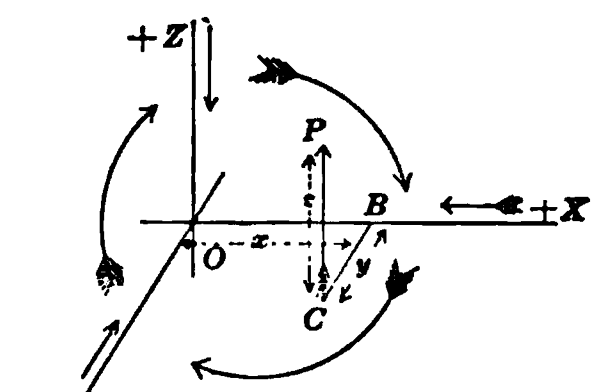


Fig. 44

An examination of Fig. 44 will disclose the fact (shown again for emphasis) that Fy is a right handed moment to an observer looking from $+X$ towards O, whenever F and y are both positive, or both negative; and that the moment is left-handed or negative whenever the factors have different signs. Accordingly, the algebraic law of signs holds for every term in $\Sigma(Fy)$.

It is just the reverse for every term in the expression $\Sigma(Fx)$. When the algebraic product is positive, the moment is left-handed and therefore negative; and when the algebraic product is negative the moment is positive.

Hence
$$\left.\begin{array}{l}M_1=\Sigma(Fy)\\ M_2=-\Sigma(Fx)\end{array}\right\}$$

This apparent anomaly of a negative product from positive factors is unavoidable.

Ex. Take as an illustration an ideal system of balanced forces which are parallel and may be tabulated as follows:

The forces represented in the table are fairly shown in this space drawing, Fig. 45. The points of application are assumed to be in plane XY. The student may combine these forces by pairs according to any method already given, and then combine the partial resultant until he finds remaining two resultant forces *which will directly balance* each other. This checking process should not be neglected. The forces shown should be laid off with great accuracy.

F	x	y	Fx	Fy
+8	+2	+3	+16	+24
−6	+3	−1	−18	+ 6
+4	−1	0	− 4	0
−3	+2	+5	− 6	−15
+2	+1	−5	+ 2	−10
−5	−2	+1	+10	− 5
$\Sigma F=0$			$\Sigma(Fx)=0$	$\Sigma(Fy)=0$

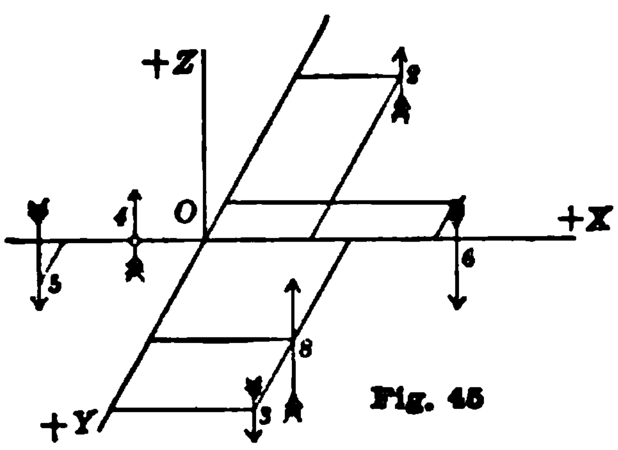


Fig. 45

53. The equations of equilibrium. It is readily understood that one may know that a system of parallel forces balances, yet be ignorant of the numerical values of some of the quantities which enter into the equations of equilibrium. The three equations of (**51**) will suffice to find three unknown quantities, provided every equation has at least one unknown quantity in it.

There are three general cases each of which will be illustrated. The three unknowns may be:—

1. Three unknown forces.
2. Two unknown forces and one co-ordinate.
3. One unknown force and two co-ordinates—one an x, and the other a y.

Case I. **Three forces are unknown.** Suppose a rigid slab, rectangular in shape, is supported in a *horizontal position* by four vertical wires connected with adjustable pins, and that one of the wires includes a spring balance which shows the tension; the other three tensions are to be found. All the points of attachment with the slab are known.

Let the length of the slab be twelve (12) feet, and the width four (4) feet, and assume that the slab weighs

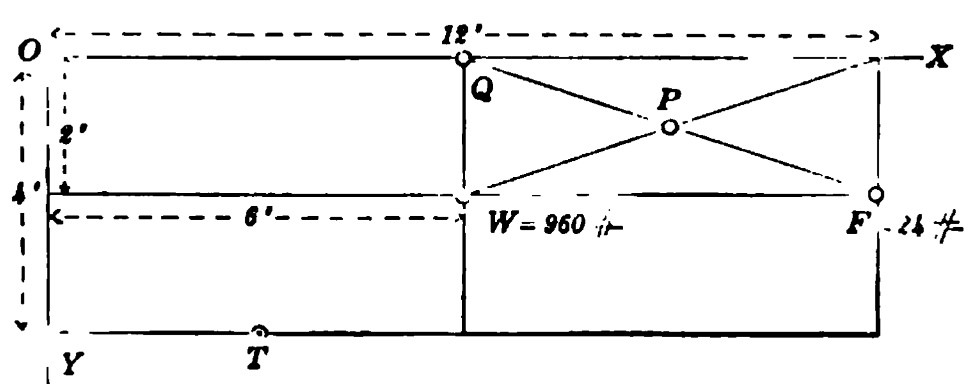


Fig. 46

960 lbs. The axes assumed, and the position of the four supporting forces are given in Fig. 46. The known force is F. The three equations of equilibrium are

$$P+Q+T+24-960=0 \tag{1}$$

$$M_1=24\times2+P\times1+T\times4-960\times2=0 \tag{2}$$

$$M_2=960\times6-24\times12-P\times9-Q\times6-T\times3=0 \tag{3}$$

Eliminate Q from (1) and (3) and get (4).

Eliminate P from (2) and (4) and get the value of T, and so on.

Ex. 1. Having fully mastered and completed the above, let the student take new axes with O at the center of the slab.

Ex. 2. Let him invent a new example with two weights and five forces, three of which are unknown.

54. Case II. **Two forces and one co-ordinate are unknown.**

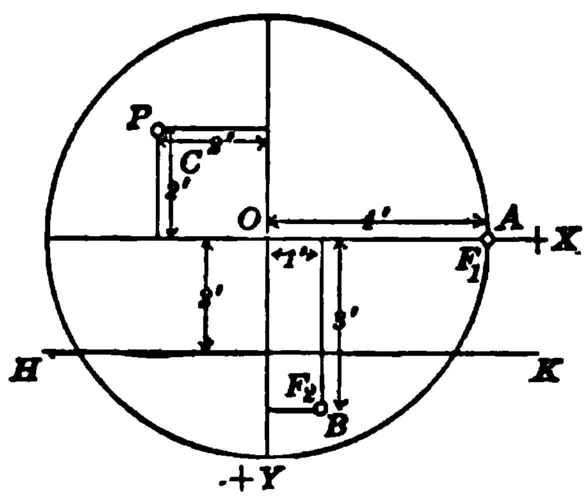


Fig. 47

Problem 1. Suppose a circular plate weighing 420 lbs., 8 feet in diameter, of uniform thickness and density, to be supported in a horizontal position by four wires connected overhead as in the last illustration. See Fig. 47. Two of the forces are known, F_1 and F_2; but two, P and Q, are unknown. $F_1=80$ lb and acts at A. $F_2=120$ lb. and acts at B. One unknown force, P, acts at C ($x=-2$, $y=-2$). The unknown force Q acts somewhere on the line HK, whose equation is $y=2$. Required the magnitudes of P and Q, and the x-co-ordinate of Q.

Ans. $P=200$ lb, $Q=20$ lb, $x=-2$.

2. A table is 8 feet square. It has three legs under the points A, B and C which are on the perimeter of a 6 ft. square as shown in Fig. 48. Two weights are on the table: one, W_1 is at the center; the other is somewhere on the line AD. The leg A supports 80 lb; B supports 30 lb.; C supports 48 lb. Find W_1 W_2 and the position of W_2.

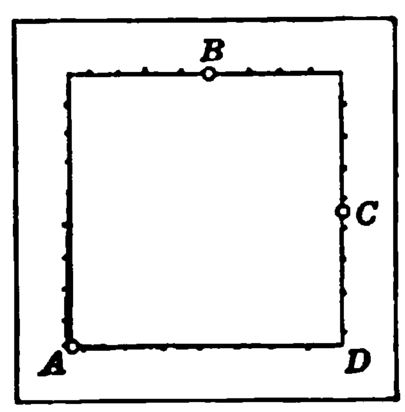


Fig. 48

55. Case III. **One force and two co-ordinates an x and a y are unknown.** (It is immaterial which x and which y are unknown). The data may be tabulated as shown on the next page.

It is *assumed* that P, x and y are all positive, tho this assumption may prove false. In addition to what is given in the Table we *know* that they *balance*; hence the three summations are separately zero.

Find P, x and y, and make a space drawing showing the five forces.

It is thus seen that the Equations of Equilibrium enable one to find the values of three unknowns. A fourth cannot be found unless there is some additional condition which may lead to a fourth equation, such as: $T=hP+kQ$, or, $x=ay+c$.

F	x	y	Fx	Fy
60	3	6		
-40	-2	8		
-104	4	$+y$		
50	-1	0		
P	$+x$	-4		
0			0	0

It will be seen later, when treating of *elastic* bodies, that some times a fourth independent equation can be found.

56. To find the resultant of a system of known parallel forces in space, which do not balance.

It is evident at once that the Equations of Equilibrium do not apply. It is also evident that the resultant force is the algebraic sum of the given forces. It is equally evident that if the positions of the forces are given in rectangular co-ordinates, the resultant will at *first* appear as a force acting thru the origin, along the axis OZ, and *two moments*: one about OX and the other about OY.

That is, we shall have:—

$$R=\Sigma F=F_1+F_2+F_3+\text{etc.}$$
$$M_1=\Sigma(Fy)=F_1y_1+F_2y_2+\text{etc.}$$
$$-M_2=+\Sigma(Fx)=+F_1x_1+F_2x_2+\text{etc.}$$

If x_1 and y_1 are the co-ordinates of a point in the line of action of R in space, we shall have

$$-M_2=Rx_1=\Sigma(Fx)\text{; so that } x_1=\frac{\Sigma Fx}{R}$$

$$M_1=Ry_1=\Sigma(Fy)\text{; so that } y_1=\frac{\Sigma Fy}{R}.$$

F	x	y	Fx	Fy
8	6	4		
-10	2	-3		
12	3	$+7$		
6	-4	$+8$		
-14	-5	-6		
Σ			Σ	Σ

This is the general case. It may be illustrated by an ideal example. The data are most conveniently given in tabulated form.

Show that the resultant is a force $+2$, parallel to OZ, whose line of action intersects the plane XY in the point:

$$x=\ 55$$
$$y=139$$

It goes almost without saying, that a force of -2 parallel to OZ, thru the same point would *balance* the above system.

57. Remarks as to the resultant. 1. If $\Sigma(Fy)=0$, there is no moment about the axis OX; this shows that the resultant force intersects that axis.

2. If both $\Sigma(Fy)$ and $\Sigma(Fx)$ are zero, the resultant force acts thru O, and along the axis of Z.

3. If $\Sigma(F)=0$, the resultant is not a single force, but is a moment (or a couple of forces).* It will be well to illustrate the last remark.

58. When the resultant of a system of parallel forces in space is a couple.

1. Ex. The Table represents a set of data, and the calculation.

F	x	y	Fx	Fy
+8	+ 6	+5	+48	+40
−6	− 2	+3	+12	−18
+3	0	−8	0	−24
−5	+10	−7	−50	+35
$R=O$			+10	+33

Hence $M_2=-\Sigma(Fx)=-10$ See (**52.**)
$M_1=+\Sigma(Fy)=+33$

Since M_2 is negative it is laid off on the *negative* part of the Y-axis, pointing towards O, see **25.** In this case θ is a negative angle, or a positive obtuse angle. (Fig. 49).

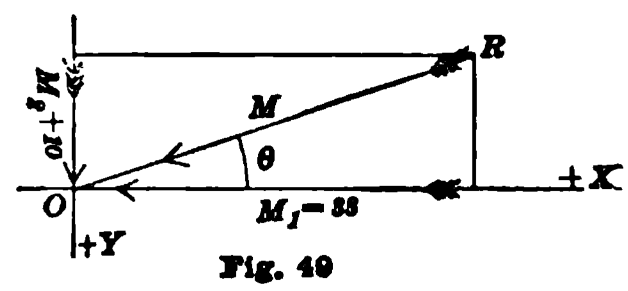


Fig. 49

The *resultant couple* acts in a plane perpendicular to the line RO, with a moment $\sqrt{(1189)}$, which is right-handed to an observer looking from R towards O.

2. A second illustration, in which both M_1 and M_2 are positive.

$$M_1=\Sigma(Fy)=+18$$
$$M_2=-\Sigma(Fx)=+6$$
$$M=\sqrt{(36+324)}$$
$\tan\theta=1/3$. See Fig. 50.

F	x	y	Fx	Fy
15	+3	+2	+45	+30
− 6	+8	+10	−48	−60
+ 4	−4	+12	−16	+48
−13	−1	0	+13	0
0			− 6	+18

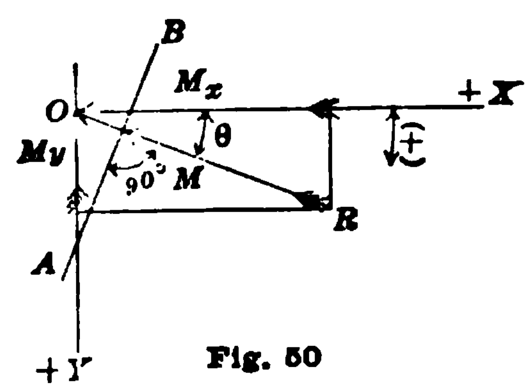


Fig. 50

The resultant couple is in the plane AB.

Exercises.

Let the student invent and solve two problems in one of which M_1 and M_2 have opposite signs; and one in which both M_1 and M_2 are negative.

* The formulas give, when $R=O$, $x_1=\infty$ and $y_1=\infty$, yet $R\times x_1=o\times\infty=\Sigma(Fx)$ which is a finite quantity. Also $R\times y_1=o\times\infty=\Sigma(Fy)$ which is *another* finite quantity. The student will see here an illustration of a truth which may have seemed obscure in Algebra.

CHAPTER IV.

Co-Planar Forces with Center Lines of Action Meeting at a Point.

59. Case I. **Three forces balance.** Since two non-parallel forces acting on a body cannot possibly balance, there must be at least three, and any one must directly balance the resultant of the other two. This fact gives rise to very important relations between the balancing forces which cannot be clearly stated till we have found how two intersecting forces may be combined, or replaced by a single force.

1. It is important that the reader picture in his mind the physical conditions of the general problem. *One* body is to be acted upon by *three* other bodies; each action is the resultant of a more or less distributed force; these force lines representing concentrated actions must lie in a plane; and the lines of action meet at a point.

2. For example, suppose a heavy solid body, B, *rests* against the combined action of the earth, the *smooth* end of a block P, and the tension in a wire rope Q. The force W is the resultant of an attractive force distributed thru a *volume*. The force P is the resultant of a pressure distributed over a *surface*. The force Q is a tension sent along a wire from an overhead hook *in the plane of* W and P; the three lines of action meet at O. The situation being *ideal*, all other actions (atmospheric and magnetic) are omitted. P is acting *towards* O; W and Q act *from* O. However, the forces are equally well represented in Fig. 52, as acting *from* their common point.

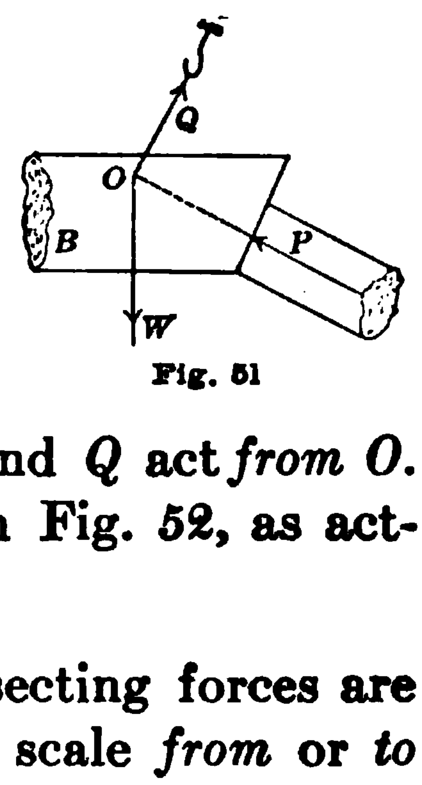


Fig. 51

60. The parallelogram of forces. If two intersecting forces are represented by two right lines drawn to scale *from* or *to* their common point, and if on them as sides, a parallelogram be drawn, the diagonal drawn *from* or *to* the common point represents their resultant. A great many proofs of this proposition have been given, some of which may be found in cyclopedias; only one is here given.

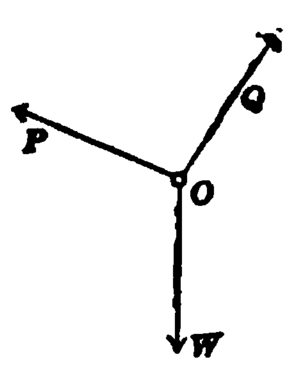


Fig. 52

Let the given forces be P and Q, both acting from O. Fig. 53. It is to be shown that the force R represented by the diagonal of the parallelogram constructed on P and Q is their resultant. Let C be

any point in the plane, and let a perpendicular to the plane at C be an "axis of moments." It will now be shown that the moment of R about the axis at C is equal to the sum of the moments of P and Q about the same axis. Draw OC, and connect the arrow heads A, B and D with C; finally draw AE and BF parallel to OC. The force lines being drawn to scale, the other lines must be considered as drawn to the scale.

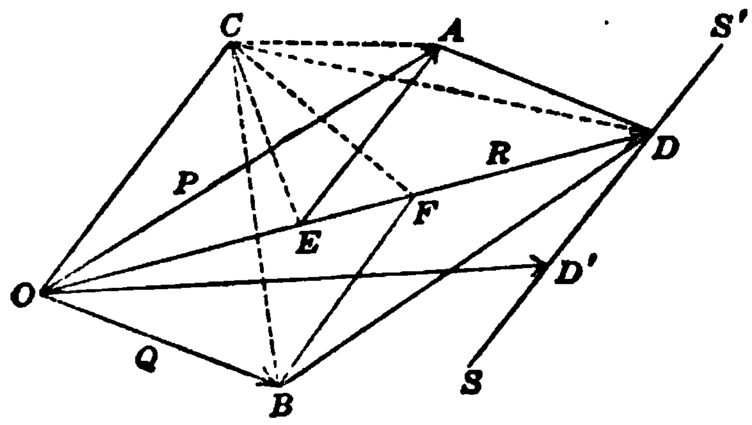


Fig. 58

The moment of R with respect to C is $R\times$ an arm from C to OD. or: moment of $R=2\times$ area of Δ OCD.

Similarly, the moment of $P=2\times$area of Δ OCA.
$=2\times$area of Δ OCE.

and the moment of $Q=2\times$area of Δ OCB.
$=2\times$area of Δ OCF.

Now, from the equal triangles OAE and BDF it is seen that $OE=FD$, and hence the area of the triangle $OCE=$ the area of the triangle FCD.

Hence the moment of $P=2\times$area of Δ FCD.

And since the area $OCD=$ area of $OCF+$ area of FCD, it follows that *the moment of R is equal to the sum of the moments of P and Q.*

But the proof that R is entitled to be called *the* Resultant is not complete, since there are any number of forces acting thru O which have the same moment. For example, draw thru D a line SS parallel to OC. Let D' be any point on SS.

A force OD' will have the same moment about O that R has, since the area of the triangle $OCD'=$ the area of the triangle OCD. Moreover the moment of OD' about any other axis taken on the line OC or OC produced, will be equal to the sum of the moments of P and Q about the same axis. If, however, we choose an axis outside of the line OC, we shall have another line $S'S'$ thru D, and all the forces OD' will be rejected except the single force $R=OD$.

Hence, since the moment of R is always equal to the sum of the moments of P and Q for the axis C, and is the *only force* which thus represents them wheresoever the axis is taken, it alone is entitled to the name "Resultant."

61. The following graphical method of finding the resultant of two converging forces, which do not meet within the limits of one's drawing, is merely a generalization of the special case with parallel forces given in (**40**). Let P and Q be the co-planar forces which do

not meet on the paper. To find the line of action and magnitude of their resultant. Fig. 54.

1. Parallel to the head-and-tail line Ab (which need not be actually drawn) draw the head-line BD, and the tail-line aO.

2. Parallel to the other head-and-tail line, draw a head-line AD and a tail-line bO.

3. The intersection of the head-lines is the head, and the intersection of the tail-lines is the tail, of the resultant required.

The analytical proof of this is left to the student.

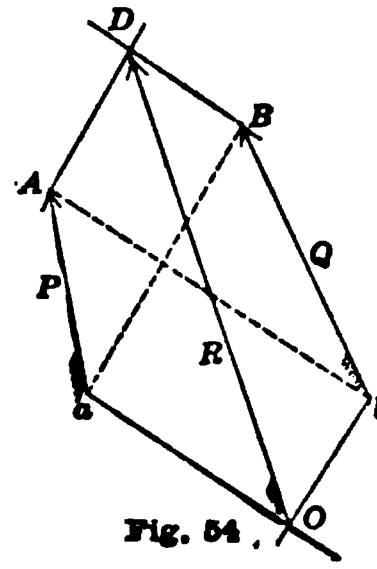


Fig. 54

62. The resolution of a given force. The given forces P and Q are known as the Components of R.

1. The converse of the proposition in **61** is, that any force may be resolved into two components by constructing a parallelogram on the line representing the given force as a diagonal.

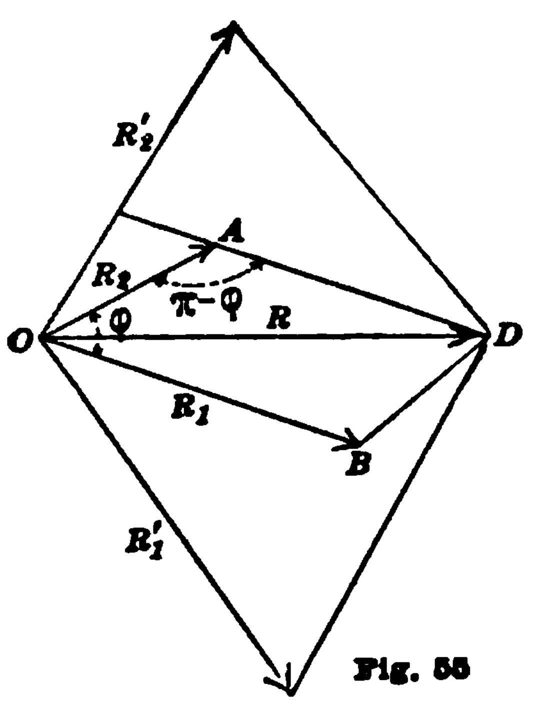


Fig. 55

Thus Fig. 55, let R be a given force; its components may be R_1 and R_2, but they may as well be R_1' and R_2'; in short, there may be any number of pairs of components. By drawing a variety of parallelograms, the student should note the important fact that the resultant of two forces *may* be less than either of them; and that one or both the components of a force *may* be greater than the force itself. The relation depends upon the size of the angle $\phi = AOB$.

2. The sides of the triangle OAD, may represent a force and its two components in magnitude and direction, but not fully in position; in fact, the position of all the forces might be changed as in Fig. 56, which means that the force R is equivalent* to the two components R_1 and R_2; and conversely, that the forces R_1 and R_2 are together equivalent to the single force R. As now understood, the lines of this triangle are frequently called VECTORS, α, β, γ and we read the equivalent equation

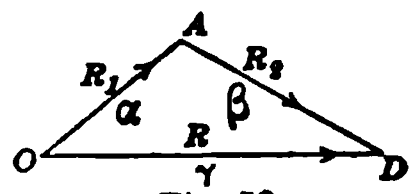


Fig. 56

$$\alpha+\beta=\gamma$$

3. Henceforth when we want the resultant of P and Q meeting at O, we shall not draw the full parallelogram, but we shall draw $OA = P$ and $AD = Q$; then OD will equal R.

---

* *Equivalent* so far as the *motion* or *rest* of the body acted upon is concerned. It is not equivalent when internal stresses are concerned.

63. The resultant by trigonometry. When P and Q and ϕ, the angle between P and Q are given, it is evident from Fig. 57 that

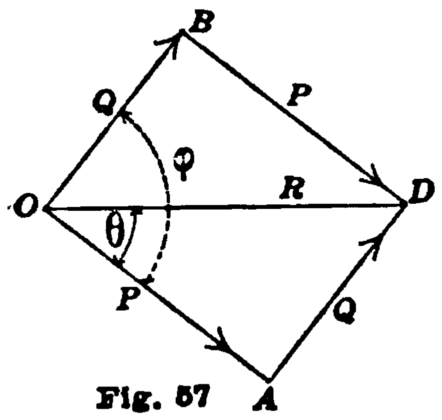


Fig. 57

$$R^2 = P^2 + Q^2 + 2PQ\cos\phi.$$

When $\phi = \pi/2$ it is at once evident that

$$R = P\cos\theta + Q\sin\theta$$

and that $P\sin\theta = Q\cos\theta$, as well as that $R^2 = P^2 + Q^2$.

It is now seen that if the direction and magnitude of one component of a given force are given, the direction and magnitude of the other component are determined—graphically and by trigonometry.

64. The static triangle or the triangle of equilibrium. We are now prepared to bring in the third force which balances two given forces. It is evident that in order to balance the two given forces, it must balance their resultant. To do this it must have the same magnitude and be directly opposite. In other words, referring to Fig. 56, and remembering that the third force, which we will call S, is the exact opposite of R, and that if R was the vector γ, S must be the vector $-\gamma$; so that

$$\alpha + \beta = -\gamma$$
$$\text{or } \alpha + \beta + \gamma = 0$$

which is the vector equation of equilibrium. The triangle (of magnitudes and directions only) becomes (Fig. 58) the ***STATIC TRIANGLE*** or the *triangle of equilibrium.*

Whenever three forces balance, such a triangle can be drawn, and the relations of sides and angles can be found graphically or by trigonometry. The sufficient evidence of equilibrium is a state of rest, or of uniform motion. Cases of absolutely uniform motion are so rare that they are for the present neglected in our thought. The great value of the static triangle will be illustrated by a few examples.

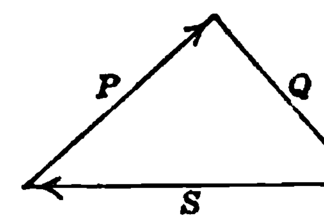


Fig. 58

65. Imponderable pins. 1. In dealing with centralized forces it is often convenient to assume that they act upon an imponderable pin, for thus we fix the attention upon a body (the pin) which is at rest under their action. This is particularly true when the forces arise from the action of wires, ropes, tie-bars, piers, posts, and struts, as at the joints of frames. In the case of an over-hanging hoist, such as is shown in Fig. 59, we may conceive of a pin of inconsiderable weight, acted upon by a tie-bar T (which may be double, consisting

of two thin bars instead of one thick one), a strut, S, or inclined post, and a chain or rope which may be connected with a pulley-block which has several plies.

2. *The pin stands still*; hence the three forces acting upon it balance, and we can draw the static triangle, since all directions are given in the figure or are known on the construction itself. If the weight, W, is known, we can draw the static triangle to scale. The values of S and T can be measured from the drawing or calculated. See Fig. 60.

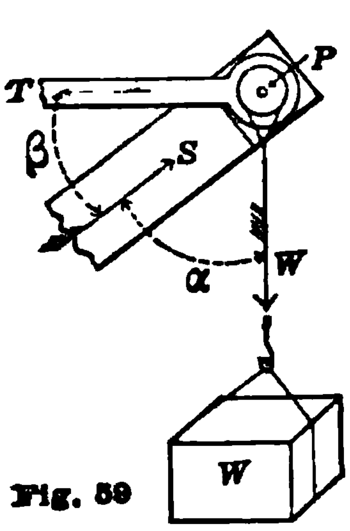


Fig. 59

If P represents the pin in Fig. 59, it is well to put the same letter within the static triangle, Fig. 60, and then follow and note the directions of the arrows which surround it. In every case the arrow shows the *direction* of the action in Fig. 59. For example: S in Fig. 60 points *upwards*; and it is therefore a strut in Fig. 59; T points towards the *left*; it therefore *acts* on P towards the left, and hence must represent a tie, or tension bar.

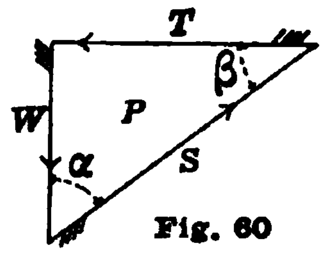


Fig. 60

3. If three non-parallel co-planar forces act upon a body at different points and *yet balance*, the three lines of action *must meet at* a point, or one could not directly balance the resultant of the other two.

66. Ex. 1. Take the ideal case of a heavy bar suspended against a vertical *smooth* wall. A weightless wire fastened to a hook or staple in the face of the wall at C supports the end B, Fig. 61, while the other end rests against the smooth wall which can act only in the direction of a normal to its surface. The bar and wire must be in a vertical plane normal to the wall. If G is the center of gravity action of the bar, the line of the wire must pass thru the intersection of the action lines of H and W at O. The *static triangle* can now be drawn showing the magnitudes of forces and angles. Fig. 61a.

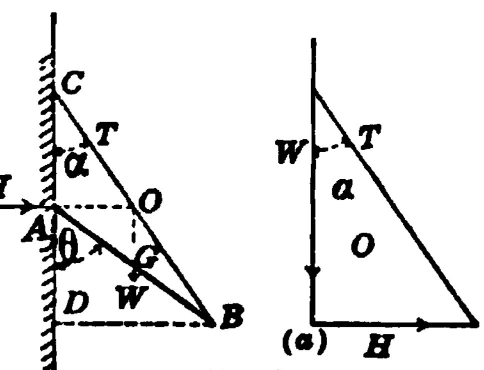


Fig. 61

2. If G be at the center of the line AB (as in the case of a uniform bar) it is seen that O is the middle point of CB, and $AD = AC$. Hence, when θ and l, the length of the uniform bar, are known, the position of C is found by laying off AC equal to $l \cos \theta$.

3. The reader must not fail to see that the above suspended body is in *unstable* equilibrium; the touch of a fourth force would cause it to fall or to collapse against the wall, even if it were prevented from swinging sidewise. If, however, the wall were rough (and all walls *are* rough), there would be a certain stability.

4. It is evident that the pull of the wire CB could be replaced by the thrust of a strut or prop SB, without in any way changing the static triangle; we would then let T stand for thrust instead of tension. It is also evident that a heavy weight may be hung at G without dislocating the bar, as it would only increase W.

67. A third example will further illustrate the principle that three balancing forces meet at a point. A slender uniform rod rests upon the inner surface and edge of a smooth hemispherical bowl. Fig. 62. We are to find the relation between l and θ.

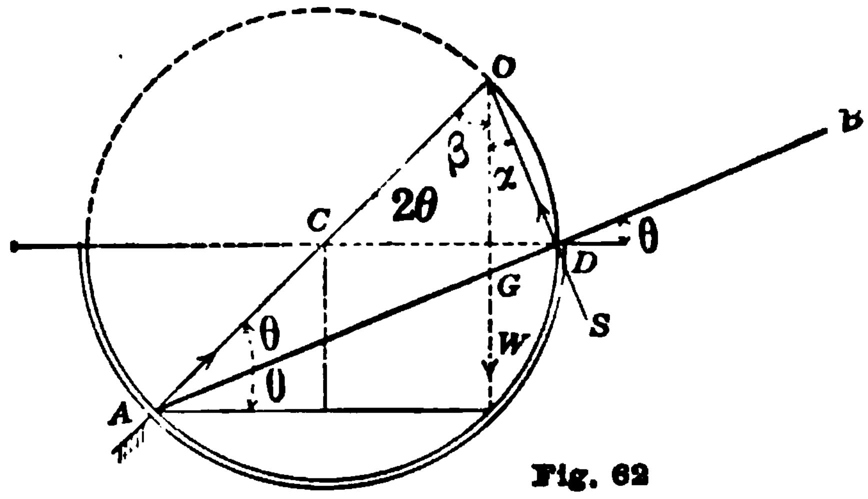


Fig. 62

The action at A must be normal to the spherical surface, and hence along a diameter; the action at D must be normal to the rod, as tho the edge were an infinitesimal round. Hence the gravity action must be in a line vertically below O, where the two normals intersect, and the length of the bar, which will rest at A and D, is determined, for its center must be at G.

1. If θ be the inclination of the rod to the horizontal, it is evident from the figure that $2r\cos 2\theta = \frac{1}{2} l\cos\theta$, which gives the necessary relation between l and θ. It is further evident that any movement of the rod due to a new force would have the effect of raising the point G, or lifting the bar, so that when relieved from the disturbing force it would slide back to its position of equilibrium. This is a case of "stable equilibrium."

2. The inferior limit to the length of the rod we have been discussing is when $l = 2r\cos\theta$. The *minimum length* of the rod is found by eliminating θ from the two equations

$$\left.\begin{aligned} 2r\cos 2\theta &= \frac{1}{2} l\cos\theta \\ l &= 2r\cos\theta \end{aligned}\right\}$$

Whence $\quad l = 2r\sqrt{(2/3)}$ and $\cos\theta = \frac{1}{3}\sqrt{6}$.

3. If l be less than $\frac{2}{3}r\sqrt{6}$. the bar will rest only when G is directly below C.

Problems.

68. 1. Given a smooth vertical wall, and at a distance s a lower parallel wall with a horizontal edge which is rounded and smooth.

Fig. 63. What is the inclination of a smooth, slender, uniform rod which will rest over the corner and against the wall? (Note where the balancing forces meet).

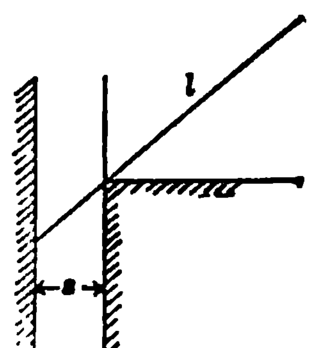


Fig. 63

Ans. $\cos\theta = \sqrt[3]{\frac{2s}{l}}$.

2. Find the stress in the members of this derrick due to the weight (8 tons). Fig. 64.

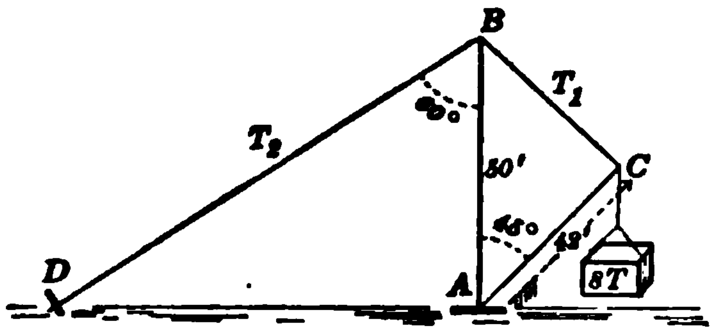


Fig. 64

Draw a static triangle for the pin at C, and then another for the pin at B.

3. If AB and AB' are equal smooth rods looped on a pin or ring at A, Fig. 65, prove that the condition, that the pair will *rest* on the surface of a smooth horizontal cylinder whose radius is r, is

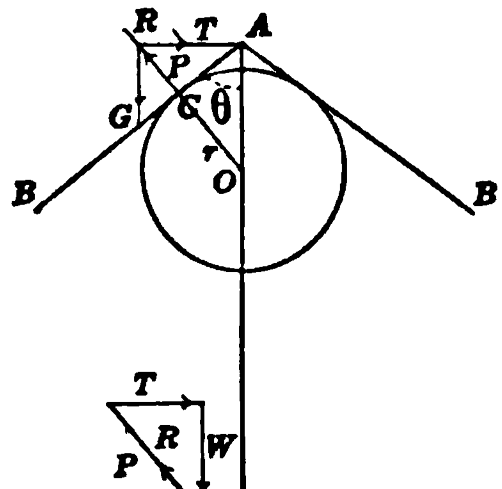


Fig. 65

$$l = \frac{2r}{\sin^2\theta \tan\theta}$$

4. A smooth bar rests upon two smooth inclined planes whose intersection at O is horizontal. Find the inclination of the bar to the horizontal. Fig. 66.

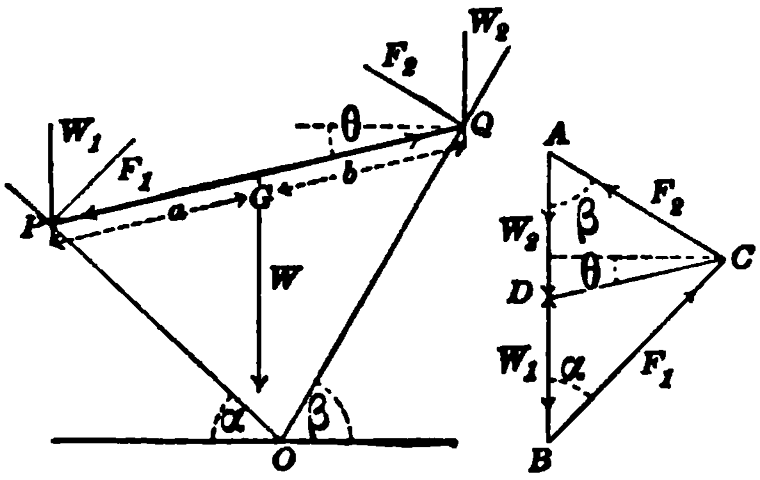


Fig. 66 (b)

The weight is centered at G. First draw the static triangle of external forces W, F_1 and F_2; Fig. 66, b; then resolve the weight into vertical components, W_1 at P, and W_2 at Q, and complete the static triangles for the points P and Q, thereby finding CD, the "*thrust*" in the bar, and its inclination.

From the diagram (b) prove that

$$\tan\theta = \frac{a\cot\alpha - b\cot\beta}{l}$$

5. A heavy cylinder is held against a smooth wall by a smooth rigid bar which is hinged upon a pin at A (Fig. 67), and held by a tie T to a hook at H, as shown. Given the weight and radius of cylinder, the length of bar, and all angles, to find the tension T, the pressures P_1 and P_2, and the magnitude and direction of the hinge action, due solely to the weight of the cylinder.

Solution. Three forces act upon the cylinder, hence the static triangle KEF whereby P_1 and P_2 are known. Fig. 67.

Since we omit the weight of the bar AB, only three forces are acting upon *it*, viz.: P_1, T and the hinge. We already have P_1, and we know the direction of T, and as the three must meet, say at D, we know the direction of the hinge action, P_3, namely in the direction AD. Its static triangle is therefore FEH. Thus the magnitude of T and both the magnitude and direction of the hinge action due to the cylinder are found.

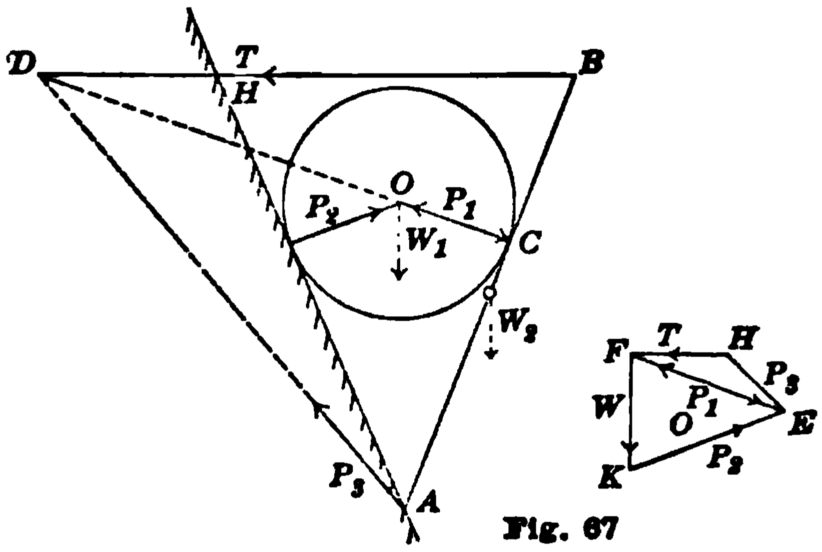


Fig. 67

69. How to determine the tension or compressive force in a bar of a loaded frame from the static triangles of its pins.

Let the members of a cantilever frame lie in a vertical plane with a load at the end. Fig. 68. We are to find the actions of the bars BA and CA upon the pin at A; and the actions of the bars AB, CB and DB upon the pin at B.

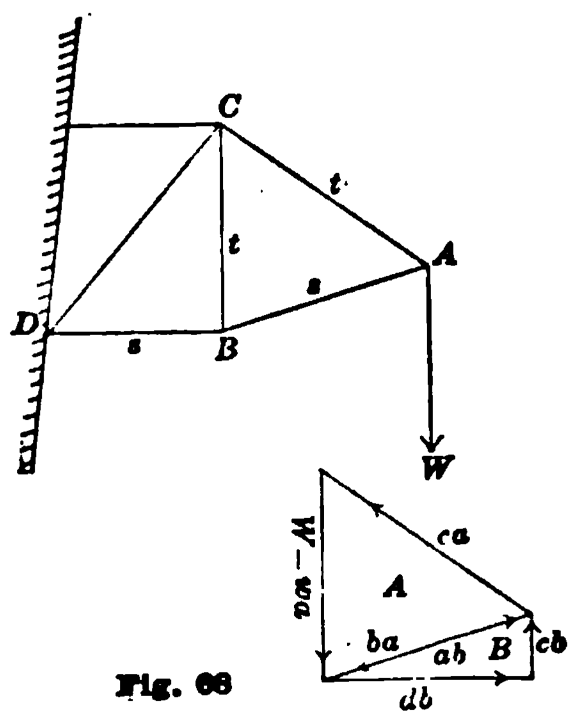


Fig. 68

Solution. The static triangle for A. The case is similar to that in **59**. Knowing one side of the triangle and the directions of the other two, it is quickly drawn. W acts *down*, BA acts *up*, parallel to the bar, and CA acts *up* to the left parallel to the bar CA, closing the triangle. The static triangle for A shows magnitudes and *character*. AB acts up on the pin A, and is therefore a *strut*; the bar CA also acts *up* and *away from* the pin, and therefore is a tie.

Going now to the pin B, three bars act on it and *balance*; hence B has a static triangle. The action of AB is known, being a strut it acts *down* on B, and the triangle already drawn gives it magnitude. We therefore use the line ba, or ab, and draw db for the bar DB, and cb for the bar CB, and mark the arrows of the static triangle for B. We note that the arrow db points to the *right*, showing that DB acts against the pin B, while the arrow cb points *up*, showing that CB acts *away from* the pin B. DB is therefore a strut, and CB is a tie, and they are lettered accordingly.

This is all very simple, but it cannot be too clearly seen. Occasions will surely arise when students will find it necessary to refer back to this section and go over it again.

70. Case II. **The polygon of forces and the static polygon.** When more than three co-planar forces are given with a common point in their lines of action, their resultant, or their necessary relations, if they form a balanced system, are readily found by an extension of what has already been given. Fig. 69.

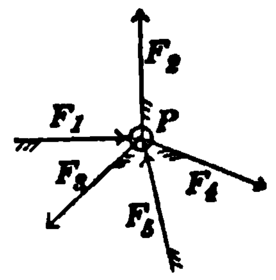


Fig. 69

Given five forces acting upon a pin at P. To find their resultant.

Graphical Solution.

Take any convenient point O as an origin; draw OA equal and parallel to F_1; draw AB equal and parallel to F_2; this gives OB equal and parallel to the resultant of F_1 and F_2.

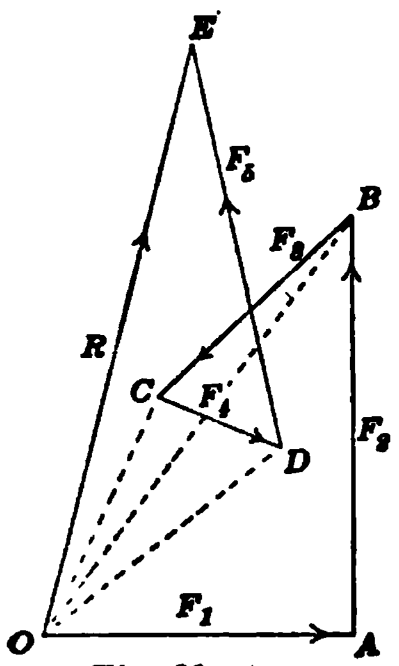


Fig. 69 (a)

Draw BC equal and parallel to F_3; OC is then equal and parallel to the resultant of F_1, F_2 and F_3.

Draw CD equal and parallel to F_4; OD is then equal and parallel to the resultant of $F_1 \ldots F_4$.

Draw DE equal and parallel to F_5; OE is then equal and parallel to the resultant of all the forces, $F_1 \ldots F_5$.

The *order* in which the forces are taken is not important, the resultant is still the same. This statement should be put to the test by a careful drawing. It is evident that the lines OB, OC, etc., are of no use excepting always *the last one* which is the ***resultant*** sought. The line representing their resultant is always drawn *from* O to the *end* of the last component force.

Had the line of the last component ended in O, there would have been no resultant, as we should have had

$$R=0$$

which is the evidence of a balanced system of forces. In all practical statical problems R is zero; and every polygon drawn as above, *which closes*, is called a Polygon or Equilibrium, or more conveniently a "Static Polygon."

71. Static triangles and polygons are useful in finding unknown magnitudes and directions. In every case where a resultant is to be found, or a static triangle or polygon is to be drawn, the data must be geometrically sufficient to that end. In most of the problems thus far solved, the magnitudes of two forces were found; all angles being given or determined by the geometrical necessity of convergence of the lines of action. In the last general problem several forces were fully given, and a single angle and force were to be found. The following problem is in a new class.

Ex. Given one force in position, a second force in magnitude only, and a third in direction only.

This may be illustrated by a special device using weights (Fig. 70) F_1, F_2 and F_3.

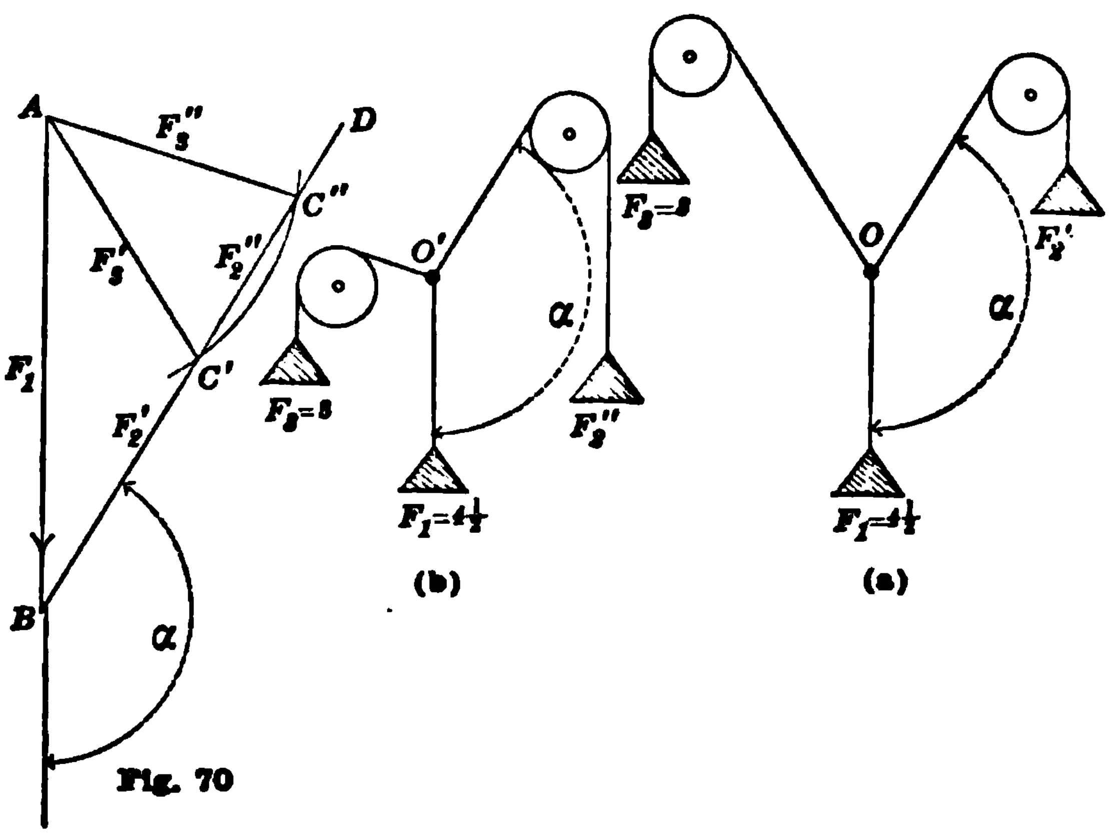


Fig. 70

Let O be the pin or body acted upon.

Let F_1 represent the force fully known $=4\frac{1}{2}$ acting down.

Let α be the known *direction* of F_2, relative to F_1; $\alpha=150°$.

Let $F_3=3$ represent the magnitude of the third or balancing force.

We lay off to scale from any point as A, F_1 as AB to the point B; draw the direction of F_2 as BD; with F_3 as a radius, from A as a center, draw the *arc* $C'C''$. The intersections of this arc with the line BD determine the magnitudes of F_2' and F_2''. We get in general two solutions for the magnitude of F_2 and two directions for F_3. We see that F_1, F_2' and F_3' balance at O; 70*a*, and that F_1, F_2'' F_3'' balance at O'; (*b*).

72. Illustrations of the use of the static polygon.

Ex. 1. Four equal cylinders lie within a hollow semi-cylinder as shown in vertical end view. Fig. 71. Given the weights, to find the pressures P_1, P_2, P_3, P_4, P_5, P_6, and P_7.

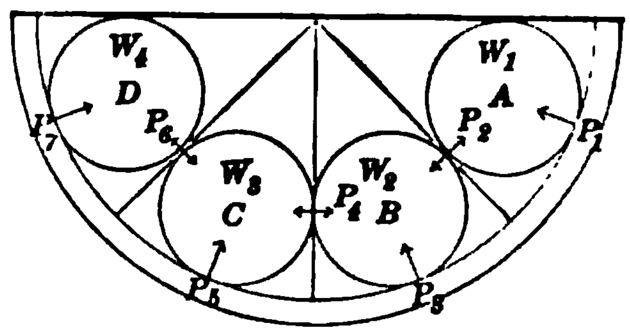


Fig. 71

Beginning with A, we draw the static triangle $W_1P_1P_2$. Fig. 72. We next draw the static polygon for B: $W_2P_2P_3P_4$. Then follows the polygon for C: $W_3P_4P_5P_6$. Finally, the static triangle for D: $W_4P_6P_7$, and the polygon closes. The reader will notice that since the four forces acting on B balance, they must, when drawn as vectors, form a *closed polygon*; hence P_4 returns to O. Having drawn Fig. 72 accurately, we should have no difficulty in checking the following:

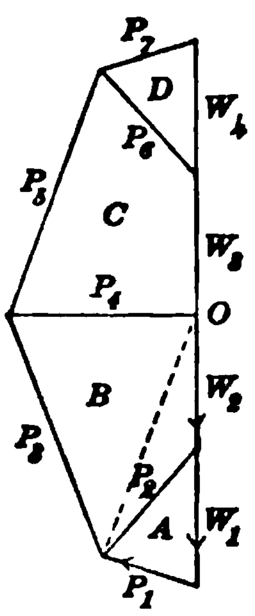


Fig. 72

$$P_7=P_1=W\sqrt{2-\sqrt{2}}$$
$$P_6=P_2=W$$
$$P_5=P_3=W(1+\sqrt{2})\sqrt{2-\sqrt{2}}$$
$$P_4=W\sqrt{2}$$

Ex. 2. A heavy horizontal cylinder, solid or hollow, rests upon two unequal cylinders which rest upon a horizontal bed with guard walls. Fig. 73. Weights being given, find the pressures at all points. The horizontal thrusts in AB and AC being balanced by retaining walls.

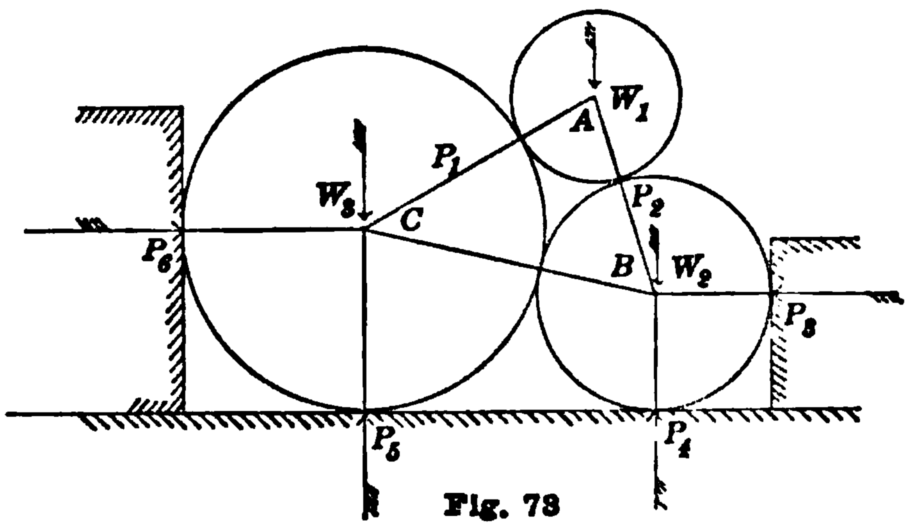


Fig. 73

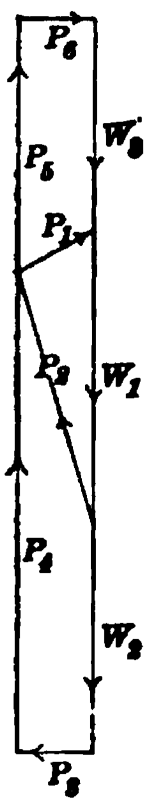


Fig. 74

The diagram (Fig. 74) is exceedingly simple, and values may be calculated when weights and radii are known. Of course it was evident without the drawing that

$$P_4+P_5=W_1+W_2+W_3,$$

and that $P_3=P_6$, as they were the external forces acting on the group.

73. The static polygon in "graphical statics." The Static Polygon enables us to find the magnitude of two forces when all other "parts" are known. This property is of the greatest value in finding the stresses in the members of frames when both frames and external forces may be regarded as co-planar. This deserves a somewhat extended illustration, as it will bring to us the very elegant method of "Reciprocal Polygons"* given in Chapter XI.

74. Equations of equilibrium and their uses. In every problem thus far solved by means of the static triangle or the static polygon, the data (forces and angles) were assumed as known, and the accuracy of the drawing was readily checked by trigonometry applied to the diagrams. When the known angles are the inclinations of forces to a given direction, that direction should be taken as the axis OX with OY perpendicular to it, and the rectangular components found by the method already given. Since the forces balance, their components along OX and along OY must balance independently, and their moment about *any axis* must be zero. Hence the Equations of Equilibrium

$$\Sigma F\cos\theta = 0$$
$$\Sigma F\sin\theta = 0$$
$$\Sigma Fl = 0$$

Here F is the representative of a force, known or unknown, and θ is

* So named by the eminent scholar Clerk—Maxwell.

its inclination to the axis X. For brevity we may write $F_1 = F\cos\theta$, and $F_2 = F\sin\theta$, so that the equations become

$$\Sigma F_1 = 0$$
$$\Sigma F_2 = 0$$
$$\Sigma Fl = 0$$

The angle θ is measured positively from $+X$, clockwise, to the head of the force arrow drawn *from* O.

The arc θ is always positive and its valve lies between $0°$ and $360°$.

75. Equations for a resultant. When the given forces do not balance, the *resultant* is readily found by means of the following equations, which, after the above, are almost self-evident.

$$\Sigma F_1 = R_1$$
$$\Sigma F_2 = R_2$$
$$R = \sqrt{R_1^2 + R_2^2}$$
$$\tan\theta = R_2/R_1$$

All these quations are useful, especially those of **74.** The third equation $\Sigma Fl = 0$, is convenient when l is the perpendicular from a point on the line of one force to the line of action of another force.

Ex. 1. A rigid prismatic rod of small diameter stands on a rough horizontal plane leaning against a smooth vertical wall. Find the pressure against the wall. Fig. 75. Three forces are acting on the rod, *and they balance.* Hence their lines of action must meet at a point. Given the position and length of the rod, the point of convergence is seen to be O where H and W meet. If we take B as the axis for moments we see at once, since $\Sigma Fl = 0$,

Fig. 75

$$Hl\cos\theta = \tfrac{1}{2}Wl\sin\theta$$
$$H = \tfrac{1}{2}W\tan\theta$$

But this value can be found from the equations $\Sigma F_1 = 0$ and $\Sigma F_2 = 0$.

$$H - P\sin\beta = 0$$
$$W - P\cos\beta = 0$$

Hence $$H/W = \tan\beta$$

But from the figure $\tan\beta = \tfrac{1}{2}\tan\theta$, hence

$$H = \tfrac{1}{2}W\tan\theta, \text{ as before.}$$

This shows that *the three equations of* **74** *are not independent. They serve for the discovery of but two unknown quantities.*

Ex. 2. Find the resultant of the forces given in the table, both by graphics and by analytics; the forces all act at a point in OX.

76. Comparison of graphical and analytic methods. All of the problems we have had thus far in this chapter may be solved by equations of equilibrium. We shall have frequent use of these equations later on, and the student should finally be equally free to use the graphical method or what is called the analytical method. In earlier works on Mechanics, graphical methods were little used. For the sake of comparing the method by equations alone with the graphical method with trigonometry applied to the force polygon, we quote from good authorities solutions of two problems already solved.

F	θ	$\cos\theta$	$\sin\theta$	F_1	F_2
16	$\pi/4$				
20	140°				
10	210°				
25	$\frac{3}{2}\pi$				

$R=$ $\theta_r=$

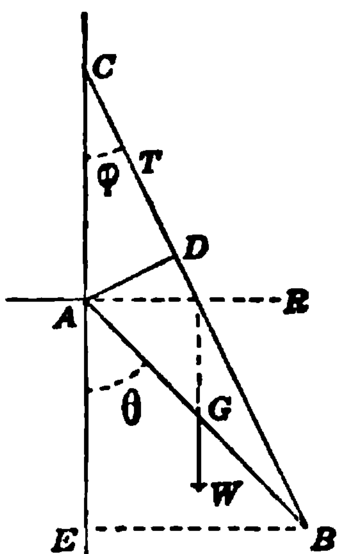


Fig. 76

Ex. 1. "A heavy uniform beam, Fig. 76, AB, rests with one end, A, against a smooth vertical wall, and the other end, B, is fastened by a string, BC, of given length, to a point, C, in the wall. The beam and the string are in a vertical plane; it is required to determine the pressure against the wall, the tension of the string, and the position of the beam and the string.

"Let $AG=GB=a$, $AC=x$, $BC=b$,

weight of beam $=W$, tension of string $=T$, pressure of wall $=R$.

$$BAE=\theta,\ BCA=\phi$$

then we have

for horizontal forces, $$R-T\sin\phi=0 \qquad (1)$$

for vertical forces, $$W-T\cos\phi=0 \qquad (2)$$

for moments about A, $$Wa\sin\theta-(T.AD=Tx\sin\phi)=0 \qquad (3)$$

$$\therefore\ a\sin\theta=x\tan\phi \qquad (4)$$

and by the geometry of the figure

$$\frac{b}{2a}=\frac{\sin\theta}{\sin\phi} \qquad (5)$$

$$\frac{x}{2a}=\frac{\sin(\theta-\phi)}{\sin\phi} \qquad (6)$$

Solving (4), (5) and (6), we get

$$x=\left[\frac{b^2-4a^2}{3}\right]^{\frac{1}{2}}$$

$$\cos\phi=\frac{2}{b}\left[\frac{b^2-4a^2}{3}\right]^{\frac{1}{2}}$$

$$\sin\theta=\frac{1}{2a}\left[\frac{16a^2-b^2}{3}\right]^{\frac{1}{2}}$$

from which R and T become known." (Price's Anal. Mech's, Vol. I p. 69.)

Ex. 2. For the second example, showing the occasional simplicity of the analytic method, I quote a problem and solution from Professor Edward A. Bowser, Analytic Mechanics, 1884.

"A heavy beam, AB, rests on two given smooth planes which are inclined at angles, α and β, to the horizon; required the angle θ which the beam makes with the horizontal plane, and the pressures on the planes.

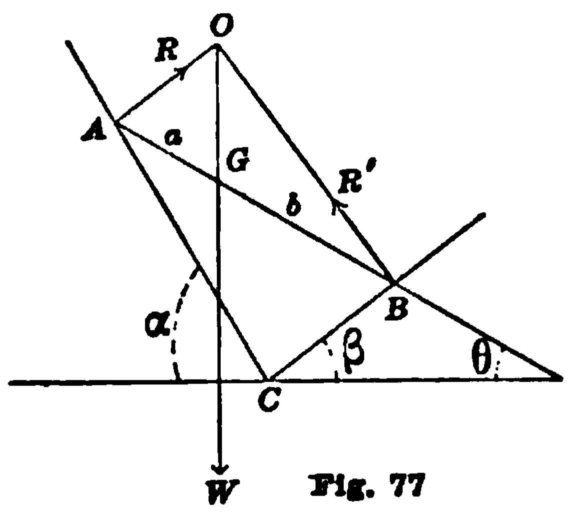


Fig. 77

"Let a and b be the segments, AG and BG, of the beam, made by its center of gravity, G; let R and R' be the pressures on the planes, AC and BC, the lines of action of which are perpendicular to the planes since they are smooth, and let W be the weight of the beam. Then we have

for horizontal forces, $\qquad R\sin\alpha=R'\sin\beta \qquad (1)$

for vertical forces, $\qquad R\cos\alpha+R'\cos\beta=W \qquad (2)$

for moments about G, $\qquad Ra\cos(\alpha-\theta)=R'b\cos(\beta+\theta) \qquad (3)$

Dividing (3) by (1), we have

$$a\cot\alpha+a\tan\theta=b\cot\beta-b\tan\theta$$

$$\tan\theta=\frac{a\cot\alpha-b\cot\beta}{a+b}$$

and from (1) and (2) we have

$$R=\frac{W\sin\beta}{\sin(\alpha+\beta)}; \qquad R'=\frac{W\sin\alpha}{\sin(\alpha+\beta)}.$$

77. The elegance and simplicity of the force diagram. Two perforated balls, A and B, whose weights are W_1 and W_2, slide down on two *straight smooth* rods until further motion is prevented by a weight-

less thread between them. The rods are stretched in a vertical plane from the same pin with equal inclinations α. Prove that

$$\tan\theta = \frac{W_1 - W_2}{W_1 + W_2} \cot\alpha. \quad \text{(See Fig. 78)},$$

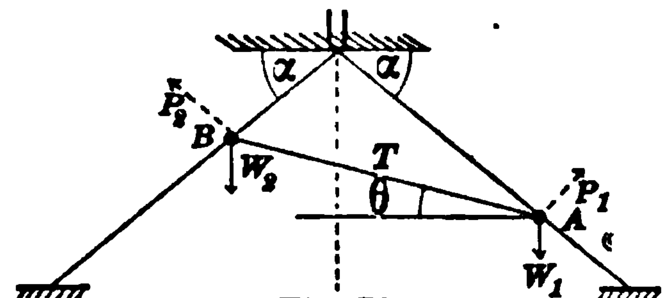


Fig. 78

and that the tension in the thread is

$$T = \frac{W_1 - W_2}{2\sin\theta}$$

Problems.

1. A triangle frame with pin-joints, whose sides are 5′, 6′, 7′, stands vertically with its shortest side resting on a smooth level floor. What stress does a weight of 269 lbs. placed at the vertex cause in the lowest side? Draw the static diagram and measure the force line.

2. A camp stool (Fig. 79), consisting of four equal legs arranged in two oblique crosses, as AK and CH (with a horizontal connecting pin at O, and parallel bars connecting corresponding tops of the legs at A and C) carries a flexible band ABC from bar to bar as shown. The band supports a heavy rod of small diameter, B. Assuming that the stool stands on a *smooth* horizontal floor; that all the members of the stool are without weight, and that there is no friction at joints, we are to find the position of equilibrium, when the weight of the rod, and the lengths of legs and band are given.

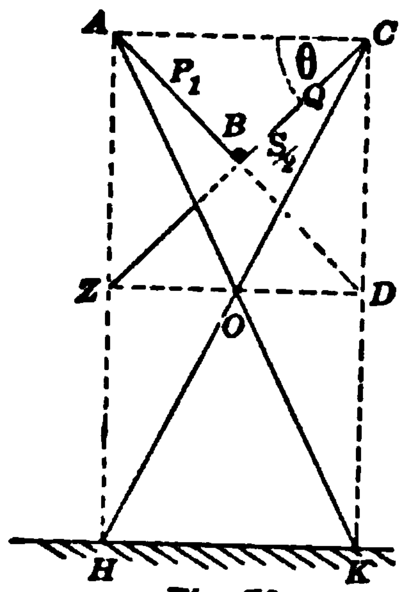


Fig. 79

78. The concentration of loads upon the pins of a frame.

RULE. WHEN A MEMBER OF A FRAME HAS WEIGHT, AND WHENEVER IT CARRIES CONCENTRATED LOADS, BOTH THE WEIGHT OF THE MEMBER AND THE LOADS ON IT MUST BE RESOLVED INTO PARALLEL COMPONENTS ACTING ON THE TWO SUPPORTING PINS.

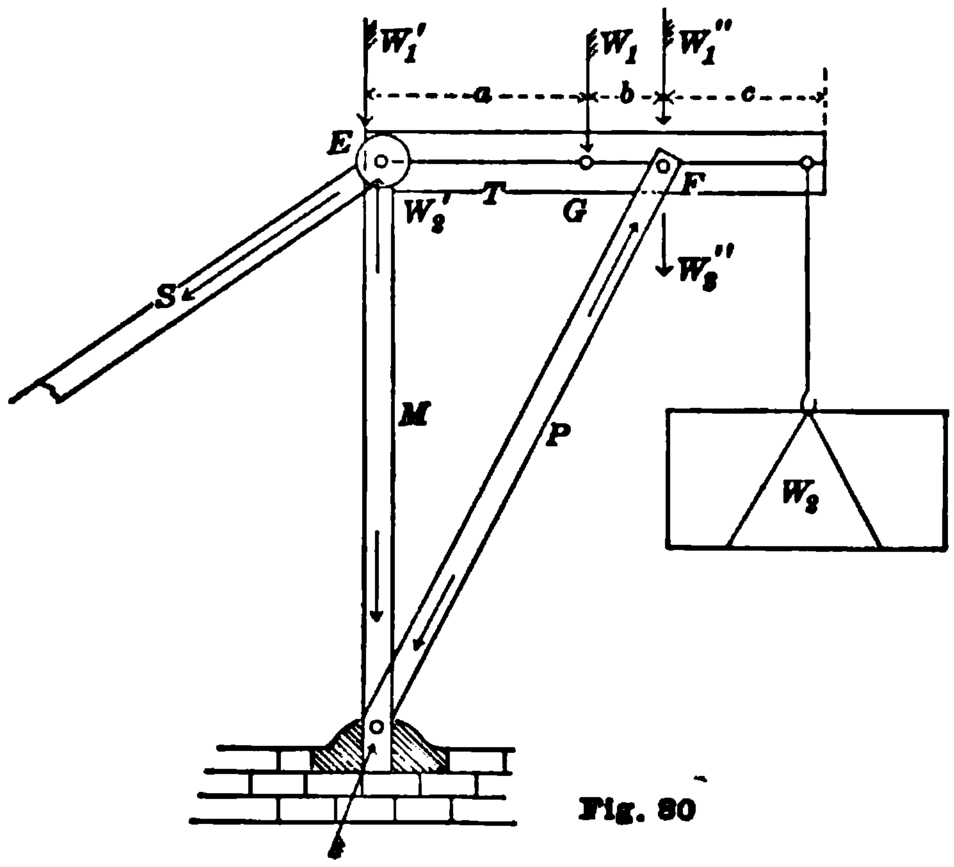


Fig. 80

The derrick shown in Fig. 80 has a cantilever or overhanging beam with a concentrated load, W_2, at the outer end. The weight of the beam, W_1, and W_2 are to be resolved into components at the two

pins, F and E. Then the static diagram can be drawn. G is the center of gravity of the beam; let W_1 be its weight, W_1' its component at E; and W_1'' the component at F.

$$W_1' = \frac{b}{a+b} W_1, \qquad \text{and } W_1'' = \frac{a}{a+b} W_1$$

In like manner W_2 is resolved into

$$W_2' = -\frac{c}{a+b} W_2 \text{ at } E$$

$$\text{and } W_2'' = +\frac{a+b+c}{a+b} W_2 \text{ at } F.$$

The *transformed derrick* now is as follows: The load at F is $W_2'' + W_1''$ (Fig. 80*a*), and the static triangle for pin F is ABC (Fig. 80*b*), which gives the *thrust* P in the boom, and the *tension* in the beam T. The forces now known as acting upon the pin E are the horizontal tension T and the *negative* load $W_2' - W_1'$; the resultant of these two forces is shown in Fig. 80*c* to be the force DH acting upwards. The static triangle for pin E is now readily drawn, viz., DH *up* (already found); HK *down*, parallel to the *stay* S; and KD *up*, parallel to the mast N; giving for E the triangle of forces DHK; and all the forces or stresses are found.

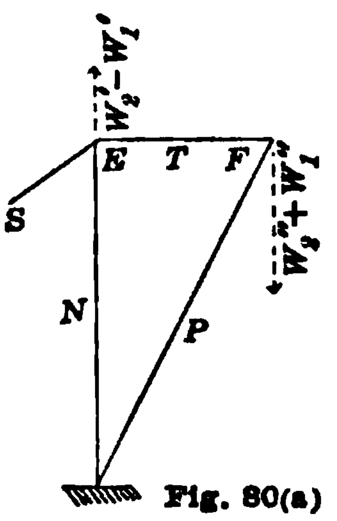


Fig. 80(a)

If instead of the resultant DH we had used its components T and $(W_2' - W_1')$, see Fig. 80*c*, we should have had the *Static polygon* of four sides, $DJHK$. We shall soon have static polygons in abundance. If all angles are given, the magnitude of the forces, in stay, mast and boom can be calculated by trigonometry.

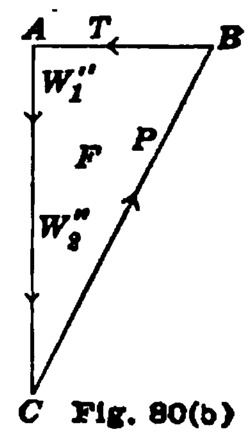


Fig. 80(b)

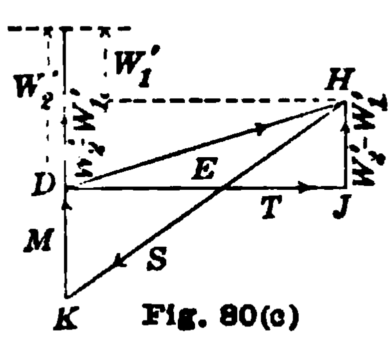


Fig. 80(c)

79. Stability due to friction. 1. When a light bar, like a cane, a ladder, prop, or strut of any kind, transmits a thrust against a rough plane surface, the roughness of the plane tends to hold the end of the bar against slipping. Fig. 81. If the thrust is normal to the plane, there is no tendency to slip and the roughness is not needed to keep the bar in position. When, however, the thrust is inclined to the normal like P, Fig. 81, the resistance R must also be inclined, so that if it also be resolved into rectangular components, one of which is normal and one tangential, *along*

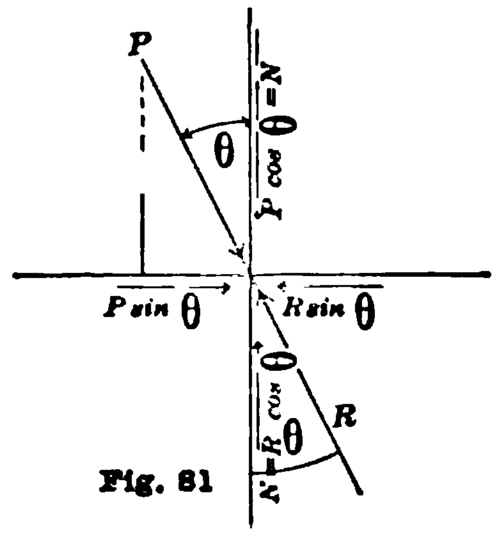


Fig. 81

the plane, the latter is $P\sin\theta$, and the normal component is $P\cos\theta$. The ratio of the tangential to the normal component is

$$\frac{P\sin\theta}{P\cos\theta}=\tan\theta$$

2. If the inclination (to the normal) be gradually increased, a value of θ will soon be reached *beyond which* the roughness of the plane will not suffice to balance the tangential component of P, so that if θ is *made any larger,* the bar will slip and fall. The *static angle* is therefore limited, and the limiting value of θ is called ϕ. The force thus exerted by a rough plane in the direction of the plane itself is called *Friction,* and ϕ is called the "Angle of Repose" or the "Angle of *Friction,*" or perhaps the "Static Angle." The roughness of surfaces both of bars and planes varies greatly.

3. Another way to illustrate friction is to place a rough body upon a rough inclined plane, Fig. 82. If it stands still, the forces balance. The normal component of the weight is balanced by the normal action of the plane; $W\cos\theta$. The tangential component of W is $W\sin\theta$, and this must be balanced by the friction.* If θ is small, very little friction is utilized or developed; as θ is increased more friction is needed. When $\theta=\phi$, the limiting value of friction is reached for that body, and the body is upon the point of sliding. If $\theta>\phi$ it *will* slide if a little initial sticking is overcome by a slight disturbing force. At the "static angle"

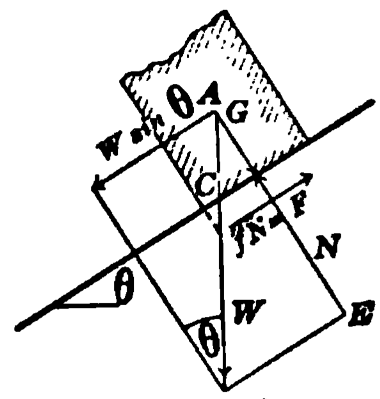


Fig. 82

$$\textit{Friction}=W\sin\phi=\tan\phi\ (W\cos\phi)=N\tan\phi=fN$$

When $\theta>\phi$ the component of W down the plane becomes greater, while the total friction usually becomes less, since $N=W\cos\theta$ is less. Hence there is no longer a static body.

4. The factor $\tan\phi$, standing with or before N, is called f, the "Co-efficient of Friction," which is used when the body is actually moving. When a body is at rest and in no immediate danger of moving, the frictional action is *less* than $N\tan\phi$, and it may be zero. It is, however, of the highest importance that the *direction* of the oblique force which is to be balanced by friction be known and provided for. The following problem will illustrate this point.

---

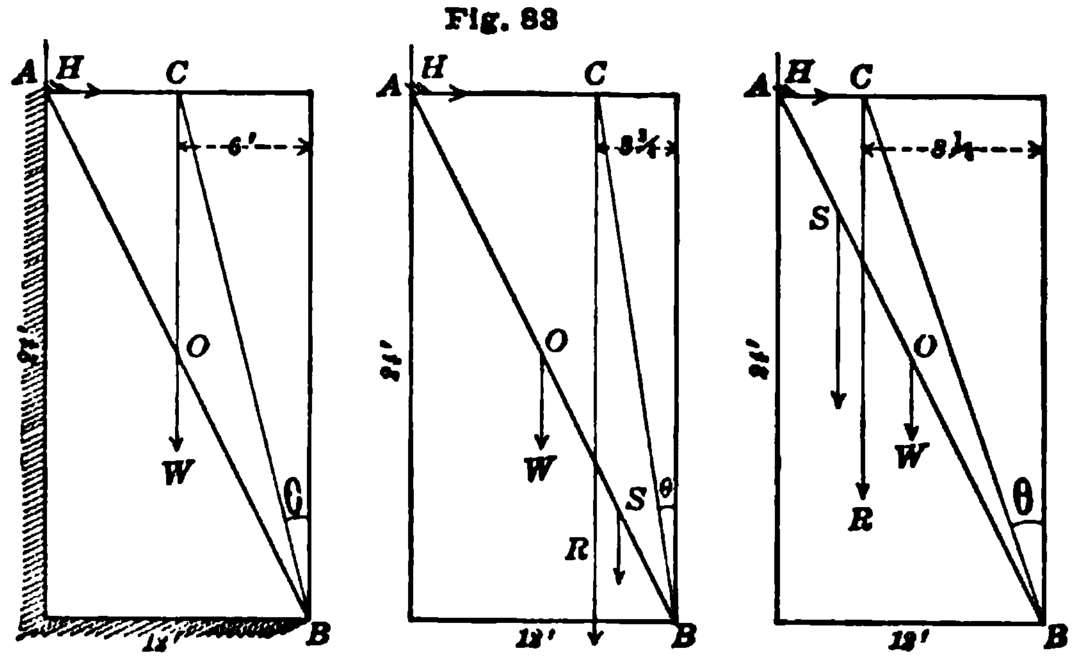
* The reader will observe that the component of W parallel to the plane acts thru G, and forms with the friction, fN, a couple which when combined with $W\cos\theta=N$ brings the *center of pressure* to the point C.

Problem.

80. A ladder, weighing 60 lbs, stands on a rough level floor, leaning against a smooth wall at a point 24 feet above the floor, with its foot 12 feet from the wall. Fig. 83. If $\phi = \text{arc}\tan 1/3$ be the "angle of friction" between the ladder's feet and the floor, how far up the ladder can a man weighing 180 lbs. go before the ladder will slip and fall?

1. Before the man steps on the ladder, the line of action against the ground is not straight down the ladder; it is nearer the normal, as seen in Fig. *a*, where $\tan\theta = 6/24$. The three forces acting on the unloaded ladder are: a horizontal force AH; a vertical force $OC = 60$ meeting AH at C; the *third* force, CB, namely the action of the floor, *must pass* thru C; its obliquity is θ.

Fig. 83



(a) (b) (c)

2. When the man steps on the ladder, the point C is over the center of gravity of the two weights, the ladder and the man. Hence, it follows that, *at first* C has moved out, and the ladder is less likely to slip. So long as the man is on the *lower half* of the ladder, it is stable. Thus, when the man stands on a rung six feet from the floor, at S (Fig. *b*) the center of gravity of the two weights is 8.25 feet from the wall, and

$$\tan\theta = \frac{3.75}{24}$$

3. When the man is at the middle, $\tan\theta = 6/24$, as when the ladder was unloaded.

As the man mounts higher, the center of gravity moves to the left as does the point C, where the three forces always meet.

When the man reaches the point S (Fig. *c*), three-fourths of the way to the top (supposing that possible), the center of gravity of the two weights (and therefore the point C) has moved to a point only 3.75 feet from the wall, and $\tan\theta = \dfrac{8.25}{24}$, which is *more than* $\tan\phi$. Hence the ladder slips and falls just when the man is *about to reach* a height of 18 feet.

4. Experienced workmen know well that ladders are apt to slip if a man (and the heavier he is the greater the danger) goes too near the top, and by sad experience they have learned to secure the foot of the ladder, either by a block or tie, or by putting another man on the ladder near the foot so as to bring the center of gravity of the whole weight away from the wall thereby diminishing the angle θ.

5. Workmen know, too, that a light ladder is more likely to slip than a heavy one. If we were to make the unreasonable assumption that the ladder has no weight worth considering, the point C in the line of action of the supporting reaction would always be directly over the head of the man as he mounts the ladder, and in a horizontal line from the top of the ladder.

6. If the student will draw the static triangle for each of the three positions of the man on the ladder, he will see how H increases as the man mounts.

81. While friction is a great hindrance to the motion of bodies, it is a great aid to the stability of bodies at rest.

1. It was assumed above that the wall was absolutely smooth. Hence it could offer no resistance to the slipping down of the top of the ladder. In reality, as all walls are rough, a tendency to slide down would have developed an *upward force* equal to $H \tan \phi_2$, (ϕ_2 being the "Angle of Friction" for the ladder and wall), which would have had the effect of moving the center of gravity of the weights *away from* the wall thereby diminishing θ (and perhaps saving a fall).

The student should clearly see how much $H \tan \phi_2 = H/6$ would move C *away* from the wall provided the resultant load was acting at a distance of 3.75 feet from the wall. This matter is worth a bit of careful study. In fact, the value of H depends *somewhat* upon the friction between the ladder and the wall.

2. The center of the ladder (Fig. 84) is at G with a man at S. The resultant load is at R, 3.75 feet from the wall. When the wall is smooth the point C is 3.75 feet from the wall and the reaction of the floor is along the line BC so that

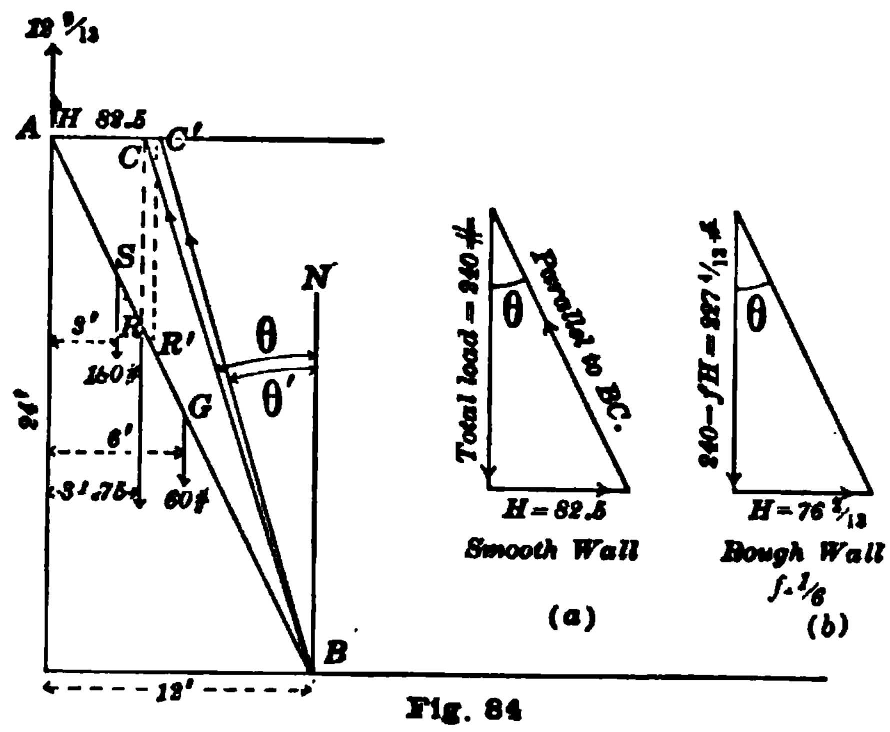


Fig. 84

$$\tan \theta = \frac{8.25}{24} = 0.344$$

Figure (84) (*a*) is the static triangle for the ladder, the value of H being

$$240 \times \frac{8.25}{24} = 82.5 \text{ lbs.}$$

If, now, we suppose the wall is rough, we shall have the co-efficient of friction $f = \tan\phi_2$. If we suppose the ladder is upon the point of slipping, the frictional action of the wall, which is *up*, is fH', so that we now have a new force involving H', acting upon the ladder. To find the value of H' under new conditions, take moments about B

$$24H' + 12fH' - 6 \times 60 - 9 \times 180 = 0$$

from which we get

$$H' = \frac{1980}{24 + 12f} = 76\tfrac{2}{13} \text{ lbs.}$$

if f is 1/6.

This gives the value of H' less than former H, so that the static triangle is new, the base being the new value H', and the vertical line being $240 - fH'$ the new θ' is seen to be less than the former value of θ.

$$\tan\theta' = \frac{76\tfrac{2}{13}}{227\tfrac{4}{13}} - 0.335$$

The movement of the point C to C', due to the friction on the wall, is readily seen if we get the resultant of the two vertical forces, namely 240 lbs at R, acting *down*, and H' acting *up* at the point A. By what was given in a previous chapter, the resultant is the force acting at the point *below* R, with a magnitude of 240 lbs. $-fH'$, giving the new point C' graphically. In fact, it is enough to prove the point stated above, namely, that the point C is carried *further away from the wall by the roughness of the wall*, since the resultant is sure to be beyond or below the point R. *We thus see that the stability of a body, like the leaning ladder, may be secured by means of friction at its base, assisted by friction at its top.*

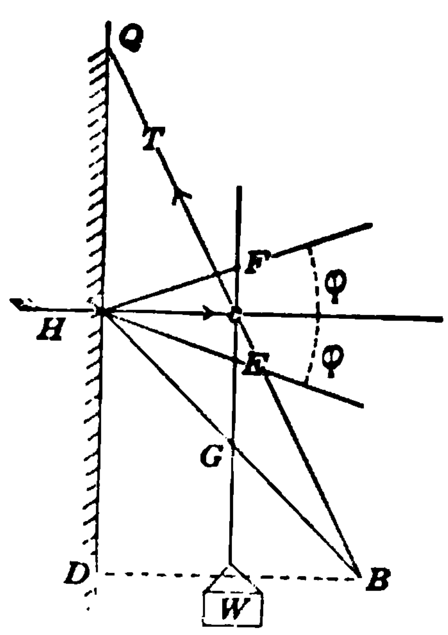


Fig. 85

82. Again, to use a modified form of the problem already solved twice (66) of a uniform bar suspended against a vertical wall. It was found that when the wall was smooth, the cord or wire should be made fast to a hook as much above H as H was above D. Fig. 85. If now the wall be rough with an "angle of friction" ϕ, the bar (and its load hung at G) will be in equilibrium if C (where the three force lines meet) be *anywhere between E and F*; tho the *safest place* is at the center, C.

83. The concentration of distributed forces and the resolution of concentrated forces. When a heavy homogeneous and uniform timber has a support directly under or over its geometrical center, Fig. 86, we say it is in equilibrium, because its center of gravity is supported. We may say more explicitly that the resultant of the earth's attraction, which is distributed thru the whole mass of the beam, acts thru the center, and that the resultant is balanced by the support. But one may ask, how does the attraction on the vertical end layer get to the center? We answer that the first layer hangs by cohesion on the second layer; and that upon the third; and so on, each layer holding up all between it and the end, and being held up with its load by the one next to it on the inside. This vertically-acting force, made possible by cohesion, has been called "**shearing stress.**" Like friction, it is tangential, but unlike friction, it is far more intimate and continuous, the molecular interlocking being nearly perfect instead of imperfect and haphazard. By means of this **shearing stress,** the weights of all parts of the timber are brought to the center. The shearing stress is evidently the greatest at the center. The fact that this transfer of weight develops other stresses within the timber does not now concern us, tho it will concern us later on.

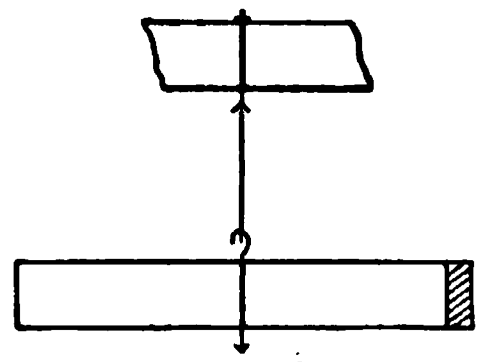


Fig. 86

84. If the timber has two supports, Fig. 87, the weight is concentrated at two points. The simplest method of making this distribution is by first concentrating the whole at the center of gravity, and then resolving W into two parallel components at A and B, in accord with the principles of Chapter III.

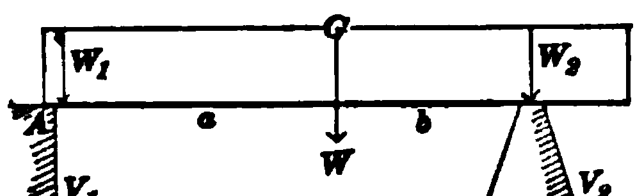


Fig. 87

Since the moments of the components are together, equal to the moment of their resultant about any axis, take an axis at A to find the component at V_2; then

$$Wa = W_2(a+b)$$

hence V_2, which must directly balance W_2, is found to be (in magnitude)

$$V_2 = \frac{a}{a+b} W$$

In like manner
$$V_1 = \frac{b}{a+b} W$$

85. Sometimes the method of support brings into a member of a frame a direct stress which is independent of the shear and bend-

ing stresses, as in this simple case. Suppose this heavy timber is supported by two separate cords or chains which are connected to a common hook. Consider the chains and timber as forming a triangular frame, with the weight of the timber (and whatever it may carry) concentrated at the ends into W_1 and W_2 with a single support at the hook. The static diagram can be drawn in the now familiar way, so arranged that only four straight lines exhibit the static triangles for the pins A and B. The cords are subject to direct tension, the timber is subject to a *direct thrust*, and to shear. Fig. 88(a).

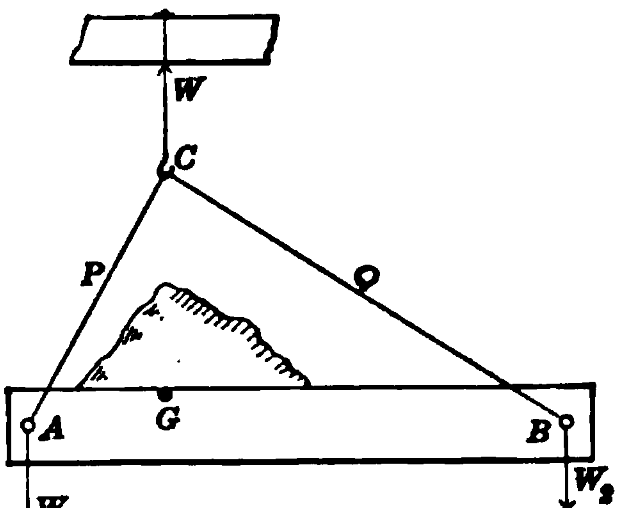


Fig. 88

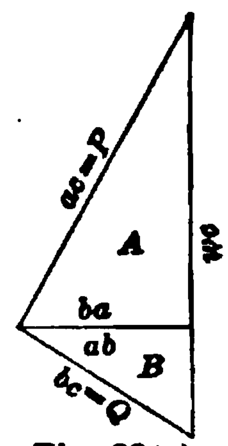


Fig. 88(a)

Had the timber and its load been supported by two chains to *separate* hooks, the timber *might have been* subject to direct tension.

The student may so represent a timber and draw its static (or stress) diagram.

CHAPTER V.

Non-Concurrent, Co-planar Forces.

86. In order to determine fully the force or couple required to balance a system of co-planar forces which act in different directions and at different points of a rigid body or frame, the resultant of the given forces is first found. There are four rather distinct methods of doing this. Every force must be actually or ideally given in magnitude, direction and position.

I. **Graphical solution.** Fig. 89. Combine the forces in pairs by producing their lines of action, if need be, till they intersect, and using the parallelogram, or by the methods of IV., 2 and (3). Combine the partial resultants in pairs, and so on till the single resultant is found. A little forethought in forming pairs will generally keep the work within bounds. The student should take his data from what is *given* in Fig. 89, as to the forces (1), (2), (3) and (4), their directions and their relative positions. The axes OX and OY are not needed in this method.

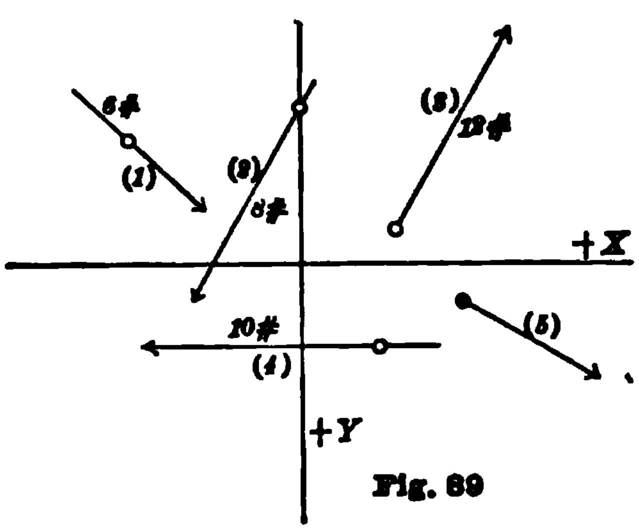


Fig. 89

If the final pair should consist of two equal and directly opposite forces, the resultant is zero, and the given forces balance.

If, in any such problem, the last pair should chance to form a couple, the combination can go no further; *the resultant is that couple,* and the system cannot be balanced by any single force; only an opposite couple can prevent the body from turning.

87. II. **Analytic solution.** Choose a pair of co-ordinate axes which can conveniently be used in locating points of action and directions of forces. Let the direction be determined by the angle θ, as shown in Fig. 90, and let a point of application be given by x and y.

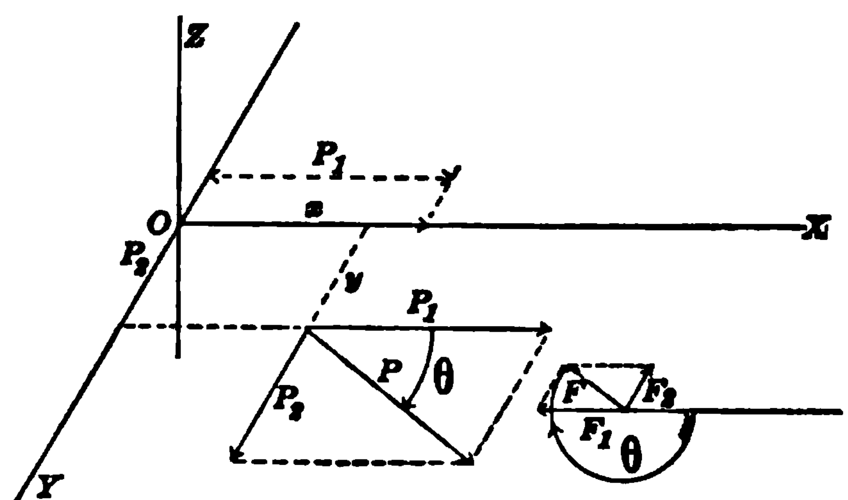


Fig. 90

(*a*) Resolve every force into its components $P\cos\theta = P_1$, and $P\sin\theta = P_2$. See Fig. 90.

(*b*) Resolve every *component* into an equal and parallel force at O, and a couple whose moment (about OZ) is $+xP_2$ for one component, and $-yP_1$ for the other. (Fig. 90.) Thus the *resultant* moment for a force P is $xP_2 - yP_1$. In the same way every given force, in general, is resolved into two forces, P_1 and P_2, at O, and a moment about OZ.

(*c*) Summing results

$$R_1 = \Sigma P_1 = \Sigma P\cos\theta.$$
$$R_2 = \Sigma P_2 = \Sigma P\sin\theta.$$
$$M = \Sigma(xP_2 - yP_1) = \Sigma xP_2 - \Sigma yP_1.$$

(*d*) Combining R_1 and R_2 gives R in *magnitude* and *direction,* but not in position.

$$R = \sqrt{R_1^2 + R_2^2}, \qquad \tan\theta_r = \frac{R_2}{R_1}.$$

(*e*) Combine M and R by **33**, and find the *position* of the resultant *force*, using the equation $p = M/R$. The balancing force is of course the above resultant reversed.

(*f*) The point of application cannot be explicitly found, but the *line* of action can, and any point in that line can be taken as the point of application provided it be a point in or on the body upon which the given forces act.

If x' and y' are the co-ordinates of a point in the action line of R, we have the equation of that line

$$M = x'R_2 - y'R_1$$

whose intercepts on the axes are readily found and the action line can be drawn and the value of p be checked.

This method can be made clear by two examples.

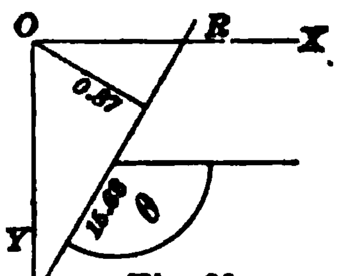


Fig. 91

Ex. 1. Find the resultant of the forces shown in the following table which gives magnitudes, co-ordinates of points of application, and directions of forces. The table also shows how to tabulate the work of calculation. Only two decimals are used in the natural sines and cosines. The forces shown in Fig. 91 are fairly represented in the table.

F	x	y	θ	F_1	F_2	xF_2	yF_1
10	-5	-4	45°	$+\ 7.07$	$+\ 7.07$	-35.35	-28.28
12	0	-6	120°	$-\ 6.00$	$+10.39$	0.00	$+36.00$
8	2.5	-1	300°	$+\ 4.00$	$-\ 6.93$	-17.33	$-\ 4.00$
16	2	2	180°	-16.00	0.00	0.00	-32.00
6	4	1	40°	$+\ 4.62$	$+3.84$	15.36	$+\ 4.62$
				$R_1=-6.31$	$R_2=14.37$	-37.32	-23.66

Results: see Fig. 91

$$R=(R_1^2+R_2^2)^{\frac{1}{2}}=15.63 \qquad \theta=\text{arc tan}\,\frac{R_2}{R_1}=113°40'.$$

$$M=-37.32+23.66=-13.66. \qquad p=M/R=-0.87.$$

Ex. 2. Since both R_1 and R_2 are zero, the resultant is not a single force, but a couple whose moment may be found.

F	x	y	θ
24	3	-1	90°
14	-8	$+10$	300°
40	4	4	135°
45.5	$+6$	-6	298°2′

88. III. **Semi-graphic method.** This requires an accurate drawing of the force lines so that in each case its perpendicular distance from some convenient central point O can be measured. If possible, O should be at the intersection of two force lines. Call the perpendiculars p_1, p_2, etc.

(a) Imagine that every force F is resolved into an equal parallel force at O, and a couple whose moment is Fp. Measure the perpendiculars and calculate the value of $M=\Sigma Fp$.

(b) Beginning at O (or some more convenient point), draw the force polygon of the transferred forces which meet at O, thereby finding the magnitude and direction of the resultant R.

(*c*) Combine the resultant force R with the resultant moment M, as already pointed out in the last method, and find p.

The student should use again the data shown in Fig. 89, and in the Table under the II Method, and so check up all former results.

If in any new problem $M=0$, and $R>$ or <0, the final resultant is a force R acting thru O.

If $R=0$ and $M=0$, the given forces balance and the body acted upon is in equilibrium.

The above three methods of solving problems of co-planar forces should be well mastered. The student will see that each method has at times special advantages.

89. IV. **The funicular, or chain polygon.** *Definition*: A **chain polygon** is a device for finding graphically a point in the resultant of a system of co-planar forces acting at different points on a solid body or frame, by the introduction of a series of *internal forces* which successively balance the given forces, and which are themselves balanced by two forces whose lines of action intersect and thus determine a *point in the resultant required.* The full meaning of this definition will be seen if a solution is closely followed.

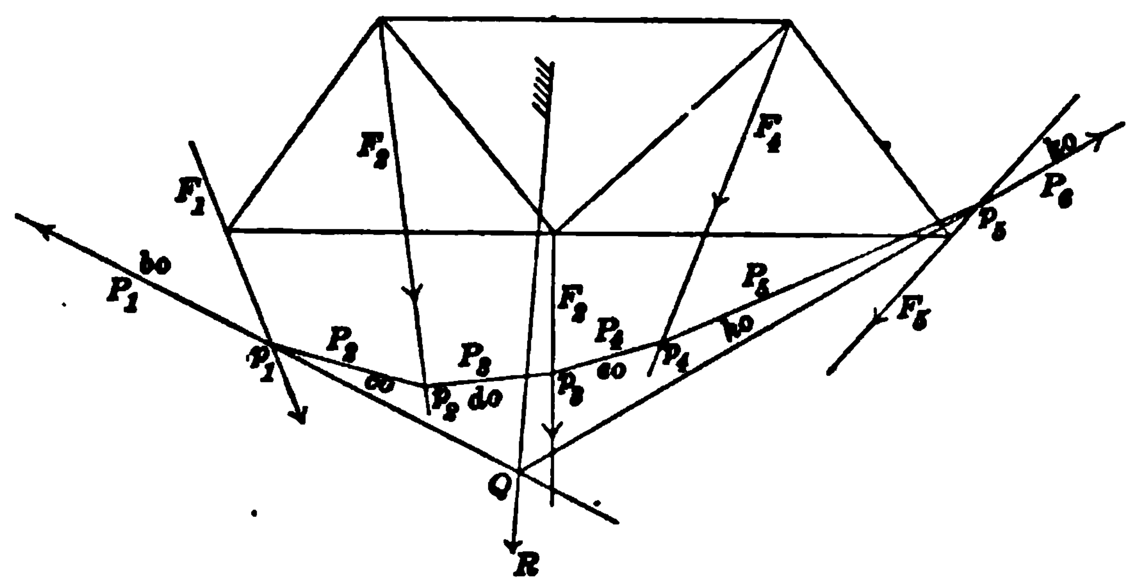


Fig. 92 a

1. Let a trapezoidal frame, consisting of seven bars with pin-joints, be acted upon, as shown in Fig. 92(*a*), by five co-planar forces $F_1 \ldots\ldots F_5$.

The resultant of the system is to be found.

The magnitudes and directions are shown in (*b*), as in the last section. The open force polygon is *BCDEHK*, and the *Resultant*, *BK*, is found in *magnitude* and *direction*.

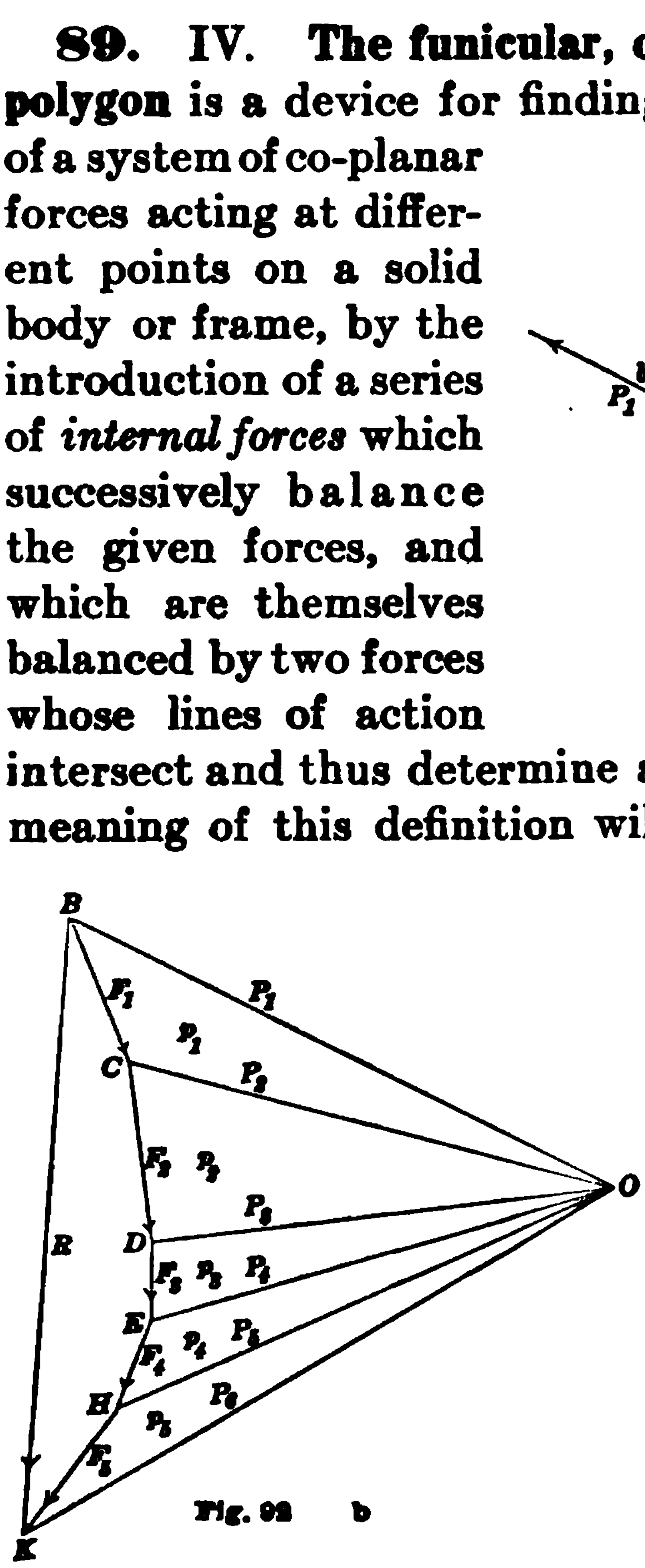


Fig. 92 b

2. We are now to find its position.

From any convenient point O on the right of the force polygon, Fig. 92*b*, draw right lines to B, C, D, E, H and K. These lines represent *tensions* in the links of our proposed

chain. The force, F_1, acts between the two links whose tensions are BO and CO. The force lines of F_1, and the two tensions form a *static triangle* for the pin at p_1, which point is taken at random on the action line of F_1, in Fig. *a*. If parallel to BO, we draw *bo* so as to intersect the line of action of F_1, and then thru the point of intersection draw *co* parallel to CO, we have the lines of action of the three forces (viz.: F_1 and the first and the second links of the chain) which balance.

Producing *co* till it intersects the line of action of F_2, and drawing *do* parallel to DO, we have the action lines of the forces whose static triangle is COD. In the same way drawing *do*, *eo*, *fo* and *ko*, we have all the forces balanced except *bo* and *ko*. The tensions in *bo* and *ko* must balance them all, and this is shown by the static triangle BOK, and hence *bo* and *ko* must balance the *resultant*. It follows that the point where they intersect, Q, is in the resultant R. Thru Q, parallel and equal to BK, the resultant of F_1, F_2, F_3, F_4, F_5 can now be drawn.

3. The point O is called the "Pole"; the lines meeting at the pole are called "rays"; Fig. 92 (*a*) is called the "space diagram"; Fig. 92(*b*), is the force diagram; and the polygon in (*a*) formed by the *interior* links and the lines P_1Q and P_5Q, is called the *Equilibrium Polygon*.

90. Different chains. The student can choose a second "Pole" for O, and so have a different chain, and find a different Q, but if the figure is carefully drawn the new point will still be on R.

Great freedom is allowed in drawing a chain polygon just as there is in a force polygon; the forces may be taken in any order, and it is quite possible that one or more links would be in compression, *i. e.*, be struts instead of ties.

1. When the given forces are all vertical and acting down, the force diagram should take the forces in order from *left* to *right*, and the pole should be on the *right*, if the links of the chain are to be in tension. If the pole be taken on the *left*, or if the forces be taken in the *reverse* order, the links will be *struts* and they will form an arch. It is not necessary that the points P_1, P_2, etc., are actually on the body. Since the shape and extent of the body are indefinite, the pins of the imaginary chain may be off or on.

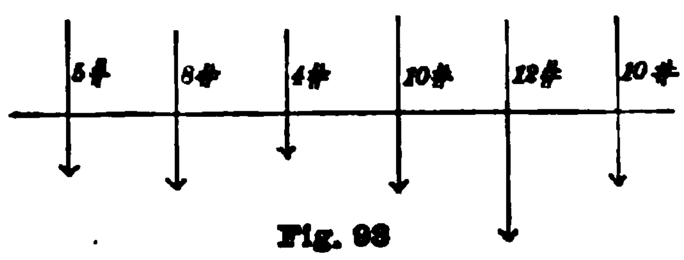


Fig. 93

Examples.

Ex. 1. Find the *chain* (made of tie bars and pins) which will balance the above forces, assuming that the end links are anchored.

Ex. 2. Find an arch which will balance the above forces acting on a body, assuming that the end struts have suitable supports.

2. If the force polygon closes, thereby showing that the resultant is not a single force, the first and last rays coincide, and the first and last links in the chain will represent equal, opposite and parallel forces, thus forming a *balancing* couple. The system shown in Fig. 94 gives a resultant couple whose moment is balanced by P_1p, see Fig. (*a*).

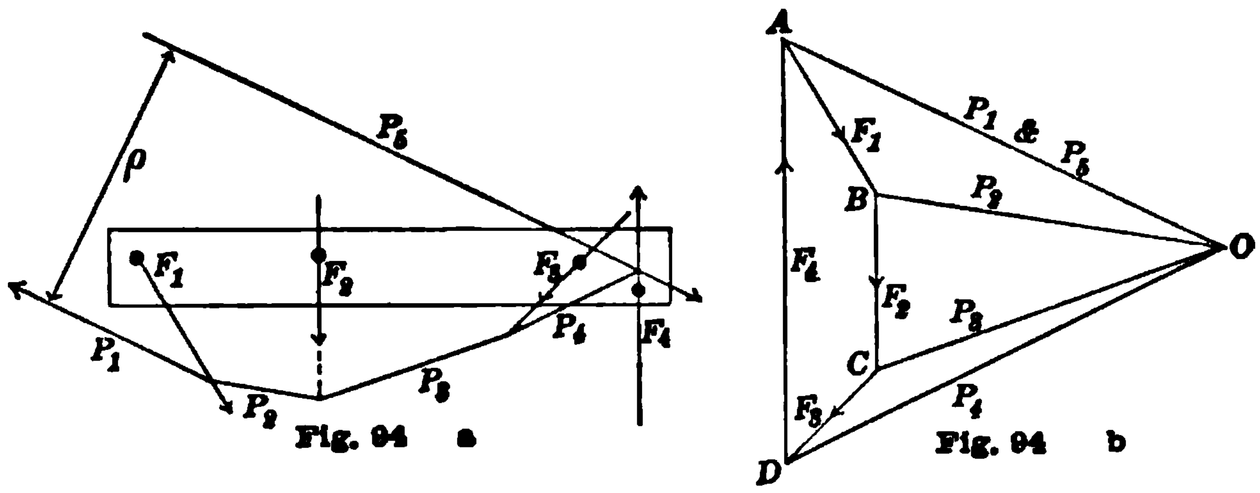


Fig. 94 a   Fig. 94 b

3. Should the lines of action of P_1 and P_5, in Fig. (*a*), coincide, there would be no couple, and as both $R=0$ and $M=0$, and the given forces would be completely self-balanced.

91. The closing link and vertical supports. Having found the resultant of a system of forces by the IV. method, it is often necessary to resolve it into parallel components acting thru given points. This is readily done by the device of inserting an *additional link* connecting the first and last links of the chain, or the first and last bars of an arch. The stress of this new link combined with the old terminal links separately gives the parallel components of the resultant. This method will be illustrated.

Let the given forces be vertical, and let one of the required parallel components, which will, of course, be vertical, pass thru the point A. Fig. 95. Draw the force polygon $BCDK$, with the resultant BK.

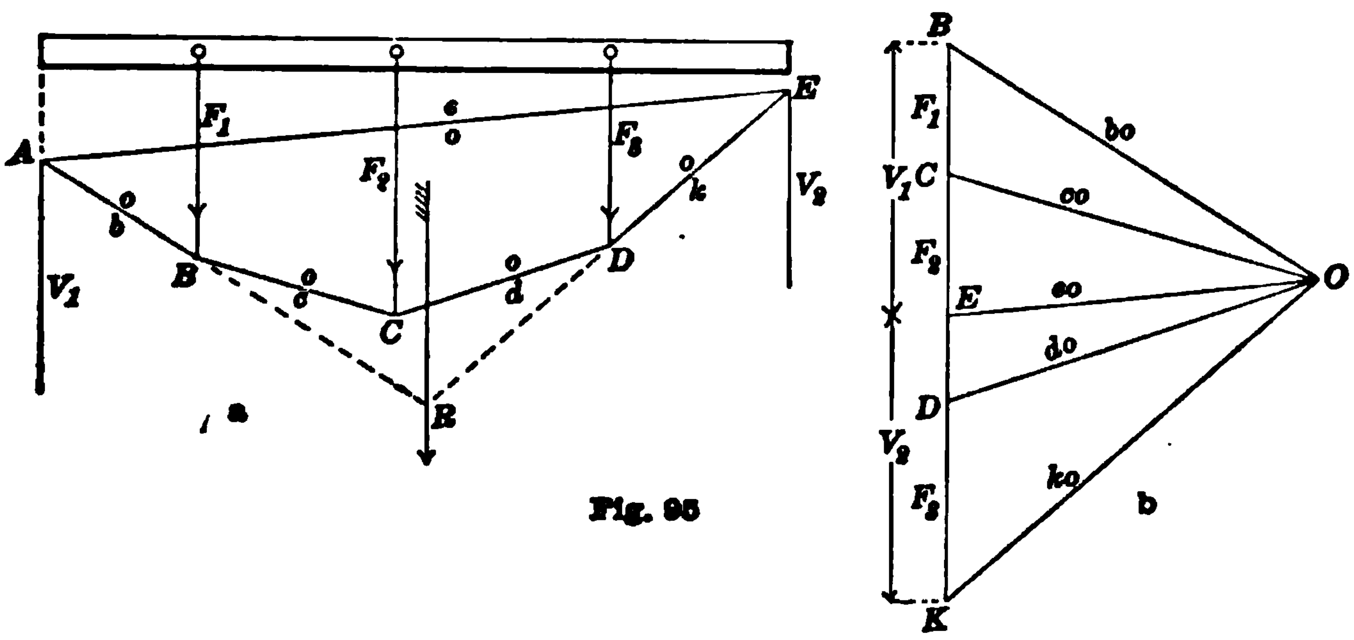


Fig. 95

Choosing now a pole for the stresses in the chain, draw the "rays" and construct the chain polygon as before, starting at A, and find the position of the resultant R.

Now, let us suppose that the other parallel component of R is to pass thru the point E, in the last link of the chain. We then insert a new link from A to E, and draw the "ray" parallel to it from O in the force diagram. The length eo gives the stress of this link and the two components of R, which are BE and EK in the stress diagram,

are determined. If these components are individually supported and balanced by direct forces, V_1 and V_2, we have parallel supports which would completely balance a given system of forces.

If now we consider V_1 and V_2 as additional forces acting upon the given body, we have a set of vertical forces which balance, and the complete closed chain forms the Equilibrium Polygon. It is, in fact, the "static polygon" of the chain.

The reader should take note of the static triangles of the several points or pins, A, B, C, D and E, in Fig. 95(b).

92. When both vertical and inclined forces act on a structure, as in the case of vertical loads and wind pressures on a roof, the resultant is in general not vertical. If now one of the supports is to be vertical, the other must be inclined, and the resultant must be resolved into two components, one of which is vertical and acting from the point where the support is to be vertical, the other inclined and acting through the point where the inclined support must act.

In such a case, the line of the resultant already found, should be produced until it intersects the vertical support-line; a line drawn from their intersection to the other support will give the direction of the inclined support.

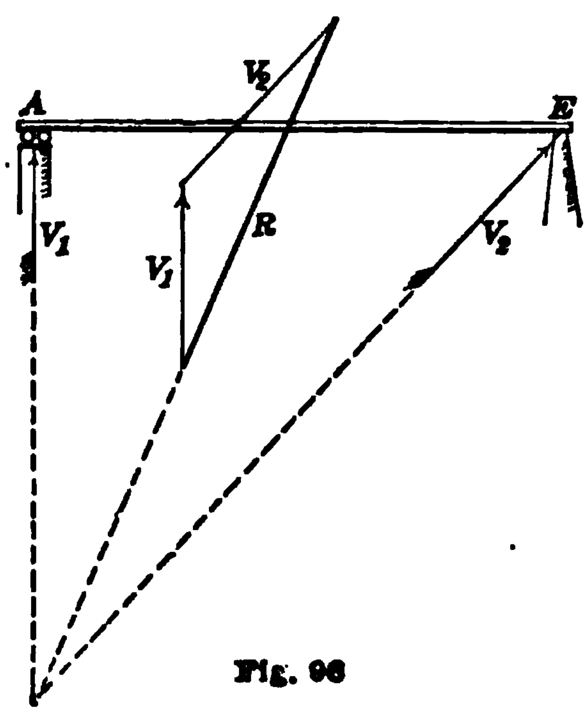


Fig. 96

Suppose the resultant of a system of vertical and inclined forces, acting on a roof, has been found, and is represented by R. Fig. 96. The two supports are at A and E. The roof is anchored at E, and rests on rollers at A, thereby making the support V_1 at A vertical.

The above furnishes a method of finding the supports for the problems solved in Graphical Statics, Chapter XI.

93. If both supports are to be inclined, any point in the line of action of the resultant may be taken as the point from which the two supporting action-lines shall pass, and their inclinations may be chosen at will. In such a case the supports, V_1 and V_2, are readily found.

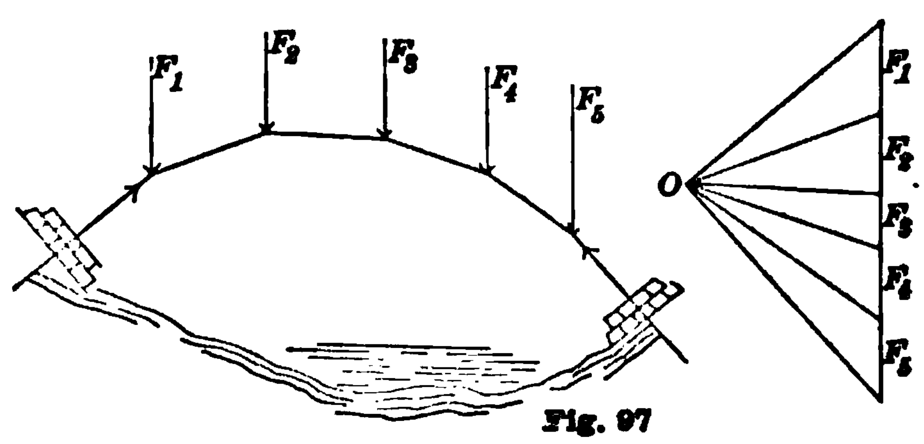


Fig. 97

94. An equilibriated arch. In the case of an equilibriated arch, the supports are always inclined as illustrated below where the given forces are all vertical. Fig. 97.

The student who later makes a special study of **arches** will find another and most important use of the equilibriated arch.

Problem.

Let the student assume a series of equal vertical loads, placed at equal intervals, and support them by an equilibriated *arch*; and again by a *chain*; and see how closely the balanced pins lie in parabolic arcs.

CHAPTER VI.

Forces in Space.

95. Case I. **When three forces meet at a point. Graphical.** When three forces not in the same plane meet at a point, they cannot possibly balance; they can, however, be balanced by a fourth force acting thru the common point. The fourth force must be equal and directly opposite to the resultant of the three. Hence, to find the equilibriating force, we first find the resultant of the given forces. The resultant can be found graphically by drawing in different planes. Thus the resultant of two can be combined with the third in a new plane. The resultant vector of three forces is evidently the diagonal of a parallelopiped.

R_1, Fig. 98, is the resultant of P_1 and P_2; and the resultant of R_1 and P_3 is R, which is seen to be the diagonal of a parallelopiped, which has the given force lines as diverging (or converging) edges. The drawing, however, is not a scale drawing, and has little value beyond giving clear mental concepts. When four forces balance, we have a static polygon (putting $P_4 = -R$), $OABC-O$, which is a warped static polygon.

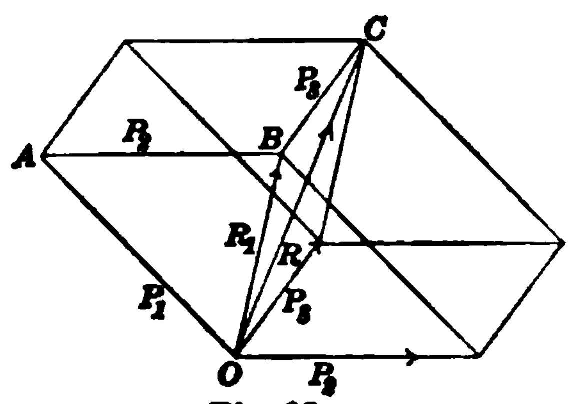


Fig. 98

96. Analytic. 1. The only convenient method of recording the data and reaching exact solutions of such space problems is the analytic, using three rectangular axes, and specifying directions by the angles which force lines make with them separately. We shall, in general, assume that OX and OY are horizontal, and that OZ is vertical. The logic of the arrangement has already been explained.

2. The angles which a force P, makes with the axes OX, OY and OZ, are respectively α, β, γ, the arrows always pointing *from* the origin.

It is evident that these angles are not wholly independent of each other. The angles lie in different planes, no one of which (except in special cases) is a co-ordinate plane.

Suppose we know, at first, only that α is 40°. At once we picture a cone of revolution with O for its vertex, OX for its axis, and a half-vertical angle of 40°. If now we learn that β is 60°, we think of another cone mounted on OY with a half vertical angle of 60°. Of course, these cones intersect in two elements, one above and one below the plane of XY. There are therefore two possible values of γ which are supplements of each other, which are both determined, and P is either in the *first* or *fifth* triedral angle.

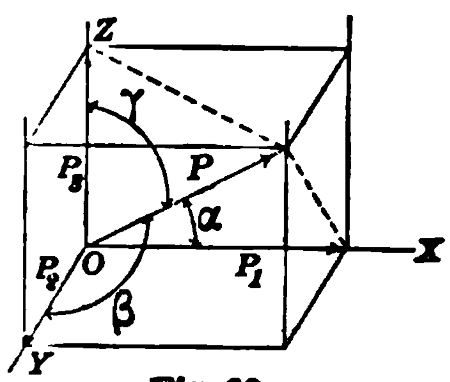


Fig. 99

3. The trigonometric equation showing the relation of the three angles is found as follows:

If the force P be resolved into its rectangular components P_1 along OX, P_2 along OY, and P_3 along OZ, we have by inspecting Fig. 99:

$$P_1 = P\cos\alpha$$
$$P_2 = P\cos\beta$$
$$P_3 = P\cos\gamma$$

Squaring and adding, and taking note of the fact that

$$P_1^2+P_2^2+P_3^2=P^2, \text{ we have}$$

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1; \tag{1}$$

hence, when

$$\alpha=40° \text{ and } \beta=60°$$

$$\cos\gamma = \pm\sqrt{(1-\cos^2 40°-\cos^2 60°)}$$

giving the two values of γ already noted.

4. When the resultant of several forces acting at the same point in space is desired, for the purpose of finding the balancing force, proceed as follows:

(*a*) Assume the common point of action as the origin, and if possible, take the co-ordinate planes so that at least one of the forces will lie in a co-ordinate plane.

(*b*) Ascertain the direction angles of every given force.

(*c*) Resolve every given force into its three rectangular components.

(*d*) Add the components on the separate axes, getting

$$\left.\begin{aligned} R_1 &= \Sigma P\cos\alpha \\ R_2 &= \Sigma P\cos\beta \\ R_3 &= \Sigma P\cos\gamma \end{aligned}\right\} \tag{2}$$

(*e*) Find R by the equation

$$R = \sqrt{R_1^2 + R_2^2 + R_3^2}$$

and α_r, β_r and γ_r by the equations

$$\begin{aligned} \cos\alpha_r &= R_1/R \\ \cos\beta_r &= R_2/R \\ \cos\gamma_r &= R_3/R \end{aligned} \tag{3}$$

97. Illustrative example. A weight of 100 lbs. is to be suspended to a small ring directly over it, and that ring is to be held in position by *three* cords. The positions and tensions of two of the cords are known, but nothing is known of the third cord and force except that it must balance the three others. The unknown force is to be fully determined.

The positions and magnitudes of the given forces are shown in Fig. 100.

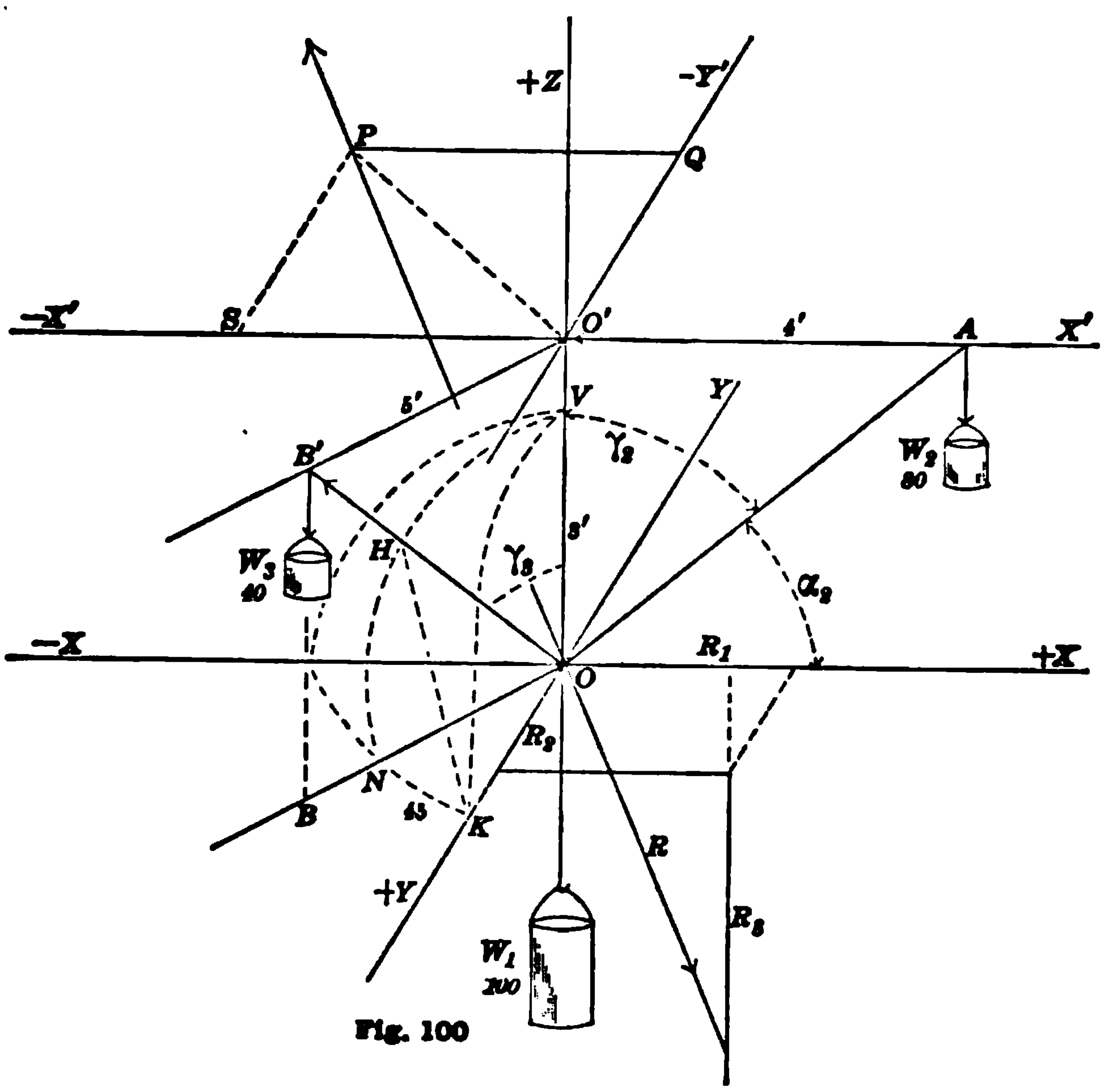


Fig. 100

Solution. Take the ring as the origin, and the plane of W_1 and W_2 as the plane XZ. A horizontal gridiron frame, three feet (3′) above the ring, supports light pulleys, over which cords are to run from the weights $W_2 = 80$ lbs., and from $W_3 = 40$ lbs. The tension line of W_2 is thru A, which is in the plane $X'Z$, and 4′ from O'; hence

$$\cos\alpha_2 = \frac{4}{5}; \ \cos\gamma_2 = \frac{3}{5}; \ \cos\beta_2 = 0$$

The tension line of $W_3 = 40$ lbs. is in a vertical plane which *bisects* the angle $(+Y)O(-X)$, and the pulley B' is 5′ from O', and 3′ above the plane XY; hence $\tan\gamma_3 = \frac{5}{3}$, and $\cos\gamma_3 = \frac{3}{\sqrt{34}}$.

To find β_3, assume a spherical surface, with radius unity and center at O, cutting OY at K, and the force line of W_3 at H.

The bisecting plane, passing thru B' and OZ, cuts the sphere in the arc HN, which is the complement of γ_3. Drawing the arc NK cut from the unit sphere by the plane XY, we have a spherical right triangle HNK, right-angled at N. Its legs are $KN=45°$, $HN=90-\gamma_3$, while $HK=\beta_3$.

Hence, by Napier's Rules,
$$\cos\beta_3=\cos 45°\sin\gamma_3=\frac{5}{34}\sqrt{17}.$$
and by **96** (1)
$$\cos\alpha_3=-\sqrt{1-\frac{9}{34}-\frac{25}{68}}=-\frac{5}{34}\sqrt{17}.$$

The reason for the negative value of $\cos\alpha_3$ is evident.

Table giving *data* and *solution*:

P	$\cos\alpha$	$\cos\beta$	$\cos\gamma$	$P\cos\alpha$	$P\cos\beta$	$P\cos\gamma$
$W_1=100$	0	0	-1	0	0	-100
$W_2=\ 80$	0.800	0	0.600	64.00	0	48.00
$W_3=\ 40$	-0.606	$+0.606$	0.515	-24.24	$+24.24$	20.60
$P=?$	α_r	β_r	γ_r	$R_1=39.76$	$R_2=24.24$	$R_3=-31.40$

$$R=\sqrt{R_1^2+R_2^2+R_3^2}=56.16$$

The position of R is seen from its three components, R_1, R_2 and R_3, in Fig. 100, to be in the 5th triedral angle.

The angles α_r, β_r, γ_r are found from their cosines, which are
$$\cos\alpha_r=\frac{39.76}{56.16},\ \cos\beta_r=\frac{24.24}{56.16},\ \cos\gamma_r=\frac{-31.4}{56.16}$$

If we represent the direction angles of the *balancing force* by
$$\alpha'=180°-\alpha_r,$$
$$\beta'=180°-\beta_r,$$
$$\gamma'=180°-\gamma_r,$$
we have
$$\cos\alpha'=-\cos\alpha_r=-\frac{39.76}{56.16}$$
$$\cos\beta'=-\cos\beta_r=-\frac{24.24}{56.16}$$
$$\cos\gamma'=-\cos\gamma_r=+\frac{31.4}{56.16}$$
and the Balancing Force is completely found.

To exhibit the position of its line of action more clearly, it can be said that the force required to balance the three given forces *acts up into the third triedral angle.* It pierces the plane $X'O'Y'$ (which is three feet above XOY) at a point P, which is:

$3\tan\gamma'$ distant from O',
$3\sec\gamma'\cos\alpha'$ distant from $O'Y'$,
$3\sec\gamma'\cos\beta'$ distant from $O'X'$.

F	α	β	γ
20	60°	< 90°	50°
85	> 90°	30°	90°
100	135	100°	> 90
60	30°	120°	?
$R=?$	$\alpha_r=?$	$\beta_r=?$	$\gamma_r=?$

Problem.

Let the student mentally picture the concrete situation whose data are given in this table, and represent them *graphically*; then derive the answers to the five questions. One square in the table is purposely left vacant.

Queries.

1. What does it mean when one of the components of R is zero?
2. What does it mean when two of the components of R are zero?
3. What does it mean when all three components of R are zero.

98. Case II. **When the forces do not converge to a point, but act at different points and in different directions.** The General Case. The analysis is an extension of the analysis for non-concurrent forces in a plane. A point of application is given by its three co-ordinates, x, y and z. The direction of a force by three direction angles, α, β, γ, with reference to three rectangular lines (parallel to the co-ordinate axes) drawn thru the point of application. The steps in the process of finding a resultant are as follows:

(*a*) *Resolve every force into its three components*:

$$P_1 = P\cos\alpha$$
$$P_2 = P\cos\beta$$
$$P_3 = P\cos\gamma$$

(*b*) *Resolve every component into an equal parallel force at O, and two moments,* with reference to the two co-ordinate axes perpendicular to it. This must be illustrated. Let P_1, Fig. 101, the component acting at the point $A(x, y, z)$ be resolved into an equal force, P_1', acting B, and a couple zP_1 (*right-handed*) about the Y axis. Next think of P_1' as acting at C, and resolve it into an equal parallel force, P_1'', at O, and a *left-handed* couple yP_1 about the axis Z. Of course P_1 produces no moment about the X axis, with which it is parallel.

We thus get for P_1 acting in space:

a force P_1 at O, along OX;
a moment about OY of $+zP_1$;
and a moment about OZ of $-yP_1$.

In like manner, for the component P_2, we shall get:

a force P_2 at O, along OY;
a moment about OZ of $+xP_2$;
and a moment about OX of $-zP_2$.

And for the component P_3:

a force P_3 at O, along OZ;
a moment about OX of $+yP_3$;
and a moment about OY of $-xP_3$.

Summing these results, we get for every force P in space:
Three components, P_1, P_2, P_3, at the origin of co-ordinates;

A moment about OX of $yP_3 - zP_2 = M_x$
A moment about OY of $zP_1 - xP_3 = M_y$.
A moment about OZ of $xP_2 - yP_1 = M_z$.

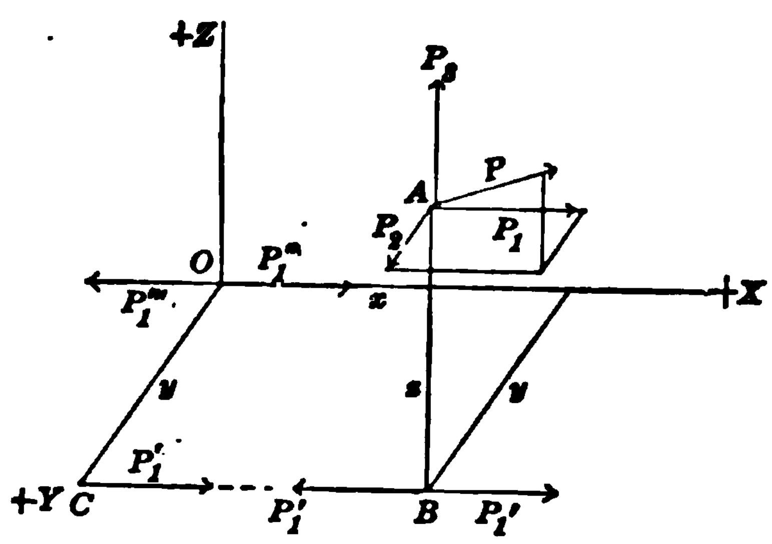


Fig. 101

A careful examination of Fig. 101 will make all these moments intelligible.

If the results are tabulated the student will notice the sequence of letters and numbers.

P	M_x	M_y	M_z
$P_1P_2P_3$	$yP_3 - zP_2$	$zP_1 - xP_3$	$xP_2 - yP_1$

Note the sequence of letters and numbers, in Fig. 102.

The student will also notice that the moment about OZ or M_3 is identical with that found for the axis OZ, when the force was in the plane XY, and there could be no moment about OX and OY as both z and P_3 were zero.

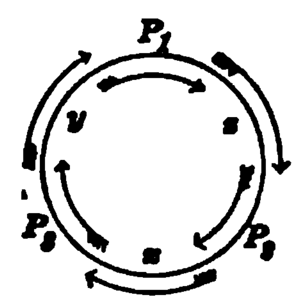


Fig 102

(c) After all the forces are resolved as above, sum the results.

$$\text{Find } \Sigma(P_1) = R_1 \qquad \text{Find } \Sigma(P_2) = R_2 \qquad \text{Find } \Sigma(P_3) = R_3$$

$$\text{and} \left.\begin{aligned} \Sigma(yP_3) - \Sigma(zP_2) &= M_1 \\ \Sigma(zP_1) - \Sigma(xP_3) &= M_2 \\ \Sigma(xP_2) - \Sigma(yP_1) &= M_3 \end{aligned}\right\} \tag{4}$$

(d)

$$\left.\begin{aligned} R &= \sqrt{(R_1^2 + R_2^2 + R_3^2)} \\ M &= \sqrt{(M_1^2 + M_2^2 + M_3^2)} \end{aligned}\right\} \tag{5}$$

This combination of three moments about three separate rectangular axes is in accord with what was shown in (II).

In general, the resultant of such a system of forces in space is a *resultant force* acting at the origin, *and* a *resultant moment* about an axis in space.

(*e*) The direction angles for R are found by the equations

$$\left.\begin{aligned}\cos\alpha_r &= R_1/R\\ \cos\beta_r &= R_2/R\\ \cos\gamma_r &= R_2/R\end{aligned}\right\} \qquad (6)$$

And the direction angles of the axis of the resultant moment are found by the equations

$$\begin{aligned}\cos\lambda &= M_1/M\\ \cos\mu &= M_2/M\\ \cos\nu &= M_3/M\end{aligned} \qquad (7)$$

2. The axis of M may be assumed to pass thru O. If the axis of M is parallel to, or coincides with, R, no further combination is possible.

If the axis of M and R make an angle θ, M may be resolved into two new components, $M\cos\theta$, and $M\sin\theta$. The component, $M\sin\theta$, can be combined with R, thereby shifting R to a parallel position distant from O by a perpendicular L.

$$L = \frac{M\sin\theta}{R}$$

The perpendicular is drawn normal to a plane which passes thru R and is parallel to the axis of M. The resultant of the system is then R in this new position and the moment $M\cos\theta$. If θ is a right angle, $M\cos\theta$ is zero, and the resultant is a single force.

If $M=0$, the resultant is a force R at the origin of co-ordinates. If $R=0$ while M is not zero, the resultant is a couple with moment M.

If $R=0$ and $M=0$, the system is self-balanced and the body acted upon is in equilibrium.

While an engineer rarely has occasion to solve a general problem in statics, it is well to embody the method in the solution of a problem with assumed data.

Problem.

Solve the problem in the table, filling out blank spaces systematically.

P	x	y	z	α	β	γ	P_1	P_2	P_3	yP_3	zP_2	zP_1	xP_3	xP_2	yP_1
8	4	5	12	75	50	<90°									
16	O	O	O	30		60									
10	−2	+3	−4	135°	60	>90									
20	8	−1	−6	>90	70°	45°									
							$R_1=?$	$R_2=?$	$R_3-?$						
			$R=?$		$M=?$					$M_1=?$		$M_2=?$		$M_3=?$	

The student will notice, when $P=8$, that $\cos\gamma = +\sqrt{\sin^2\alpha - \cos^2\beta}$, since γ must be acute.

The actual value of γ need not be found, as only its cosine is wanted.

The value of β for the second force the student must supply himself.

The above system of forces can be balanced only by balancing R, and M, either by a single force or a combination of a single force and a single couple.

CHAPTER VII.

Centroids and Centers of Pressure.

99. 1. The two most important applications of the theory of parallel forces are found in problems which require the concentration of distributed forces. In Chapter I it was assumed that such concentration was possible, and we did not hesitate to give the total magnitude of a force distributed over a surface or thru a volume; but we did not attempt to determine the exact location of the line of action of the concentrated force. This chapter will be devoted to finding the positions of concentrated or resultant forces when the law of distribution is definitely known.

100. The point in a surface where the resultant of a distributed force pierces that surface, is called the "Center of Action."

As has been said, all of our problems are more or less ideal, but they closely approximate the real. We assume, for example, that the steam or air pressure upon the vertical face of a piston is uniform, tho we know that such is not absolutely the case. We assume that the pressure between the corner stone of a building and the material below it is uniformly distributed, when it probably is not uniform. We assume that the earth's attraction upon a cubic inch of material in a block of wood, stone or metal, is uniform thruout the block, tho we have no idea that such is the absolute truth.

101. The phrase: "The intensity of the surface action at a point" means: the amount of action or force there would be on a unit of surface if thruout the unit surface the force were uniformly what it is upon the infinitesimal area surrounding the given point.

Since the *magnitude* and *direction* of a *uniformly distributed force* cannot affect the position of the Center of Action, we may assume that the direction is normal, and that the intensity is p, which is as small as one likes.

102. Centroids of plane surfaces. The point where the *resultant of a uniformly distributed force* pierces a *plane surface* is called the "*Centroid*"* of that surface.

In some simple geometrical surfaces the centroid is evident by symmetry, without demonstration. For example: The circle, the ellipse, a rectangle, all regular polygons. If a surface has an axis of symmetry, the centroid is on that axis; two such axes determine the centroid. If a surface can be divided into parts such that the centroid of every part is known, and known to lie on a certain straight line, the centroid of the surface is known to lie on that line. The number of parts may be finite or infinite. Hence the centroid of a triangle is at the intersection of its medial lines.

103. When a surface of action can be divided into finite parts whose several areas and centroids are known, the problem of finding the centroid of the entire surface falls directly under parallel forces in space. Chapter III.

It was proved in geometry that the point which we now call the centroid of a plane triangle is on a medial line two-thirds of its length from the vertex end. **The centroid of a plane trapezoid** is found graphically as shown in Fig. 103. The diagonal AB divides the surface into two triangles whose centroids determine the right line, ab. The diagonal CD, gives two new triangles, and a second line, cd, which, with ab determines G, the centroid of the trapezoid.

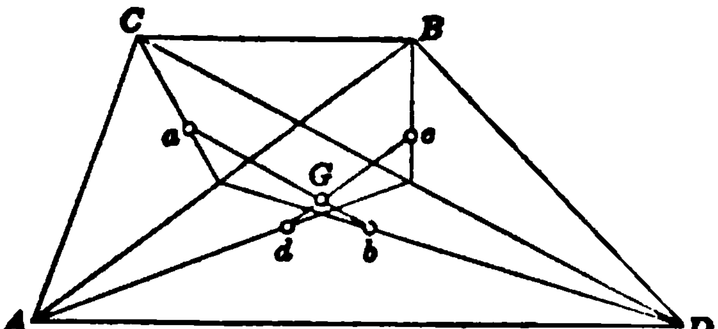


Fig. 103

The centroid of any rectilinear plane surface is found by an extension of the method just used.

104. The general case. When the outlines of a surface are curved and irregular, and there are no axes of symmetry, the *Centroid* must be found by the doctrine of moments and the use of the Calculus.

1. Let the uniform force intensity be p. If A represents the whole area of the surface of action, ΔA will represent a portion of that surface, and $p\Delta A$ will represent the magnitude of the force on that portion. As the whole is equal to the sum of its parts

$$R = \Sigma(p\Delta A)$$

* The author prefers the word "Centroid" to the phrase "Center of Gravity" when speaking of plane areas, yet the latter is in very common use.

The introduction of thin flat plates of heavy material fails to hide the absurdity of speaking of the *gravity of a surface;* and to go further and assume that the plate is half above and half below the given area in order to bring the C. G. of the plate into coincidence with the Centroid of the surface is confusing to a novice.

In this case the summation must be made by the Calculus. Instead of the sum of a finite number of finite terms, we must have the sum of an infinite number of infinitesimal terms. Hence we shall use $\int$ for Σ, and d for Δ, and the above formula becomes

$$R = \int p dA$$

It is here and elsewhere assumed that the student is familiar with the ordinary methods of integration, but he may be less familiar with their application to practical problems. Accordingly, some fundamental matters will be fully explained. The expression dA is the numerical measure of a portion of the surface A, and preferably represents the product of a length by a breadth. Both factors may be infinitesimals, as $dxdy$; or one may be finite, as ydx, or $(y_2-y_1)dx$; see Figs. 104 and 105. If $dA=dxdy$, two integrations will be required; if $dA=ydx$, only one integration is necessary; hence the second form is preferred, when it can be used.

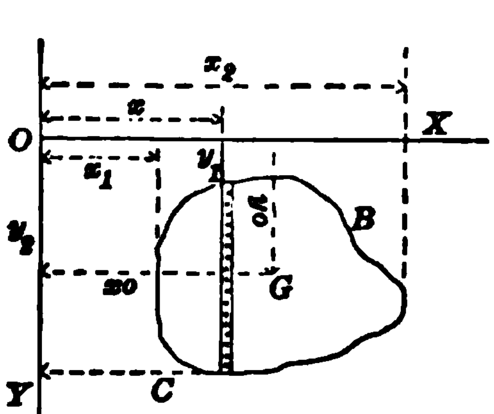


Fig. 104

2. Let BC be a surface bounded by outlines not at present defined. General formulas for the position of its centroid are to be found. Let the surface A be divided into elements parallel to the axis OY, with a constant width, dx. The elementary area of a single strip is $(y_2-y_1)dx$, according to the lettered Fig. 104. If p be the intensity of the distributed force, the force acting upon the strip is $(y_2-y_1)pdx$. The moment of this force about OY is $x(y_2-y_1)pdx$, and the sum of the moments of all the elements is

$$M_y = \int_{x_1}^{x_2} x(y_2-y_1)pdx = p\int_{x_1}^{x_2}(y_2-y_1)xdx \tag{1}$$

since p is constant during the integration.

Now, the total force acting on the surface, BC, is $p\int_{x_1}^{x_2}(y_2-y_1)dx$ which is p times the area, or pA. When this force is centralized it will act at the centroid of the surface whose co-ordinates we will call x_0 and y_0. Then the moment of the centralized or resultant force about OY will be $x_0p\int_{x_1}^{x_2}(y_2-y_1)dx = M_y$

Now follows an *Axiom of Mechanics*:

The moment of the whole is equal to the whole of the moments.

$$x_0p\int_{x_1}^{x_2}(y_2-y_1)dx = p\int_{x_1}^{x_2}(y_2-y_1)xdx$$

$$x_0R = M_y$$

Hence $$x_o = \frac{\int_{x_1}^{x_2}(y_2-y_1)x\,dx}{\int_{x_1}^{x_2}(y_2-y_1)dx} = \frac{M_y}{R} \tag{2}$$

3. **The method of integration** is determined in every case by the equations of the boundary lines of the surface.

The disappearance of p shows that x_o depends not on the absolute value of p, but upon its *uniformity*. We will therefore assume that $p=1$, and numerically $R=A$.

4. The co-ordinate, y_o, is found in a similar manner. If the same elementary strip, $(y_2-y_1)dx$, is used, we must get its moment about OX by multiplying its area by the ordinate to its central point, which is $\frac{y_1+y_2}{2}$. Hence our value of y_o will appear

$$y_o = \frac{\frac{1}{2}\int_{x_1}^{x_2}(y_2^2-y_1^2)dx}{\int_{x_1}^{x_2}(y_2-y_1)dx} = \frac{M_x}{R} = \frac{M_x}{A}. \tag{3}$$

5. If a new element is taken parallel to the axis of X, the value of y_o appears thus:—

$$y_o = \frac{\int_{y_1}^{y_2}(x_2-x_1)y\,dy}{\int_{y_1}^{y_2}(x_2-x_1)dy} = \frac{M_x}{A} \tag{4}$$

There are other general methods depending upon the form of the surface element and upon the use of circular or polar co-ordinates. A variety of illustrative examples is chosen so as to make every method clear.

105. Examples illustrating methods of finding centroids.

Ex. 1. The centroid of a parabolic half segment. Fig. 105.

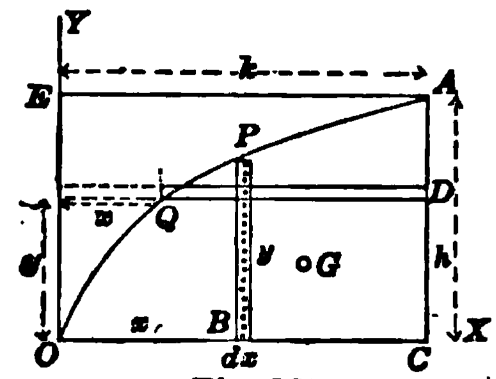


Fig. 105

Let the surface be OAC, and the element be $PB=ydx$. The Equation of the curve is $y^2=ax$. N. B.—While we are to find x_o and y_o, the co-ordinates of the *interior* point G, the student must bear in mind that x and y thruout our analysis are the co-ordinates of points P, on the *bounding curve*.

Since $p=1$, the force on our element is ydx, and its moment about the axis OY is
$$dM_y = xydx$$

Hence $$M_y = \int_0^k a^{\frac{1}{2}}x^{\frac{3}{2}}dx = \frac{2}{5}k^2h$$

It was found in the calculus that

$$\text{Area of } OAC = \frac{2}{3} kh = A = R.$$

Hence by **104**, $$x_o = \frac{M_y}{A} = \frac{3}{5} k.$$

The centroid of the strip element, BP, is at its middle point $y/2$ from OX; hence, the moment of the element about OX is

$$dM_x = \frac{y}{2} ydx$$

and $$M_x = \frac{1}{2}\int_o^k y^2dx = \frac{a}{2}\int_o^k xdx = \frac{ak^2}{4}.$$

$$y_o = \frac{M_x}{A} = \frac{3}{8} h.$$

106. The student must not be wedded to one form of a surface element. The element $QD = (k-x)dy$, would have served equally well. We see at once using subscripts 1 and 2 in place of x and y:

$$dM_1 = (k-x)ydy$$

$$M_1 = \int_o^h kydy - . \int_o^h xydy$$

$$= \frac{kh^2}{2} - \frac{h^4}{4a} = \frac{kh^2}{4}$$

$$y_o = \frac{M_1}{A} = \frac{3}{8} h$$

$$dM_2 = (k-x)\frac{k+x}{2} \cdot dy$$

$$M_2 = \frac{1}{2}\int_o^h (k^2 - x^2)dy = \frac{hk^2}{2} - \frac{h^5}{10a^2} = \frac{2}{5} k^2h$$

$$x_o = \frac{M_2}{A} = \frac{3}{5} k.$$

107. The algebraic method. This method is based upon a full set of data, and is best illustrated by an example. Suppose we wish to know the centroid of the parabolic spandril $OQAE$, knowing the centroid of the rectangle $OCAE$, and the centroid of the half segment OCA, as found above.

The three centroids are on the same straight line. Why? And the centroid of the spandril is *twice* as far from the center of the rectangle as is the G of the segment. Why?

The Axiom of Moments with respect to each axis, OX and OY, serves our purpose. Letting x_o, y_o be the unknown co-ordinates of the centroid of the spandril, and using the data of the rectangle and segment, we have the two moment equations.

$$M_2 = hk\times\frac{1}{2}k = \frac{2}{3}hk\times\frac{3}{5}k+\frac{1}{3}hk\times x_o$$

$$M_1 = hk\times\frac{1}{2}h = \frac{2}{3}hk\times\frac{3}{8}h+\frac{1}{3}hk\times y_o$$

whence

$$x_o = \frac{3}{10}k$$

$$y_o = \frac{3}{4}h.$$

This indirect method is often very convenient.

Ex. 1. Find the Centroid of the above Spandril by direct integration.

Ex. 2. Find the Centroid of a quadrant of an ellipse.

Ex. 3. In Fig. 103, let the non-parallel sides, AC and DB, be produced till they meet in O. Find graphically the centroid of the triangles OAD and OCB, and measure their distance from O. Then by the algebraic method find the distance to the centroid of the trapezoid.

108. Find the centroid of a **circular segment**, introducing circular co-ordinates.

Solution. The segment is ABC, Fig. 106. Since OX is an axis of symmetry, the centroid is on that axis and $y_o=0$, The element is $2ydx$. From the figure

Fig. 106

$$x=r\cos\theta,\ y=r\sin\theta,\ dx=-r\sin\theta\,d\theta,\ x_1=r\cos\phi.$$

$$M_y=\int_{x_1}^{r}2yxdx=-2\int_{\varphi}^{o}r^3\sin^2\theta\cos\theta\,d\theta=2r^3\int_{o}^{\varphi}\sin^2\theta\cos\theta\,d\theta,=\frac{2}{3}r^3\sin^3\phi.$$

$$A=2\int_{x_1}^{r}ydx=-2r^2\int_{\varphi}^{o}\sin^2\theta\,d\theta=2r^2\int_{o}^{\varphi}\sin^2\theta\,d\theta=r^2(\phi-\sin\phi\cos\phi).$$

The student will take note that the expression for A (the area of the segment) is explicitly the area of the sector $OACB$ *minus* the area of the triangle OAB: that is

$$A=r^2\phi-r^2\sin\phi\cos\phi.$$

Dividing M_y by A we have

$$x_o=\frac{M_y}{A}=\frac{2}{3}\cdot\frac{\sin^3\phi}{\phi-\sin\phi\cos\phi}\cdot r$$

For a *semi-circle* $\phi = \pi/2$ and

$$x_o = \frac{4r}{3\pi}.$$

NOTE. The student must not be surprised that $dx = -r\cos\theta\, d\theta$ with the negative sign. If he looks at Fig. 106, he will see that as *x increases, θ decreases*; hence, if dx is positive, $d\theta$ must be negative, and $(-d\theta)$ is positive.

109. The centroid of a circular sector. The surface element may best be defined by polar co-ordinates. Let the given surface be the sector of a circle, *OABC*, Fig. 107. As *OB* is an axis of symmetry, we have only to find the distance x_o. This could be found by algebra if we knew the position of the centroid of the circular segment *ACB*, as well as of the triangle *OAC*. But the direct method is simpler, as follows:

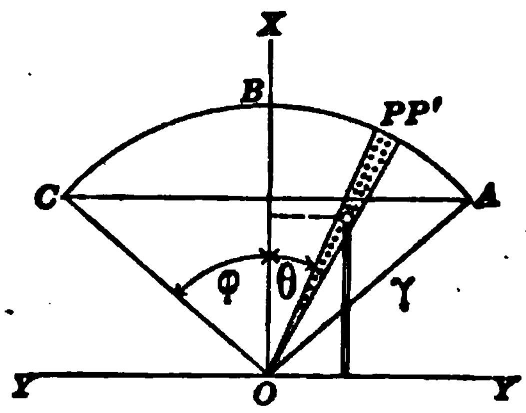


Fig. 107

Let the radius be r, and the half-angle $AOB = \phi$. Draw two consecutive radii in a general position defined by the arcs θ and $d\theta$; the elementary surface is an infinitesimal triangle whose area is $\frac{1}{2}r^2 d\theta$; its centroid is at a distance $\frac{2}{3}r$ from O, and its distance from YY is $\frac{2}{3}r\cos\theta$. Hence, taking moments about YY, and integrating, we have

$$x_o = \frac{2\int_o^{\varphi}\frac{r^2}{2}\cdot\frac{2}{3}r\cos\theta\, d\theta}{2\int_o^{\varphi}\frac{r^2}{2}d\theta} = \frac{2}{3}\cdot\frac{\sin\phi}{\phi}.r$$

Both integrals are doubled so as to include the whole sector.

Corollary.

If $\theta = \pi/2$, the sector is a semicircle and $x_o = 4r/3\pi$ as found in **108**.

Ex. Find the centroid of the "spandril" to a segmental arc.

110. DOUBLE INTEGRATION. **To find the centroid of a sector of a circular ring area,** *EABCDN*. Fig. 108. We now take an elementary part of such a sector enclosed by two cosecutive arcs, and two consecutive radii. This element being an infinitesimal of the second order is treated as a rectangle with the area $(\rho d\theta)$ $(d\rho)$. Its moment about YY is $\rho^2 d\rho \cos\theta d\theta$.

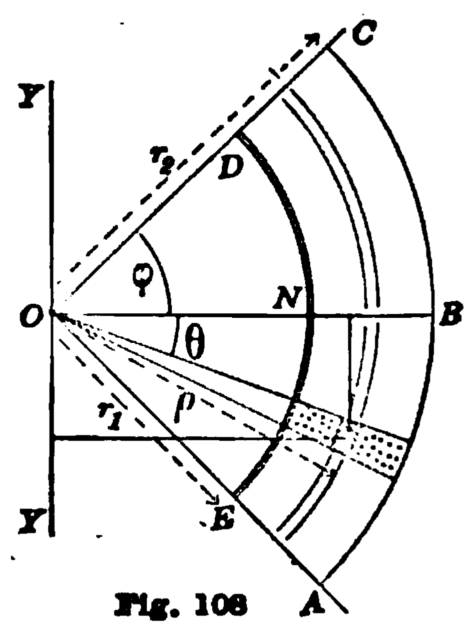


Fig. 108

Hence
$$x_0=\frac{2\int_{r_1}^{r_2}\int_0^{\varphi}\rho^2 d\rho\cos\theta d\theta}{2\int_{r_1}^{r_2}\int_0^{\varphi}\rho d\rho d\theta}$$

The order of integration is immaterial. If we integrate for θ first, we shall get the moment of a sector of an elementary ring as shown by adjacent arcs; its length is $2\rho\phi$ and its width $d\rho$. The next integration would sweep the ring outward from $\rho=r_1$ to $\rho=r_2$.

Had we integrated first for ρ, we should have had a trapezoid lying between two consecutive radii and extending from r_1 to r_2.

Taking the first order, integrating for θ while ρ and $d\rho$ are constant, we get (canceling the 2)

$$x_0=\frac{\sin\phi\int_{r_1}^{r_2}\rho^2 d\rho}{\phi\int_{r_1}^{r_2}\rho d\rho}=\frac{2}{3}\cdot\frac{\sin\phi}{\phi}\cdot\frac{r_2^3-r_1^3}{r_2^2-r_1^2}$$

This problem could have been done by the algebraic method and **109.**

If $r_1=0$, we get the centroid of a sector as in the last example.

111. The centroid of a circular arc. Fig. 109. The arc may be supposed to have a breadth of dr, the radius being r. As above, the element is $rdrd\theta$; its moment* about YY is $(rdrd\theta)(r\cos\theta)$.

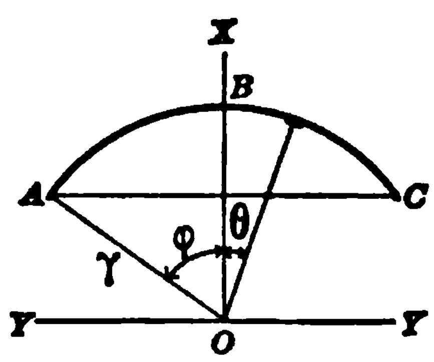


Fig. 109

$$M_y=2r^2dr\int_0^{\varphi}\cos\theta\,d\theta=2r^2\sin\phi dr$$

The area of the arc, or infinitely narrow ring is $2r\phi dr$.

$$\therefore x_0=\frac{M_y}{A}=\frac{2r^2\sin\phi}{2r\phi}=\frac{r\sin\phi}{\phi} \qquad (10)$$

Another form of this result is easily remembered: Since the chord $AC=2r\sin\phi$, and the arc $=2r\phi$, we have

$$\frac{x_0}{r}=\frac{2r\sin\phi}{2r\phi}=\frac{\text{chord}}{\text{arc}}$$

which proportion is readily put into words.

---

* Since we have obliterated the force, p, per square unit of surface, by making it *unity*, we naturally drop into the habit of speaking of a *surface* as tho it were a normal force. The "habit" is quite conventional.

Problem. What is the length of the arc whose centroid is at the center of its central radius?

112. The centroid of a cycloidal area. Suppose the area under consideration is bounded by a cycloidal arc and its base. Fig. 110. It is evident that BH, the longest ordinate, is an axis of symmetry: hence, $x_0 = \pi r$, and only y_0 is to be found.

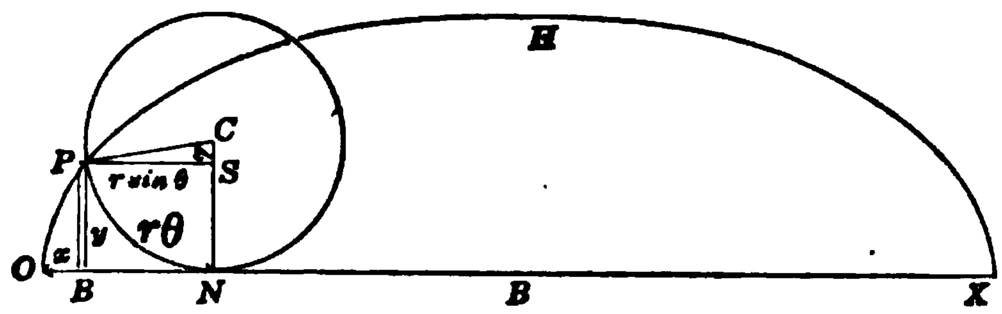


Fig. 110

From the figure, since $ON = r\theta$

$$x = r\theta - r\sin\theta$$
$$y = r - r\cos\theta$$

The element of the surface is ydx, and its centroid is distant from OX a length $y/2$. Hence

$$y_0 = \frac{\int_0^{2\pi r} ydx \cdot y/2}{\int_0^{2\pi r} ydx} = \frac{M_1}{A}$$

From $x = r\theta - r\sin\theta$, we have

$$dx = r(1-\cos\theta)\, d\theta, \text{ so that}$$

$$y_0 = \frac{\frac{r^3}{2}\int_0^{2\pi}(1-\cos\theta)^3\, d\theta}{r^2\int_0^{2\pi}(1-\cos\theta)^2\, d\theta} = \frac{\frac{5}{2}\pi r^3}{3\pi r^2} = \frac{5}{6}r$$

The limits for θ follow from the facts that when $x=0$, $\theta=0$; and when $x=2\pi r$, $\theta=2\pi$. The uncanceled denominator incidentally shows that the area of the cycloid is just three times the area of the rolling circle.

CENTERS OF ACTION OF DISTRIBUTED FORCES WHICH VARY IN INTENSITY.

113. When a surface action is **a distributed force which varies in intensity** according to a definite law, the center of the force for the whole can be found if the centers of the parts can be found. The most important case is that of a *uniformly varying force*, acting on a plane surface. The following are cases where the variation is approximately uniform:

1. The pressure of water upon the wall of a tank.

2. The normal force acting upon the cross-section of an elastic beam which is slightly bent.

3. The pressure upon the top of a column which has a rigid cap supporting an eccentric load.

4. The pressure on the material under a foundation stone which is eccentrically loaded.

All of these cases will be discussed with more or less fullness.

114. The center of hydrostatic pressure. The *intensity* of internal pressure at any point in any liquid which is at rest in an open tank or vessel is equal to the weight of a column of liquid and the air above it, having for its base a unit area at the point, and reaching to the top of the atmosphere. The weight of such a column of air, with its base at sea level, is about 2,116 lbs per square foot. This produces a pressure of 2,116 lbs. upon the surface of the level liquid, and by so much increases the internal pressure of the liquid at every point. The weight of a column of the liquid itself from the point under consideration to the surface must be added to get the intensity required. See Chapter XXIV under Buoyancy.

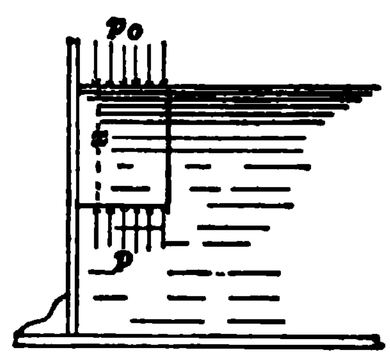


Fig. 111

If p be the required intensity at a depth of x below the surface of a liquid, let w be the *specific weight*, or the weight of a unit of volume, of the liquid, and p_o the atmosphere pressure on the surface of the water; then

$$p = p_o + wx$$

In point of fact, in problems in Applied Mechanics, the pressure p_o is more often balanced by an equal pressure on the other side of the wall or vessel, so that the pressure of still air can generally be left out of consideration. The increased pressure due to moving air, and the diminished pressure caused by rarefaction, require special examination. Accordingly, the *hydrostatic pressure varies with the depth.*

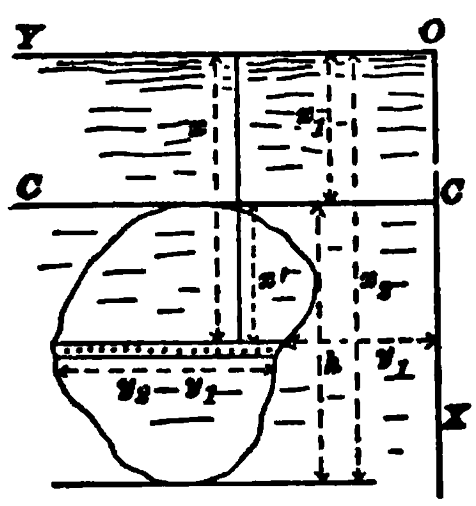


Fig. 112

115. To find the total pressure on a given vertical plane surface and the "center of pressure," we proceed as follows, assuming that the atmospheric pressure is self-balanced:—

1. Take an element of the area of action, at a depth of x below the top of the liquid. Fig. 112. The intensity being the same at all points at that depth, let the element extend across the given surface horizontally, so that its area shall be $(y_2 - y_1)dx$. The pressure at that

depth is xw per unit of surface, and the pressure on the element is $wx(y_2-y_1)dx$, and the total pressure on the entire area is

$$P=w\int_{x_1}^{x_2}(y_2-y_1)xdx. \tag{1}$$

2. The Center of Pressure involves the moment. Take for the moment axis the line CC_1 Fig. 112, tangent to the upper edge of the surface of action. Let $x'=x-x_1$ and $dx=dx'$. The moment of the pressure upon the element is

$$dM_c=w(y_2-y_1)\ (x_1+x')x'dx'$$

and
$$M_c=w\int_0^h(y_2-y_1)\ (x_1x'+x'^2)dx'$$

and
$$x_o=\frac{M_c}{P} \tag{2}$$

The general formula for y_0 is found, when needed, in a similar manner.

116. Let the surface of action A be a rectangle with vertical edges, so that y is constant, and equals b, Fig. 113, so that the area is $A=bh$.

1. Applying **115** (1), we have

$$P=bw.\frac{x_2^2-x_1^2}{2}=bwh\frac{x_2+x_1}{2}=wA\frac{x_2+x_1}{2}=wA\frac{h+2x_1}{2}$$

i. e., the *total pressure* on the rectangle $BCDE$ is equal to the area of the rectangle multiplied by the pressure at the "mean depth." This result was almost self-evident, and the same formula must be true of a circle, viz.: the area of the circle times the intensity of the pressure at the center.

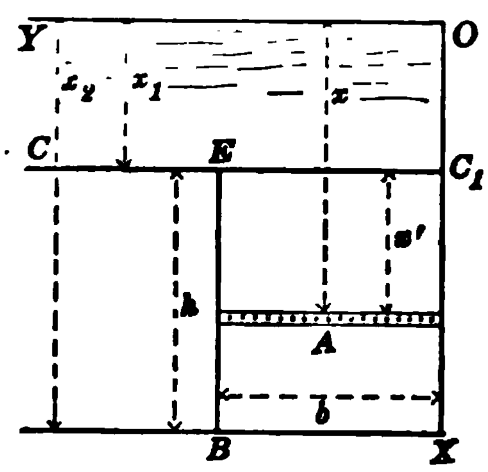


Fig. 113

2. *The moment* about CC_1 of the liquid pressure on the elementary area bdx is $bwxdx\times x'$, as shown in Fig. 113. Since $x=x_1+x'$, and $dx=dx'$, we have for the whole moment

$$M_o=bw\int_0(x_1x'+x'^2)dx'=bw\left(\frac{x_1h^2}{2}+\frac{h^3}{3}\right)$$

$$=\frac{bwh^2}{6}(2h+3x_1)=\frac{wAh}{6}(2h+3x_1)$$

$$\text{and } x_o=\frac{M_c}{P}=\frac{h}{3}\cdot\frac{2h+3x_1}{h+2x_1}$$

in which x_o is measured from the top of the rectangle, CC_1 and x_1 is the distance from CC_1 to the top of the liquid.

Corollary. If the area of action extends to the top of the liquid, $x_1 = o$ and we have

$$x_o = \frac{2}{3}h.$$

117. 1. Let the immersed area be the triangle BBD. See Fig. 114. With the lettering shown, we have

$$P = w\int_{x_1}^{x_2} xy\,dx \text{ in which}$$

$$y = \frac{bx'}{h},\ x = x_1 + x',\ \text{and } dx = dx';$$

Hence
$$P = \frac{bw}{h}\int_0^h (x_1x' + x'^2)dx'$$

$$= \frac{bwh}{6}(3x_1 + 2h)$$

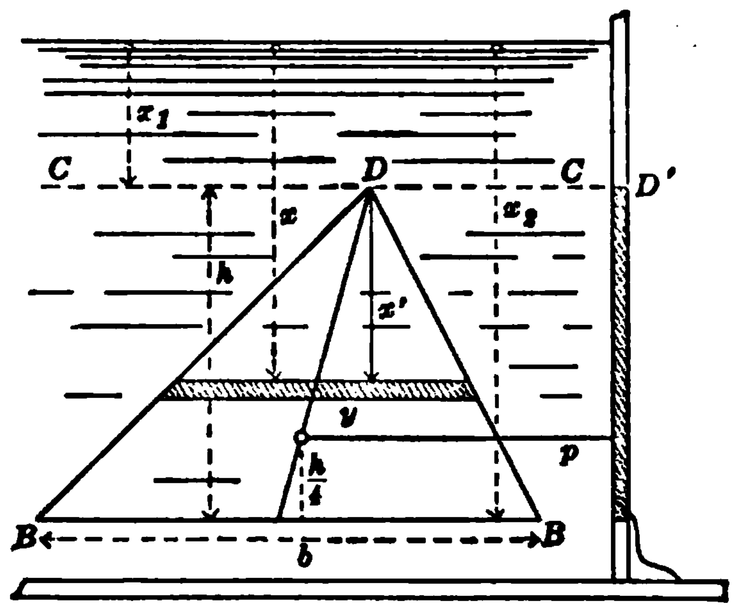


Fig. 114

2. The *moment* about CC of the pressure on the element is $wxydx' \times x'$

and
$$M_c = \frac{wb}{h}\int_0^h (x_1x'^2 + x'^3)dx'$$

$$= \frac{wbh^2}{12}(4x_1 + 3h)$$

and
$$x_o = \frac{M}{P} = \frac{h}{2}\left(\frac{3h + 4x_1}{2h + 3x_1}\right);$$

x_o is measured from CC.

Corollary. If the vertex of the triangle reaches the top of the liquid, $x_1 = 0$, and

$$x_o = \frac{3}{4}h.$$

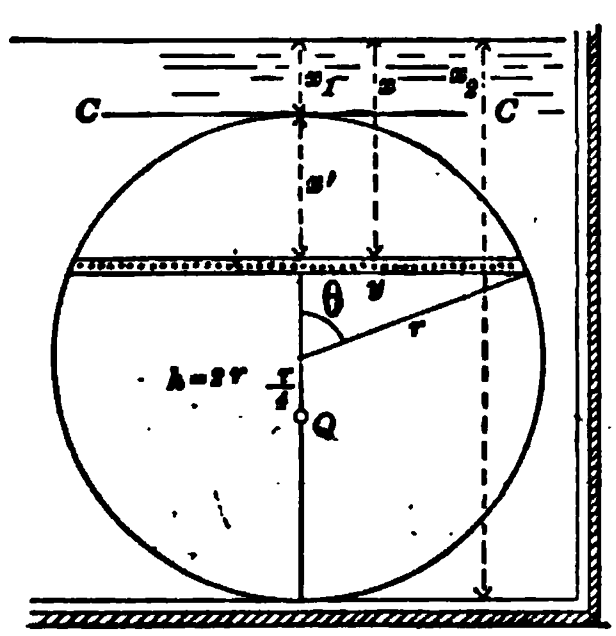


Fig. 115

The center of pressure is shown in Fig. 114, *provided the vertex D is at the surface of the liquid.*

118. Let the surface in the vertical wall of a tank be a circle. Fig. 115. Let x, x_1, x_2, and x' be defined as in the last section. In this case $y = 2\sqrt{x'(2r - x')}$. As intimated in **116**, the total pressure is equal to the area, multiplied by the mean depth.

$$P = (\pi r^2)\cdot\frac{x_2 + x_1}{2}\cdot w, \text{ hence}$$

$$P = \pi wr^2(x_1 + r)$$

and $M_c = \int (ydx')(wx)x'$

Changing the variable to θ we have

$$y = 2r\sin\theta$$
$$x' = r\,(1-\cos\theta)$$
$$dx' = r\sin\theta\,d\theta$$
$$x = x_1 + r\,(1-\cos\theta)$$

and hence

$$M_c = 2wr^3\left[x_1\int_0^\pi \sin^2\theta(1-\cos\theta)\,d\theta + r\int_0^\pi \sin^2\theta\,(1-\cos\theta)^2\,d\theta\right]$$

$$M_o = 2wr^3\left[x_1\left(\frac{\theta-\sin\theta\cos\theta}{2}-\frac{\sin^3\theta}{3}\right)+r\left(\frac{\theta-\sin\theta\cos\theta}{2}-\frac{2\sin^3\theta}{3}+\frac{2\theta-\sin2\theta\cos2\theta}{16}\right)\right]_0^\pi$$

$$M_c = 2wr^3\left(\frac{x_1\pi}{2}+\frac{r\pi}{2}+\frac{r\pi}{8}\right) = \pi wr^3\left(x_1+\frac{5}{4}r\right)$$

and $$x_o = \frac{M_o}{P} = \frac{r}{4}\cdot\frac{4x_1+5r}{x_1+r}$$

Corollary. If the circle touches the surface of the liquid, $x_1 = 0$,

$$x_o = \frac{5}{4}r$$

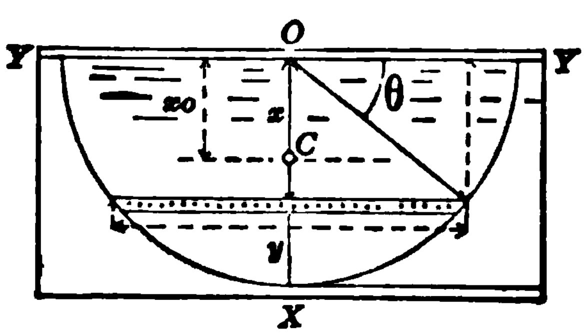


Fig. 116

119. The pressure on a semi-circle. If the circle is half above the surface of the liquid, and only a semi-circle is under pressure, the problem is wholly new.

With the notation shown in Fig. 116

$$x = r\sin\theta$$
$$dx = r\cos\theta\,d\theta$$

and $$y = 2r\cos\theta$$

so that $$P = 2wr^3\int_0^{\frac{\pi}{2}}\cos^2\theta\sin\theta\,d\theta = -\left[\frac{2wr^3\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}} = \frac{2}{3}wr^3$$

and $$M_y = 2wr^4\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\,d\theta = 2wr^4\left[\left(\frac{2\theta-\sin2\theta\cos2\theta}{16}\right)\right]_0^{\frac{\pi}{2}}$$

$$M_y = \frac{\pi wr^4}{8} = \frac{\pi r^2}{2}\cdot w\cdot\frac{r^2}{4} = w\frac{\pi r^2}{2}\cdot\frac{r^2}{4}$$

Hence $$x_o = \frac{M}{P} = \frac{3}{16}\pi r$$

120. The student must not fail to see that the *Center of a varying Pressure* is a different point from the *centroid* of the surface. The "Center of Hydrostatic Pressure" is always *below the centroid* of the surface of action, but the distance below depends upon the depth of the liquid. The deeper the liquid, the less the distance between the two points.

The results in all the above cases apply to surfaces in inclined planes which are immersed—if the axis of X is a line of greatest declivity in such a plane. Just as the value of w (the specific weight of the liquid) disappeared from the final formulas for vertical planes, so $w \sin \phi$ will disappear for the inclined plane. Fig. 117.

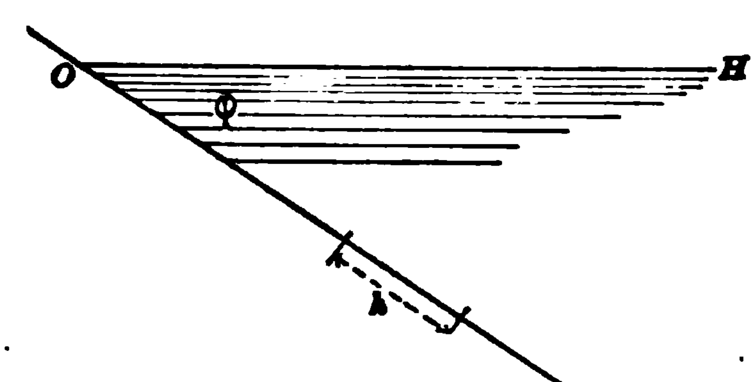


Fig. 117

The practical application of these formulas for hydrostatics is readily seen. If a gate or a valve in the vertical wall of a dam, flume or tank, is to be supported by a single prop, or a pair of shaft bearings, the prop or shaft must be so adjusted as to act normally at the *center of pressure* and *not* at the centroid of the surface. If the gate or valve is to revolve freely on a horizontal axis, *i. e.*, so that the moment pressure will balance on the axis, that axis must pass thru the *center of pressure.*

The problems of this chapter are by no means purely abstract, given simply as exercises. Some of the formulas giving the value of x_0 will appear several times in later chapters.

I.

121. The theorems of Pappus. The general expression for the volume generated by the revolution of a plane area about an axis in the plane, not intersecting the area revolved, is (Fig. 118,)

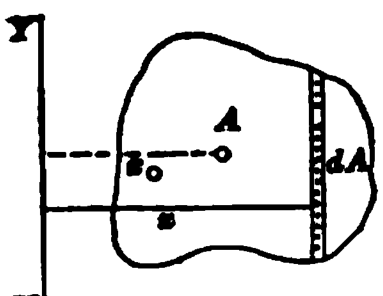


Fig. 118

$$V = 2\pi \int x dA$$

integrated between proper limits; dA is a differential of the given area taken parallel to the axis of revolution. The integral

$$\int x dA = x_0 A$$

by the *axioms of moments.*

Hence $$V = 2\pi x_0 A$$

In other words the volume of a solid of revolution is equal to the area of the generating surface, multiplied by the path described by the centroid of that surface. This is known as the *Theorem of*

Pappus. It may be used to find a volume when x_0 is known, or, conversely, to find x_0 when V is known. For example, if we know that the volume of a sphere is $\frac{4}{3}\pi r^3$, we can find the centroid of a semicircle by the equation

$$\frac{4}{3}\pi r^3 = 2\pi x_0 \cdot \frac{\pi r^2}{2}$$

$$x_0 = \frac{4r}{3\pi}$$

II.

The surface of a solid of revolution is found by multiplying the length of the generating curve by the length of the path of its centroid during the revolution. Fig. 119.

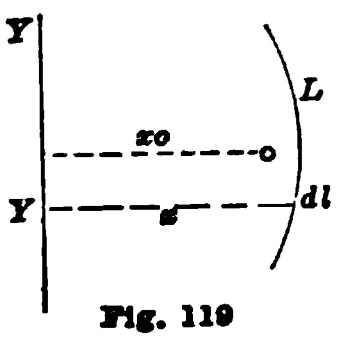


Fig. 119

Since $S = 2\pi\int x dl$, in which S is the surface of a solid of revolution, and l is the length of the generating line, co-planar with the axis, and not intersecting it; and since $x_0 l = \int x dl$ we have

$$S = 2\pi x_0 l$$

which is the above theorem.

Problems.

1. Find the surface generated by revolving the cycloid about its base.

2. Find the surface and volume of a "torus" generated by revolving a circle about an exterior line in its plane.

3. Find the volume of the solid generated by revolving the spandril of a parabolic half-segment about the axis of the parabola. Fig. 120.

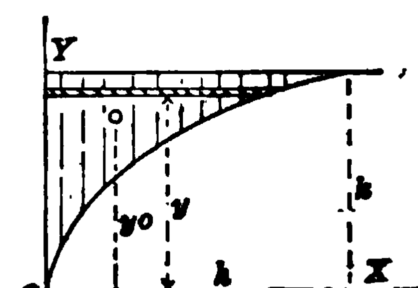


Fig. 120

4. Find the volume of an oblate spheroid whose polar diameter is $2b$, and whose major diameter is $2a$.

CHAPTER VIII.

Centers of Gravity.

122. In Chapter III we studied the relations of parallel forces, both co-planar and in space, without considering the source or nature of the forces. In this chapter we shall deal with parallel forces which are distributed through volumes, such as the earth's attraction

upon all bodies at or near its surface. In Applied Mechanics, we have no occasion to discuss the problems of physical Astronomy. In everything relating to structures and mechanism, the lines of action of the earth's attraction are so nearly parallel that they are assumed to be parallel, and in all cases the arrows representing gravitational forces point vertically downward.

The magnitude of the earth's attraction upon a body depends upon the mass of the body. Since such attraction is known as the *weight** of the body, the weight is proportional to the mass. In a body of uniform density, the masses, and therefore the weights, of equal volumes are equal.

Definition of center of gravity. The earth's attraction upon all of the minutest parts of a body constitutes an infinite number of constituent parallel forces, whose resultant is the weight of the entire body.

It will soon be shown that there is within the enveloping surface of every body a certain point thru which the resultant always passes, no matter how much that body be turned over and about in space. That point is called the Center of Gravity, or the Center of Mass of the body. † This chapter will be devoted to finding the C. G. of typical solids, which closely approximate the forms of real or imaginary bodies.

123. The center of gravity found by experiment. If a body can be suspended by a cord or wire so that the axis of the wire coincides with the resultant line of action of the weight which it balances, we have one line which must contain the C. G. of the body. If, then, the body can be suspended differently, giving a second line of action, the C. G. must be the point where the two lines intersect. This method is frequently used for finding the C. G. of thin plates of irregular shape; for example, a piece of zinc or drawing paper representing the cross-section of a steel rail. In such a case the C. G. of the thin solid is anent the "centroid" of the surface.

124. General formulas. When the bounding surfaces of a solid are given in mathematical language, the co-ordinates of the C. G. are determined as follows:

1. Let the position of the body be given with reference to a set

* Under the subject of Deviating Forces in Chap. XVI, it is shown that a component of the earth's attraction is employed in compelling the mass to revolve about the earth's polar axis, and that accordingly the observed *weight* of a body does not measure the entire attraction. But the *weight* is still proportional to the mass.

† In the case of such bodies as are considered in this book, the center of gravity coincides with the center of mass.

of three rectangular axes. Fig. 121. An elementary portion of the body is represented by the rectangular solid

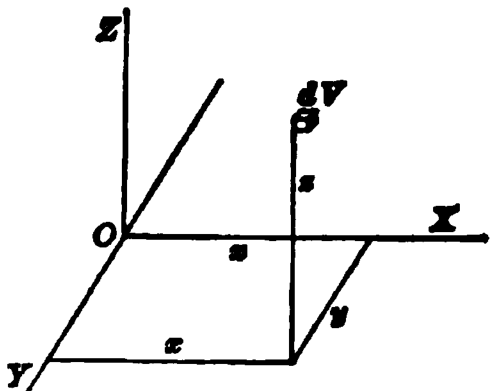


Fig. 121

$$dV = dxdydz.$$

The weight of a unit volume of a homogeneous material is known as its "specific weight." If the specific weight of the material is w, the weight of dV is

$$dW = wdxdydz = wdV$$

If the plane XY is horizontal, the moment of the weight of the element with respect to the axis, OX, is

$$dM_1 = wdV.y = wydxdydz.$$

and

$$M_1 = \int\int\int wydxdydz$$

The successive integrations are to be taken between the limiting surfaces of the solid. If w is constant, it comes before the integral signs. The entire weight of the body is

$$W = w\int\int\int dV = w\int\int\int dxdydz.$$

If the co-ordinates of the C. G. are x_o, y_o z_o, we have by the *axiom of moments*

$$y_o = \frac{M_1}{W}$$

which becomes, if w is constant,

$$y_o = \frac{\int ydV}{V} = \frac{\int\int\int ydxdydz}{\int\int\int dxdydz.}$$

2. Similarly, finding moments about OY we get

$$x_o = \frac{M_2}{W},$$

and

$$x_o = \frac{\int xdV}{V} = \frac{\int\int\int xdxdydz}{V}$$

3. If now the body and the co-ordinate axes be so turned that the plane of XZ is horizontal, and the moments about OX are again considered, we get

$$z_o = \frac{\int zdV}{V} = \frac{\int\int\int zdxdydz}{V}.$$

and the C. G. is found.

If w varies, it does not disappear from the formulas. It is generally assumed to be constant unless the law of its variation is definitely given.

125. Center of gravity by inspection. The center of gravity of a homogeneous body is evidently at its geometrical center, if it has one; like the center of a sphere; the center of a parallelopiped; and the center of a cylinder of revolution, thin, thick or solid, with parallel bases. It is further evident that if a cylinder or prism have parallel bases whose centroids are known, the center of gravity of the solid is at the central point of the right line connecting the centroids of the bases. Moreover, when a body of uniform density has an axis of symmetry, we know that the center of gravity must lie on that axis, and the problem of finding it is rendered simple.

When a body having no geometrical center can be separated into parts each of which has a geometrical center, the center of gravity may be found by the method of moments and "centers of action," as already given in Chapter III. If the number of parts is finite, and the weight and the center of gravity of each part is known, the center of gravity of the whole can readily be found by repeated application of the "axiom of moments."

126. The C. G. of a group. To find the center of gravity of a group of bodies whose weights, and the co-ordinates of whose centers of gravity are respectively known.

N. B. In choosing co-ordinate planes, it is generally best to so take them that one, and if possible two, of the (C. G.)'s lie in the co-ordinate planes.

1. Let a typical body, whose weight is W_1, have a center of gravity at x_1, y_1, z_1. Now assuming that the axis of X is horizontal, and letting x_o be the X-co-ordinate of the center of gravity of the group, we have by the axiom

$$x_o\Sigma W = \Sigma(Wx)$$

Hence
$$x_o = \frac{\Sigma(Wx)}{\Sigma W} = \frac{W_1x_1 + W_2x_2 + W_3x_3 + \text{etc.}}{W_1 + W_2 + W_3 + \text{etc.}}$$

Similarly
$$y_o = \frac{W_1y_1 + W_2y_2 + W_3y_3 + \text{etc.}}{W_1 + W_2 + W_3 + \text{etc.}}$$

$$z_o = \frac{W_1z_1 + W_2z_2 + W_3z_3 + \text{etc.}}{W_1 + W_2 + W_3 + \text{etc.}}$$

The point z_o, y_o, z_o, is, of course, the point thru which the resultant of the forces supporting the entire system against the earth's attraction must act.

2. When integration is necessary, let dV be a differential element of the first order if possible.

127. To find the center of gravity of a solid cone of revolution.

1. Take the axis of the cone as the axis of X with the vertex as the origin. Fig. 122. Take as an element a *lamina** lying between two consecutive planes perpendicular to the axis.

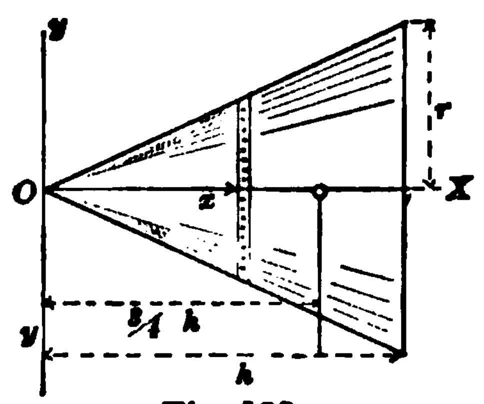


Fig. 122

The center of gravity of the element is in the axis at a distance x from O.

Its volume is $dV = \pi y^2 dx$;

Its weight is $dW = w\pi y^2 dx$;

Its moment about YOY is $dM_y = w\pi y^2 x dx$.

Since $y = \dfrac{rx}{h}$

$$W = \frac{\pi w r^2}{h^2}\int_0^h x^2\,dx = \frac{\pi w r^2 h}{3}$$

and
$$M_y = \frac{w\pi r^2}{h^2}\int_0^h x^3 dx = \frac{\pi w r^2 h^2}{4}$$

Hence
$$x_o = \frac{M_y}{W} = \frac{3}{4}h.$$

2. The center of gravity of a frustum of a cone of revolution is readily found, by using the limits h_1 and h_2, to be

$$x_o = \frac{3}{4}\cdot\frac{h_2^4 - h_1^4}{h_2^3 - h_1^3}$$

128. An oblique cone. Suppose we have an oblique *cone* or *pyramid*, with a *base of any figure* whose centroid is known, and we wish to find its center of gravity. Let the point C, Fig. 123, be the centroid of the base, and let $VC = x_1$. The line VC will contain the centers of gravity of all elements made by consecutive planes parallel to the base; hence the center of gravity required lies on that line. Let VX and VY be horizontal axes. Let EF be an element parallel and similar to the base, so that we have:

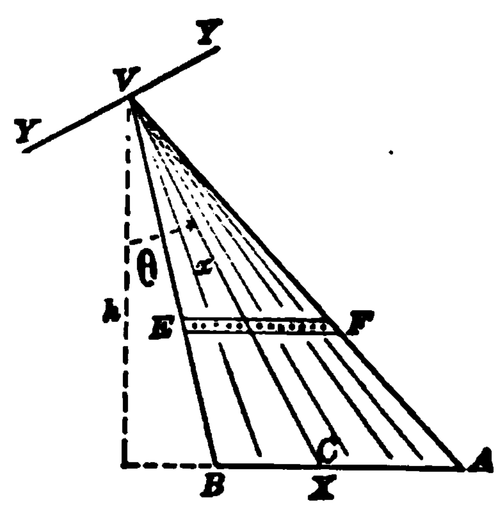


Fig. 123

$$dV = (\text{Area } EF)\ (dx \cos\theta)$$

But
$$(\text{Area } EF) = (\text{Area } AB)\ \frac{x^2}{x_1^2}, \text{ by geometry.}$$

* An infinitely thin plate.

Hence $$W = w\,\frac{\text{Area } AB}{x_1^2}\cdot\cos\theta\int_o^{x_1} x^2dx = \frac{h}{3}(\text{Area } AB)w$$

and $$M_y = \frac{hx_1}{4}(\text{Area } AB)w$$

so that $$x_o = \frac{3}{4}\cdot x_1$$

129. Find the center of gravity of a spherical segment. The center of gravity must lie on the axis of X if axes are assumed as shown. Fig. 124. The element is a thin circular plate. As before, we have the general formulas; the segment is $ABDH$.

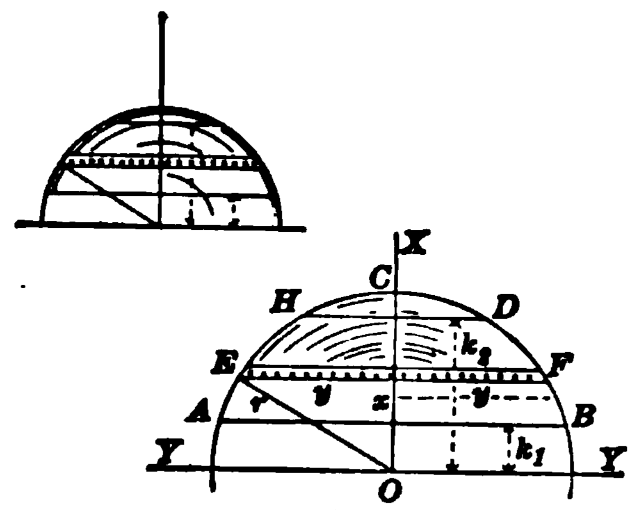


Fig. 124

$$dV = \pi y^2dx$$

$$dW = \pi wy^2dx$$

$$dM_y = \pi wy^2xdx$$

$$W = \pi w\int_{k_1}^{k_2}(r^2 - x^2)\,dx = \frac{\pi w}{3}\cdot\left[3r^2(k_2 - k_1) - (k_2^3 - k_1^3)\right]$$

$$M_y = \pi w\int_{k_1}^{k_2}(r^2 - x^2)\,xdx = \frac{\pi w}{4}\left[(r^2 - k_1^2)^2 - (r^2 - k_2^2)^2\right]$$

$$x_o = \frac{3}{4}\cdot\frac{(r^2 - k_1^2)^2 - (r^2 - k_2^2)^2}{3r^2(k_2 - k_1) - (k_2^3 - k_1^3)}$$

If the segment have but one base, and $k_2 = r$,

$$x_o = \frac{3}{4}\,\frac{(r^2 - k_1^2)^2}{2r^3 - 3r^2k_1 + k_1^3}$$

If the segment be a hemisphere, $k_1 = o$

$$x_o = \frac{3}{8}r\cdot$$

It is now obvious that the position of the Center of gravity is in no way dependent upon the value of w if it is constant. It will therefore for the present be considered as unity.

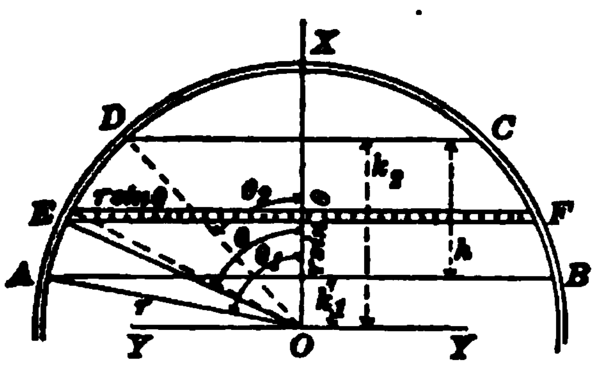


Fig. 125

130. *Problem.* **To find the center of gravity of a segment of a thin spherical shell.** Fig. 125. Let the element be a ring generated by the revolution about OX of the

cross-section $rd\theta dr$, or $trd\theta$. Its center of gravity is on OX at a distance $r\cos\theta$, from YY, and its radius is $r\sin\theta$. Hence

$$dV=(trd\theta)\ (2\pi r\sin\theta)$$

$$W=2\pi r^2t\int_{\theta_2}^{\theta_1}\sin\theta\,d\theta=2\pi r^2t\ (\cos\theta_2-\cos\theta_1)$$
$$=(2\pi r)\ (t)\ (k_2-k_1)=2\pi rht,$$

where $k_2-k_1=h$;
which agrees with a proposition of elementary geometry, which says, that "the area of a segment of spherical surface is equal to the circumference of a great circle $2\pi r$, multiplied by the height of the segment h."

$$M_y=2\pi r^3t\int_{\theta_2}^{\theta_1}\sin\theta\cos\theta\,d\theta=\pi r^3t\,(\sin^2\theta_1-\sin^2\theta_1)$$
$$=\pi r^3t\,(\cos^2\theta_2-\cos^2\theta_1)$$

hence $$x_o=\frac{r}{2}\,(\cos\theta_1+\cos\theta_2)=\frac{k_2+k}{2}\ ;$$

that is, the center of gravity of a spherical zone is always in the plane midway between the bounding planes.

Corollary. The center of gravity of a thin hemispherical shell is at $x_o=r/2$, the middle of the central radius.

131. A segment of a thick spherical shell may be made by two parallel planes which cut both spheres; or by two cones which are coaxial and have their vertices at the center of the sphere.

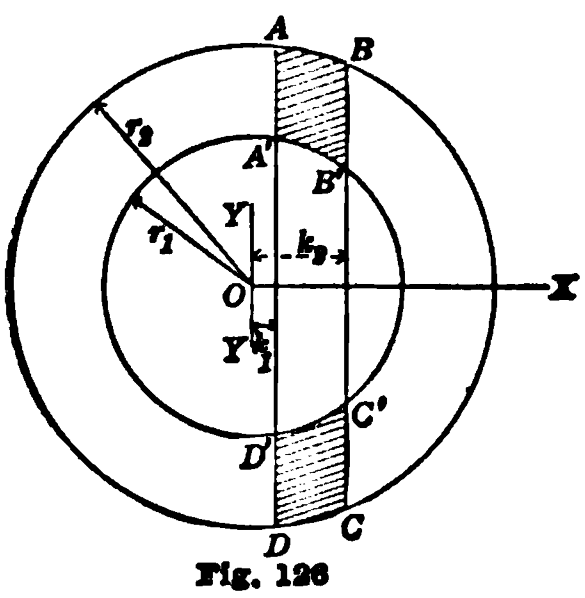


Fig. 126

The first case is seen in Fig. 126. To find the center of gravity of the segment suggested, find W_2 and M_2 for the segment of the larger sphere, as given in **129** above; then W_1 and M_1 for the segment of the smaller spheres. The value of x_o is found by algebra.

$$x_o=\frac{M_2-M_1}{W_2-W_1}$$

which involves nothing new.

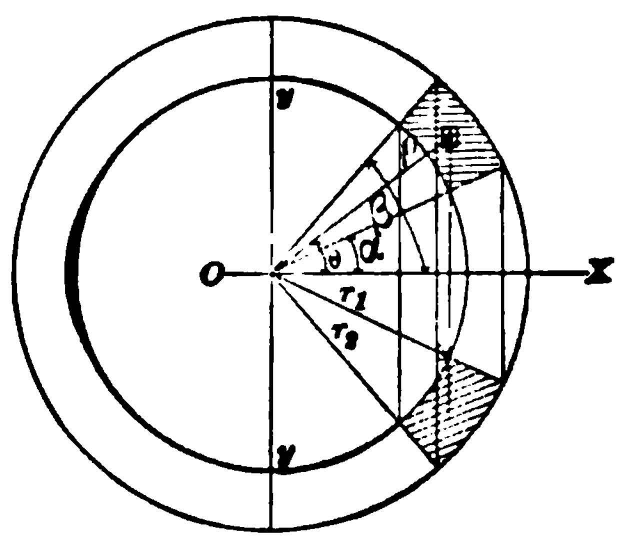


Fig. 127

132. In the second case, Fig. 127, when the segment is cut from the thick shell by two cones, we must integrate an element which has two differential factors. The shaded areas represent the section of the

segment made by an axial plane. The element is a ring whose cross-section is $(\rho d\theta)$ $(d\rho)$, and whose circumference is $2\pi\rho\sin\theta$. Hence we have, assuming that $w=1$.

$$dV = 2\pi\rho^2 d\rho \sin\theta\, d\theta$$

$$W = 2\pi\int_{r_1}^{r_2}\int_{\alpha}^{\beta}\rho^2 \sin\theta\, d\rho d\theta$$

The order of integration is immaterial since the limits of one variable are independent of the other variable.

$$W = 2\pi\left(\frac{r_2^3 - r_1^3}{3}\right)(\cos\alpha - \cos\beta)$$

Also
$$M_y = 2\pi\int_{r_1}^{r_2}\int_{\alpha}^{\beta}\rho^3 \sin\theta\cos\theta\, d\rho d\theta$$

$$= 2\pi\left(\frac{r_2^4 - r_1^4}{4}\right)\left(\frac{\cos^2\alpha - \cos^2\beta}{2}\right)$$

Hence
$$x_o = \frac{3}{4}\cdot\frac{r_2^4 - r_1^4}{r_2^3 - r_1^3}\cdot\left(\frac{\cos\alpha + \cos\beta}{2}\right)$$

Corollary I.

If $\alpha = 0$ and $r_1 = 0$, we have the center of gravity of a *spherical cone.*

$$x_o = \frac{3}{8}r\,(1 + \cos\beta)$$

Corollary II.

If now $\beta = \frac{\pi}{2}$, the cone becomes a *hemisphere* and $r = \frac{3}{8}r$ as before.

128. *Problem.* To find the center of gravity of the solid cut from the sphere O by the cone V-ABC. Fig. 128. This problem could easily be solved by taking it in two parts: the cone VAB, and the spherical segment ABC (since both volumes and their centers of gravity have been found), and then proceeding by the algebraic method. It may, however, be more simple to proceed directly by integration.

Fig. 128

For an element, pass two co-axial and co-vertical cones whose half-vertical angles are θ and $\theta + d\theta$. Pass also two consecutive axial planes which make with each other an angle $d\phi$. The solid enclosed will be an oblique pyramid whose base is a rectangle $(r\sin 2\theta d\phi)$ $(rd2\theta)$, and whose altitude is $2r\cos^2\theta = h$.

Hence $$dW = \frac{1}{3}\cdot 4r^3\cos^2\theta\sin 2\theta\, d\theta\, d\phi$$

$$W = \frac{8}{3}r^3\int_0^a\int_0^{2\pi}\cos^3\theta\sin\theta\, d\theta d\phi$$

The limits for ϕ are 0 and 2π; for θ they are 0 and a.

$$W = \frac{16\pi r^3}{3}\int_0^a\cos^3\theta\sin\theta\, d\theta$$

$$W = \frac{4\pi r^3}{3}\,(1-\cos^4 a) \tag{19}$$

The center of gravity of the element is $\frac{3}{4}$ of the slant height from V, or $\frac{3}{4}\cdot 2r\cos^2\theta$ from YY. Hence

$$dM_y = \left(\frac{8}{3}r^3\cos^3\theta\sin\theta\, d\theta\, d\phi\right)\frac{3}{2}\,r\cos^2\theta$$

and $$M_y = 8\pi r^4\int_0^a\cos^5\theta\sin\theta\, d\theta$$

$$= \frac{4}{3}\,\pi r^4\,(1-\cos^6 a)$$

Hence $$x_o = \frac{M_y}{W} = \frac{1-\cos^6 a}{1-\cos^4 a}\cdot r$$

Corollary.

If $a = \frac{\pi}{2}$, the solid becomes the sphere, and as $\cos\frac{\pi}{2} = 0$ we have $W = \frac{4}{3}\pi r^3$, and $x_o = r$ which serves as a check on the formula.

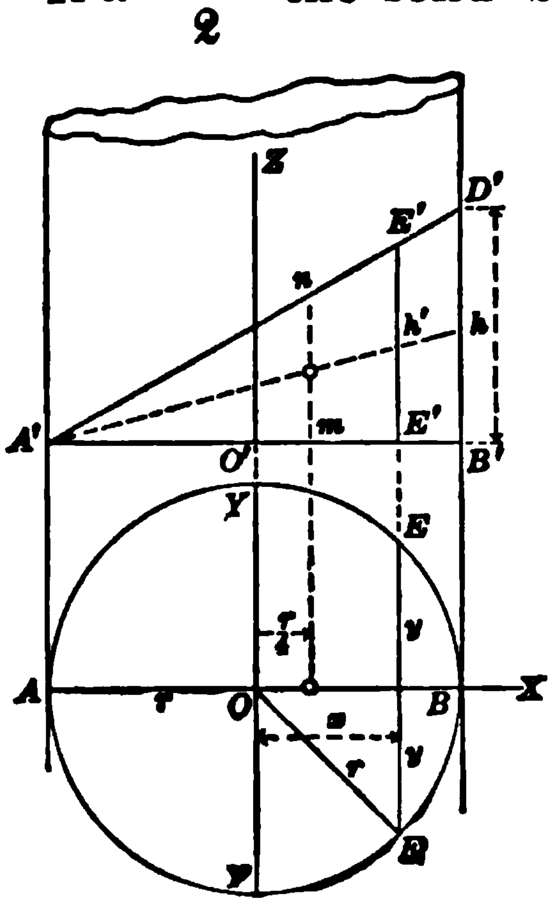


Fig. 129

136. *Problem.* **To find the center of gravity of a wedge cut from a circular cylinder.** The wedge is shown in plan and elevation in Fig. 129. Let the plane of the base be the XY-plane, with the center as the origin, and the plane XZ a plane of symmetry; hence $y_o = 0$. The weight of the wedge is known by inspection to be $W = \frac{1}{2}\,\pi r^2 h$, if $w = 1$. Taking an element $(2yh'dx)$ its moment about YY is

$$dM_y = 2yxh'dx;$$

but $$h' = \frac{h}{2r}(r+x); \text{ and } y = (r^2 - x^2)^{\frac{1}{2}},$$

hence $$M_y = \frac{h}{r}\int_{-r}^{+r}(r^2-x^2)^{\frac{1}{2}}(r+x)x\,dx = \frac{\pi h r^3}{8}$$

Hence $$x_o = \frac{M_y}{W} = \frac{r}{4}.$$

as was found in Chapter VII for the center of hydrostatic pressure.

1. When we consider the center of a uniformly varying pressure, which is geometrically represented by the wedge-shaped solid ADB (Fig. 129); and the center of gravity of the geometric solid, we find that the center of gravity of the solid and the center of pressure are in the same vertical line.

2. To find Z_o, note the fact that a plane passing thru a tangent to the base at A, and bisecting the altitude h, contains the centers of gravity of all vertical elements, and hence would contain the center of gravity of the entire solid. Hence we have, when $x = \frac{r}{4} = x_o$

$$z_o = \frac{\overline{mn}}{2} = \frac{h}{4r}\left(r + \frac{r}{4}\right) = \frac{5}{16}h$$

This problem is closely analagous to that of finding the center of action of a uniformly varying force. In fact, some writers picture a distributed force whose elements are parallel, as a solid, and find the center of action by finding the center of gravity of the ideal solid.

3. The center of gravity of a truncated cylinder may now be found by the algebraic method.

Problems.

Ex. 1. Find the center of gravity of a semi-circular cylindrical wedge, as shown in Fig. 130. Let $y = r\cos\theta$, $x = r\sin\theta$, $dx = r\cos\theta\, d\theta$.

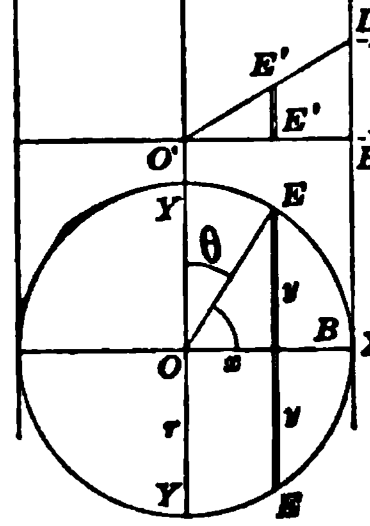


Fig. 130

Ans.
$$\left.\begin{aligned} x_o &= \frac{3}{16}\pi r \\ z_o &= \frac{3\pi}{32}\cdot h. \end{aligned}\right\}$$
$$y_o = 0$$

Ex. 2. Find the center of gravity of an annular wedge which has for a base one half of a ring cut from a thick cylinder, and whose upper surface is

an inclined plane, which passes thru the diameter of the ring as shown in Fig. 131.

Ans. $$x_o = \frac{3\pi}{16} \cdot \frac{r_2^4 - r_1^4}{r_2^3 - r_1^3},$$

$$z_o = \frac{3\pi}{32} \cdot \frac{r_2^4 - r_1^4}{r_2^3 - r_1^3} \cdot \frac{h}{r}$$

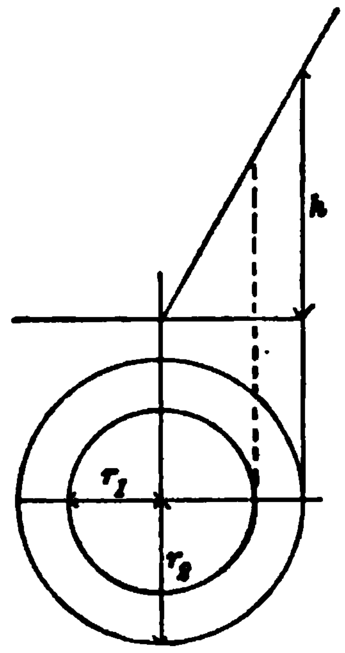


Fig. 131

137. Composite figures. In the case of a composite figure whose components have geometrical forms, the center of gravity may be found by the algebraic process of addition. It often happens that certain of the components are negative, and that their weight and moment must both be treated as negative quantities. A single example will now be given to illustrate the method when some of the components are negative.

Figure 132 shows a cube whose edges are 3 feet long. All four upper corners are beveled, the bevel being cut away one foot on each of the three edges. A central vertical cylinder is cut out from top to bottom, the diameter being 1.5 feet. The student is to find the volume and center of gravity of the solid.

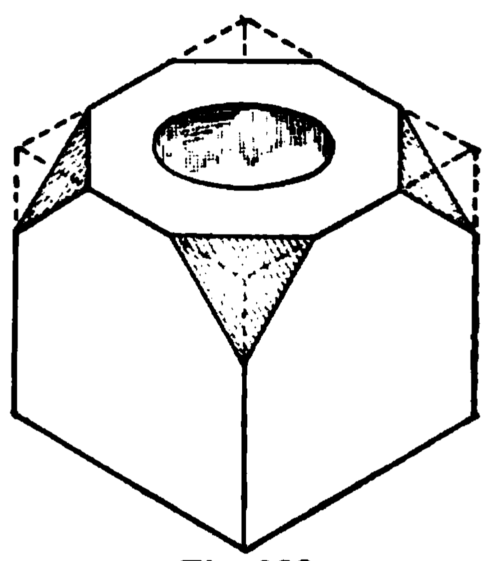


Fig. 132

Suggestions: In finding the volume of one of the pyramids cut off, the student should know that the perpendicular dropped from the vertex of the pyramid upon the equilateral triangular base makes equal angles with the edges, and if θ is the angle which the perpendicular makes with one of the edges, $\cos\theta = \sqrt{\frac{1}{3}}$. See **96.** The student of solid analytic geometry may very readily find the length of a perpendicular upon a plane which has equal intercepts on three rectangular axes.

Problems.

1. Prove that C. G. of a triangular pyramid is on the right-line from the vertex to the centroid of the base, and one-fourth of the length of that line from the base.

2. Find the C. G. of a fine wire having the form of an arch of the cycloid generated by a rolling circle whose radius is r.

3. Find the center of gravity of a solid generated by revolving a circle about one of its tangents, thru an angle of 180°. Make use of the solution for finding the centroid of a circular arc.

4. Suppose a hemispherical bowl, radius r, of infinitesimal thinness has a plane circular cover, radius r, equally thin, but of twice the density (*i. e.*, its w is twice as great). Find the C. G. of the combination.

CHAPTER IX.

VARYING STRESS IN BEAMS; MOMENTS OF INERTIA.

138. Stress and strain. Modulus of elasticity. 1. All materials used in construction are elastic to a greater or less degree. A metallic bar is stretched by tension, shortened by compression. The same is true of wood, stone and cement.

Think of a steel rod of uniform section, A square inches, and length l, subjected to a tension P. Fig. 133. When the bar or rod is well made it is *assumed* that the tension at the cross-section A is uniformly distributed, so that each *square inch* sustains a "*stress*" of $\frac{P}{A}$ lbs. If we write $\frac{P}{A}=p$, we read p as the *unit stress.*

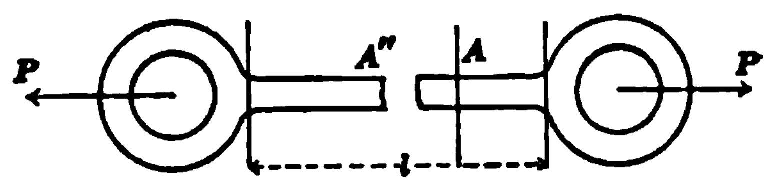


Fig. 133

2. Under this stress the length l becomes $l+\lambda$, if λ is the elongation of the rod, and $\frac{\lambda}{l}$ is the elongation of a unit of length. The elongation λ is called the *strain*, and $\frac{\lambda}{l}$ is called the *unit strain.*

3. Now, so long as p does not exceed a certain limit, called the *elastic limit*, the numerical ratio between p and $\frac{\lambda}{l}$ is constant. This ratio is called "The Modulus of Elasticity" of the steel, and is represented by the letter E, so that

$$E=\frac{\textit{unit stress}}{\textit{unit strain}}=p\div\frac{\lambda}{l}=\frac{pl}{\lambda};$$

hence $$\frac{p}{E}=\frac{\lambda}{l}=\epsilon$$

The last form of the definition shows that if there were no limit to the elastic law, a unit stress of E would double the length of the rod.

139. How E is found. The value of E is found by actual tests (see **49**), in which p has such values as 6,000 lbs., 10,000 lbs. or 30,000 lbs. A sound bar will sustain such values of p, and when relieved, will recover its exact original length. If, however, a stress

of 80,000 lbs., or 100,000 lbs. were applied, the bar, when relieved, would not fully recover; it would be permanently lengthened, or it would have taken a "permanent set." The greatest value of p without "set" is called the "Elastic limit."* The elastic limit is therefore defined as the stress at which the ratio p/ε ceases to be constant. With greater values of p, the ratio p/ε generally becomes less than E.

The Elastic Limit, or limiting value of p for sound steel varies from 30,000 lbs. to 40,000 lbs. The value of E varies from 28,000,000 lbs. to 30,000,000 lbs. See Appendix.

140. Compressive stress. When a bar or block is subjected to compressive stress, it is shortened, and if the forces are not excessive, the *deformation* λ, will be proportional to p, and we have again a value

$$\frac{\lambda}{l} = \frac{p}{E_c}, \text{ therefore } E_c = \frac{pl}{\lambda}$$

which is the **Modulus of Elasticity for Compression.**

Thus there are two Moduli, E_t and E_c. Under moderate values of p, the two E's for steel are assumed equal, tho their ***elastic limits*** may vary.

141. Normal stresses. When a cantilever beam is broken by an excessive load, its torn fibers in the top of the beam show that they were pulled in two by a destructive tension; also the crushed and bent fibers in the bottom of the beam show that they suffered from a destructive compression. These features are easily seen if the beam be of wood. A steel beam may not break, but may bend out of shape; still it would show *stretching* on one side and *upsetting* on the other.

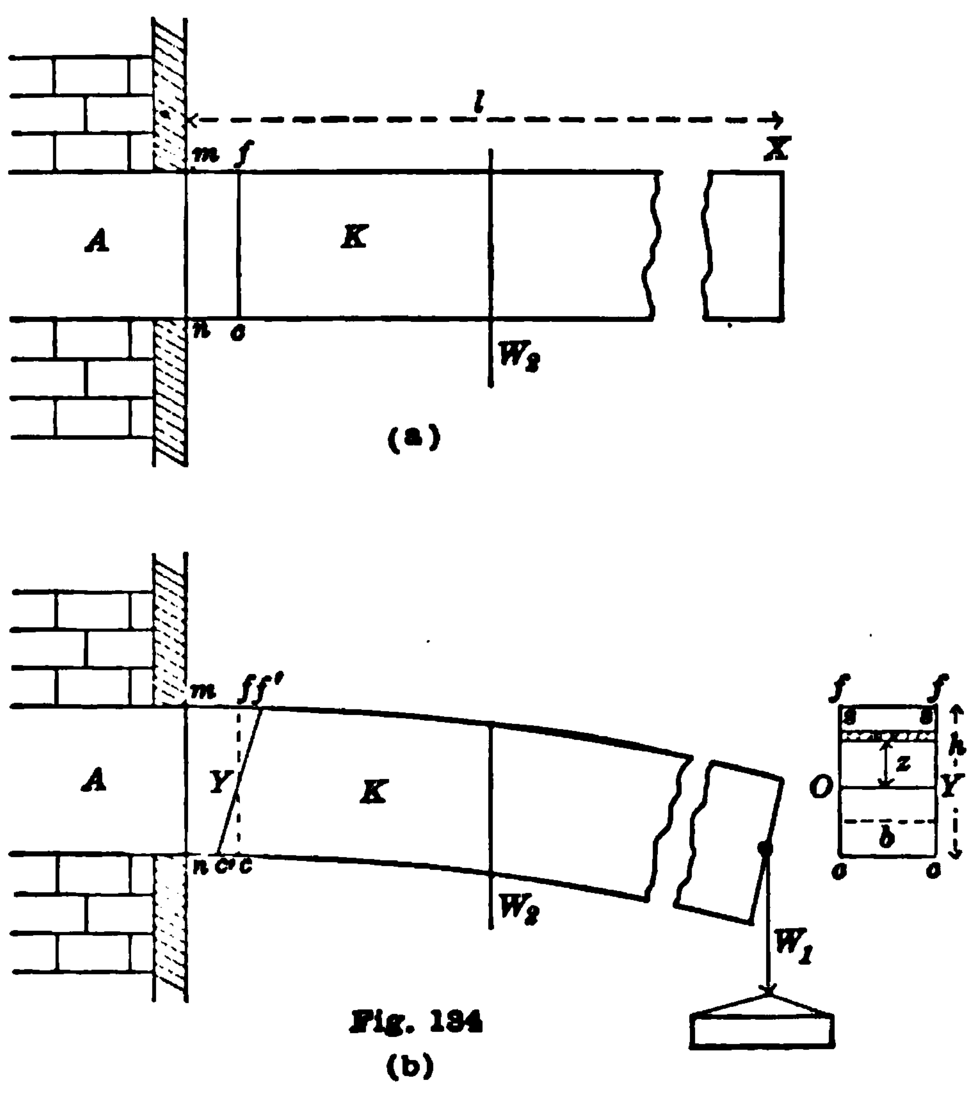


Fig. 134
(b)

Let us suppose now that such a beam, Fig. 134 *a*, is not broken, but is slightly bent, and that a thin layer, or block, *mncf*, is distorted

* A *very slight* "set" may appear upon the first application and removal of a stress below the elastic limit, which set will not be increased by subsequent applications tho repeated many times with even greater stress.

or deformed into *mnc'f'*, Fig. *b*, by the internal stresses. The fibers in its top are lengthened, and those at the bottom are compressed, and the "*strain*" varies on each side of the center (where it must be zero) in such a way that what was a vertical plane face, *fc*, becomes an inclined plane face *f'c'*, and the beam actually bends. This is what happens under a load, and as the *stress* is proportional to the *strain*, it is seen that the *stress varies uniformly* both ways from OY, being tension on the upper half, and compression on the lower, while the horizontal diameter, across the beam, is a "*neutral axis.*" OY is the neutral axis. See cross-section enlarged.

142. The moment of resisting stress. In the cross-section, the rectangle *ffcc* represents the outer face of the thin layer. All the fibers in the area $OffY$, have been subjected to tensile stress; all in the area $OYcc$ have been subjected to compressive stress. This distributed action by the portion of the beam, K, has had a tendency to turn the thin block round *right-handed.* In fact, it *has* turned it a little. It is plain that the portion A must *resist* and *balance* that tendency of K to turn the thin block.

Let us suppose that the intensity of the stress at a unit's distance from the neutral axis OY is a; then the intensity at a point z distant must be az; and the *amount* of *stress* on an element of the surface,

$$dA = bdz, \text{ is } dF = abzdz.$$

And the moment of that stress about the axis OY is

$$dM = abz^2dz$$

so that

$$M = ab\int z^2dz = a\int z^2dA.$$

In the integration the limits of z are $-\frac{h}{2}$, and $+\frac{h}{2}$; so that the integral becomes $\frac{bh^3}{12}$ and

$$M = a\cdot\frac{bh^3}{12}$$

which is known as the **moment of resistance.**

When OY is an axis of symmetry the integration could have been

$$2\int_0^{\frac{h}{2}}.$$

143. The neutral axis passes thru the centroid of the section. It is readily seen that for every element of the cross-section, dM is

positive, since the only factor in it that can be negative (z), is *squared.* It is otherwise with

$$dF = abz(dz)$$

for z is negative for all points below OY, and it is seen without other proof that, if OY is an axis of symmetry of the surface (the cross-section of the beam), the algebraic sum of the normal stress, tensile and compressive, is zero. In point of fact, F is always zero if the beam be horizontal, and is acted upon by vertical forces alone, no matter what the shape of the cross-section may be, since the pull and the push upon the thin block in the direction of OX must have equal magnitudes. In that case, since

$$F = ab\int zdz = 0$$

we must have

$$\int zabdz = 0$$

which means that the moment of a *uniform* stress on the surface about the axis OY is zero. Hence, the *neutral axis must pass thru the centroid of the surface of action.* This is a very important fact.

144. The bending moment. It should be evident from the above that the tendency of the *external* forces, acting on the beam, to turn it to the right, is, after some bending, fully resisted by the stresses of the fibers *within* the beam. The load tends to turn the thin block about OY one way, while the distributed stress on the rear face, mn, tends to turn it the opposite way, and finally to actually *hold it.* Hence the action of the portion A we have called the holding or "Resisting Moment."

Having found an expression for the resisting moment of the beam in terms of the *internal* stresses of the fibers, let us find the resultant moment of the *external* forces which act upon the beam and so call into action the balancing moment already found. The external forces are W_1, the pull of the chain at the end, and W_2, the weight of the beam whose line of action is vertical thru its center. Their moment about the *same axis*, OY, is

$$M = W_1 l + \frac{l}{2} W_2 \qquad (1)$$

This is called the *Bending Moment.*

As the beam, when loaded and slightly bent, is at rest, the two moments must balance, that is

$$a. \frac{bh^3}{12} = W_1 l + W_2 \frac{l}{2} \qquad (2)$$

from which a, the internal stress at a unit's distance from the neutral axis, can be found as all other quantities are known.

The intensity of stress is greatest at the extreme fibers. In the case of a beam with rectangular cross-section, the greatest stress is

$$p_1 = a.\frac{h}{2}$$

hence
$$a = \frac{2p_1}{h}$$

and the equation of moments becomes

$$p_1\frac{bh^2}{6} = W_2\frac{l}{2} + W_1 l, \qquad (3)$$

$$\text{and } W_1 = \frac{bh^2p_1}{6l} - \frac{W_2}{2}$$

from which the maximum allowable load, W_1, can be found if p_1 is the maximum allowable intensity of stress in the fibers of the beam.

Examples.

1. Find the extreme fiber stress in a cantilever steel beam, 24′ long, 2″ wide and 12″ deep if a weight of one ton is hung at its end. (Fig. 135.)

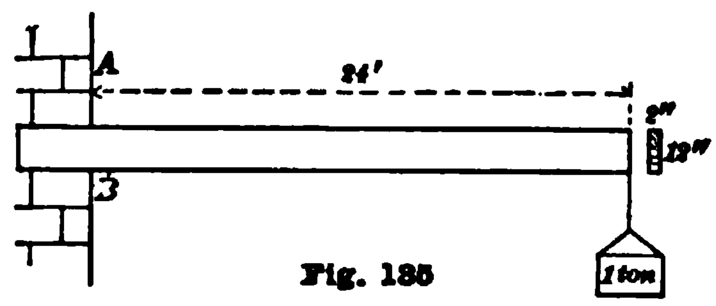


Fig. 135

N. B. Reduce dimensions to inches and the ton to lbs. Steel weighs about 490 lbs. per cubic ft. It is evident from (1) that the Bending Moment, *i. e.*, the moment of the *External* forces, will be greatest when l is greatest. Hence the cross-section we have to consider is at AB, where l is the full length of the cantilever.

Ans. $p_1 = 17{,}880$ lbs. per square inch.

2. If p_1 had been limited to 8,000 lbs., how large could W_1 have been? Ans. $353\frac{1}{3}$ lbs.

3. A beam 14″ by 6″, and 20′ long, supports a uniform load of 180 lbs. per running foot. Fig. 136. The beam is of wood which weighs 38 lbs. per cubic foot. What is the maximum fiber stress in the beam?

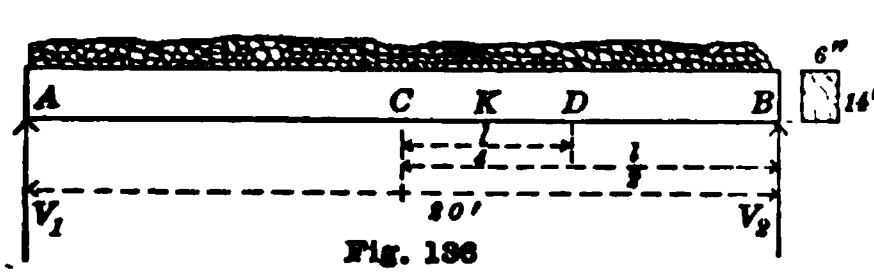


Fig. 136

By symmetry, $V_2 = \frac{1}{2}$ (weight of beam and load). The greatest

stress is evidently at the *center*. The external forces acting on the half K to make it bend about an axis at C are: V_2 = half of the *entire load, acting up at B*; and half of the *entire load*, acting *down* at D;

hence $$p_1 \cdot \frac{bh^2}{6} = \left(\frac{\Sigma W}{2}\right)\left(\frac{l}{4} - \frac{l}{2}\right), \text{ etc.}$$

145. The moment of inertia of plane surfaces.
Going back now to **142**

$$M = a\int z^2 (bdz) = a\int z^2 dA$$

it is evident that while a means intensity of stress at a unit's distance from the neutral axis, the integral is only a function of the surface; it is the sum of all the products resulting from multiplying every elementary area by the square of its distance from an axis in its plane. This integral is sometimes called "The Second Moment of the Surface," but the more common name is "The Moment of Inertia" of the surface. It is very generally designated by the capital letter I. If the axis, from which z is measured, passes thru the centroid of the surface, it will be written I_o.

Of course I will vary with the size and shape of the surface, for example, the cross-section of a beam. In the case of a rectangle, we found it to be

$$I_o = \frac{bh^3}{12}$$

We shall soon find values for I_o for triangles, circles and other figures of use in engineering. The general formula which every student should keep in mind is

$$M = aI_o$$

in which M **is the Bending Moment** of external forces with reference to the neutral axis of the cross-section of the beam where the stress is to be examined; and aI **is the Moment of Resistance** of the beam at the same cross section.

146. The radius of gyration. As the name "Moment of Inertia" was borrowed from another department of Mechanics, which is much older in history, so the name "Radius of Gyration" has been taken from the same source. It is here defined simply by the formula

$$k^2 = \frac{I}{A}$$

in which k is the "radius of gyration," and A is the area of the surface under consideration.* I. usually can be separated into two factors, one of which measures the area of a cross-section, and the other will be, by definition, the value of k^2. When I is I_o, k becomes k_o.

In the case of the rectangular beam

$$I_o = \frac{bh^3}{12},\ A = bh,\ k_o^2 = \frac{h^2}{12},\ k_o = \frac{h}{6}\sqrt{3}$$

Engineering hand-books usually contain tables giving values of I_o, A, k_o^2 for all beams in ordinary use, and p_1, the maximum allowable stress in the materials used, which greatly facilitates the work of designing or calculating the strength of beams.

147. Take a prismatic beam, whose cross-section is an isosceles triangle, suitably loaded with a uniform load, which, *with the beam itself*, weighs w lbs. per linear foot. Fig. 137. The beam is supported in a horizontal position by piers at the ends. Let us find the maximum fiber stress at a distance x from the end P. The cross-section at A gives the surface of action, which is shown enlarged. Fig. (*b*).

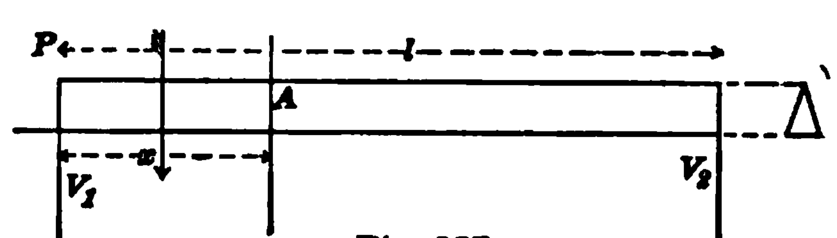
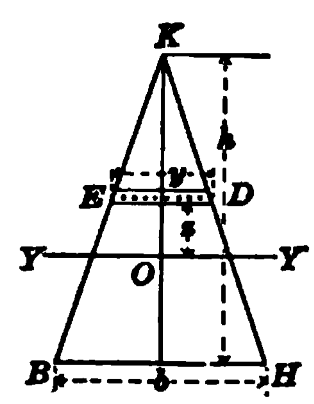


Fig. 137 (a)



(b)

1. Now we know that the greatest stress will be at the top of the beam at the point K, which is distant from the line of no stress, $\frac{2}{3}h$, inasmuch as the neutral axis passes thru the centroid of the triangle. Accordingly

$$p_1 = \frac{2h}{3} \cdot a$$

* The name "second moment" would appear to be more logical than the "moment of inertia." The expression zdA is the (so called) moment of an elementary area. When this moment is multiplied by z, we seem to get a moment of a moment, or a "second moment."

However, the term "moment of inertia" is in good use and will be retained as will the mysterious term "radius of gyration." Both will be accounted for later on in this book. All that the students need to know now about either of these terms is, what algebraic functions they represent. $I = \int z^2 dA$ and $k^2 = \frac{I}{A}$.

So we must find a. Now, we know that

$$M = aI_o$$

or

$$a = \frac{M}{I_o}$$

Hence, in order to solve our problem, we must find both M and I_o.

2. M is the moment of the external forces acting on the beam-segment, PA. Those forces consist of the "support" $V_1 = \frac{lw}{2}$ acting *up*; and the weight of PA, which is wx acting *down*, "centered" at the middle point of the segment. Hence, the moment about the *neutral axis at A* is

$$M = \frac{wl}{2}x - wx.\frac{x}{2} = \frac{w}{2}(lx - x^2)$$

We must next find I_o, the Moment of Inertia of the triangular section; see the enlarged figure in Fig. 137.

3. For finding I_o, take an element (ydz).

So that we have $$I_o = \int z^2 y dz.$$

Since $OK = \frac{2}{3}h$, we get by proportion

$$y = \frac{b}{h} \cdot \left(\frac{2}{3}h - z\right)$$

The limits for integration for z evidently are $-\frac{h}{3}$ and $+\frac{2}{3}h$, so that

$$I_o = \frac{b}{h}\int_{-\frac{h}{3}}^{+\frac{2}{3}h} \left(\frac{2}{3}h - z\right) z^2 dz$$

$$I_o = \frac{bh^3}{36}$$

Now, the extreme fiber is at the point K, distant $\frac{2}{3}h$ from YY, so that

$$p_1 = \frac{2h}{3} \cdot \frac{M}{I_o} = \frac{2h}{3} \cdot \frac{\frac{w}{2}(lx - x^2)}{bh^3} \cdot 36 = \frac{12w}{bh^2}(l - x)x$$

4. It thus appears that the extreme fiber stress varies as does the product of the two segments of the beam: x and $l - x$. It was shown in the calculus that the product of two numbers whose sum was con-

stant was a maximum when the numbers were equal. Hence, the maximum value of p occurs when $x=\frac{l}{2}$ which gives

$$(Max.)p_1=\frac{3wl^2}{bh^2}$$

148. Engineer's hand book. Of course, we knew, without mathemathics, that the greatest stress (greatest danger of breaking) was at the middle of the beam, and had we turned to the Engineer's hand book we should have found

$$Max.\text{ Bending Moment}=\frac{wl^2}{8};\ I=\frac{bh^3}{36};$$

$$A=\frac{1}{2}bh;\quad k^2=\frac{h^2}{18};$$

$$\text{and if we put } c=\frac{2}{3}h,\text{ and } M=\frac{p_1}{c}\cdot Ak^2=aI_o,$$

we should have had, as found above:

$$\frac{wl^2}{8}=p_1\cdot\frac{3}{2h}\cdot\frac{bh}{2}\cdot\frac{h^2}{18},\text{so that } p_1=\frac{3wl^2}{bh^2}$$

149. Some relative properties of the areas and stresses above and below the neutral axis of a triangle.* The reader has noted the fact that the algebraic sum of the normal stress on a cross-section of a horizontal beam is zero. This is self-evident if the axis YY (the neutral axis) is an axis of symmetry; it does not *look* quite so evident if the section is a triangle as in Figure 137. However, if we find the total stress (tension) above the neutral axis; and then the total compressive stress below the neutral axis, and find them of equal magnitude and with opposite signs, the proof of the doctrine laid down long ago (that the algebraic sum of the horizontal forces must be zero, because the beam is at rest) will be complete.

1. Taking the general formula for the stress, Fig. 137 (b)

$$F=a\int yzdz$$

Substituting $$y=\frac{b}{h}\left(\frac{2}{3}h-z\right)$$

* While the discussion which follows may not seem to be very practical, it deals with general principles which are vital, and in an emergency may be important. Engineers have been known to quarrel over matters here discussed.

and integrating for the surface above YY we have

$$F_1 = \frac{ab}{h}\int_0^{\frac{2}{3}h}\left(\frac{2}{3}h - z\right)zdz$$

$$= \frac{ab}{h}\left(\frac{hz^2}{3} - \frac{z^3}{3}\right)\Bigg]_0^{\frac{2}{3}h} = \frac{4abh^2}{81} \qquad (1)$$

2. If now we integrate for the stress below YY, we shall have

$$y = \frac{b}{h}\left(\frac{2}{3}h - z\right) \text{ as before.}$$

(It must be remembered that z is itself negative.)
Accordingly we have

$$F_2 = \frac{ab}{h}\int_{-\frac{h}{3}}^{0}\left(\frac{2}{3}h - z\right)zdz = \frac{ab}{h}\left(\frac{hz^2}{3} - \frac{z^3}{3}\right)\Bigg]_{-\frac{h}{3}}^{0}$$

$$= -\frac{4abh^2}{81}$$

and $F_1 + F_2 = 0$, as was expected.

The proposition is universally true if the neutral axis passes thru the centroid of the surface.

3. But the **Moments** of the stresses above and below **are not equal** for the triangle, as is readily shown.

Taking the formula
$$M = a\int yz^2dz$$

and integrating for the moment of the stress above YY, we have

$$M_1 = \frac{ab}{h}\int_0^{\frac{2}{3}h}\left(\frac{2}{3}h - z\right)z^2dz = \frac{4}{243}abh^3$$

4. Integrating for the moment of the stress *below* the neutral axis we have

$$M_2 = a\frac{b}{h}\int_{-\frac{h}{3}}^{0}\left(\frac{2}{3}h - z\right)z^2dz = \frac{ab}{h}\left(\frac{2h}{3}\cdot\frac{z^3}{3} - \frac{z^4}{4}\right)\Bigg]_{-\frac{h}{3}}^{0}$$

$$= \frac{11}{972}abh^3$$

so that $\dfrac{M_1}{M_2} = \dfrac{16}{11}$, M_1 being the larger.

Since a is the same for the both areas, $\dfrac{I_1}{I_2} = \dfrac{16}{11}$ (23)

The sum $M_1+M_2=M=a\dfrac{bh^3}{36}$

and $$I_1+I_2=I_o=\frac{bh^3}{36}$$
as already found.

5. The "Center" of the uniformly varying stress *above* the neutral axis is

$$z_1=\frac{M_1}{F_1}=\frac{\dfrac{4}{243}abh^3}{\dfrac{4}{81}abh_2}=\frac{h}{3}$$

which is just half-way from YY to K.

6. The "Center" of the uniformly varying stress *below* the neutral axis is

$$z_2=\frac{M_2}{F_2}=\frac{\dfrac{11}{972}abh^3}{-\dfrac{4}{81}abh^2}=-\frac{11}{48}h$$

so that, numerically,

$$\overline{C_1C_2}=\frac{h}{3}+\frac{11h}{48}=\frac{9}{16}h=L$$

It must now be evident that the internal normal stress of a beam forms a couple at every section whose force is F_1, whose lever arm is L, and whose moment is

$$F_1L=\frac{4ahb^2}{81}\times\frac{9}{16}h=\frac{abh^3}{36}=M=aI_o$$
as already found.

150. Results may be summarized.

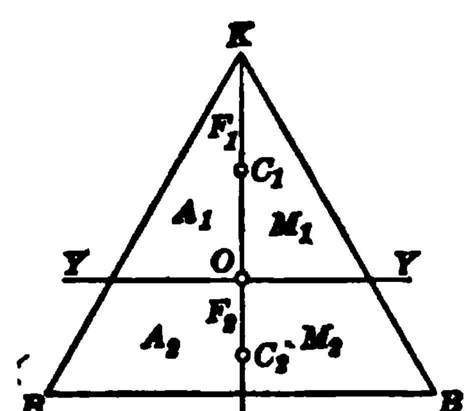


Fig. 188

1. A_1 is to A_2 as 4 is to 5.
2. $F_1=F_2$.
3. M_1 is to M_2 as 16 to 11.
4. I_1 is to I_2 as 16 to 11.
5. $OC_1=\dfrac{h}{3}$ center of stress on A_1.
6. $OC_2=\dfrac{11}{48}h$, center of stress on A_2.
7. $OC_1+OC_2=L=\dfrac{9}{16}h$.
8. $LF=M_1+M_2=M=a\dfrac{bh^3}{36}$

A similar set of results and comparisons can, of course, be made for every cross-section of a beam whenever the areas on the two sides of the neutral axis are unequal, which is often the case, because the material used is not as strong and elastic in tension as it is in compression, as, for instance, when the material on one side of the neutral axis is concrete and on the other side of the neutral axis is steel, or a combination of steel and concrete. Such cross-sections will be considered in later sections.

151. This chapter does not aim to cover the extensive subject of beams. Some simple forms have been introduced to illustrate the subject of uniformly varying stress, and to develop the function, "Moment of Inertia," and show how intimately connected it is with the study of stress and the strength of beams.

1. Before solving and proposing a variety of problems for finding values of I, it may be well to state again some assumptions we are making, and to put down in words and in the language of mathematics, the general equations we are to apply.

2. All beams and girders are assumed to be of homogeneous material, whatever may be the cross-section, and to have uniformly varying normal stress at every cross-section, when they are slightly bent under moderate loads. In all cases this important formula holds:

The Bending Moment of external forces acting to bend the beam at any section is numerically equal to the Moment of Resistance of the internal stresses acting at that section.

If p is the stress in the extreme fiber, c the distance of the extreme fiber from the neutral axis, and M the Bending Moment, the above dictum is more accurately expressed as follows:

$$M = aI_o = \frac{p}{c} I_o = \frac{pA}{c} \cdot k_o^2$$

3. The general formula for the Moment of Inertia when YY is the axis of reference is

$$I = \int\int z^2 dz dy.$$

When, however, the element of the surface can be a differential strip parallel to the Y-axis, the formula is greatly simplified by assuming the element ydz, so that we have

$$I = \int z^2 y dz.$$

152. Moments of Inertia, fundamental examples.

1. Find I_o for a circle whose radius is r. Fig. 139. The element of the surface is ydz

$$I_o = 2\int_o^r z^2 y\,dz$$

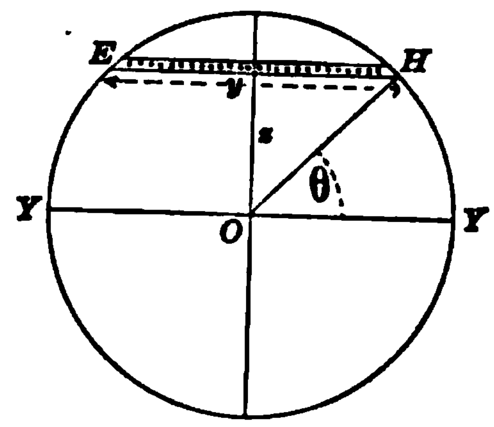


Fig. 139

Introducing θ, we have

$$y = 2r\cos\theta. \quad z = r\sin\theta. \quad dz = r\cos\theta d\theta$$

$$\text{and } I_o = 4r^4\int_o^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta d\theta$$

$$= \frac{r^2}{2}\int_o^{\frac{\pi}{2}} \sin^2 2\theta d(2\theta)$$

$$I_o = \frac{\pi r^4}{4} = A\frac{r^2}{4} = Ak_o^2, \text{ so that } k_o^2 = \frac{r^2}{4} \qquad (25)$$

For a *semi-circle* about its base

$$I = \frac{\pi r^4}{8}$$

2. The I_o of a Circular Ring, radii r_1 and r_2.

$$I_o = \frac{\pi r_2^4}{4} - \frac{\pi r_1^4}{4} = \frac{\pi}{4}(r_2^4 - r_1^4) = A.\frac{r_2^2 + r_1^2}{4}$$

3. Find I_o for an ellipse.

Let the equation of the ellipse be $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and let the axis of reference be YY.

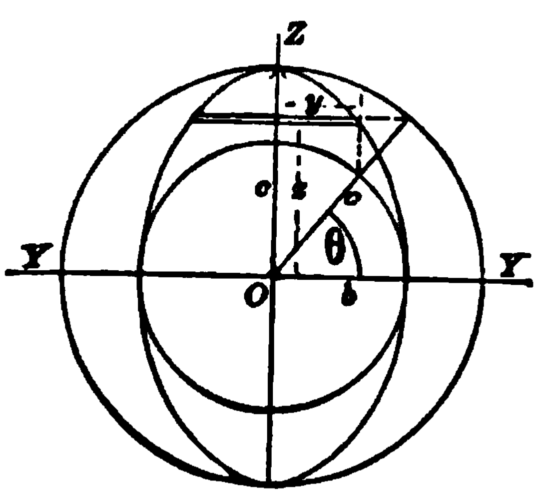


Fig. 140

From Fig. 140, it is evident that

$$I_o = 4\int_o^c z^2 y\,dz$$

Introducing θ, we have

$$y = b\cos\theta. \quad z = c\sin\theta. \quad dz = c\cos\theta d\theta$$

so that

$$I_o = 4c^3 b\int_o^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta d\theta$$

$$I_o = \frac{\pi c^3 b}{4} = A.\frac{c^2}{4} = Ak_o^2 \therefore k_o = \frac{c}{2} \qquad (26)$$

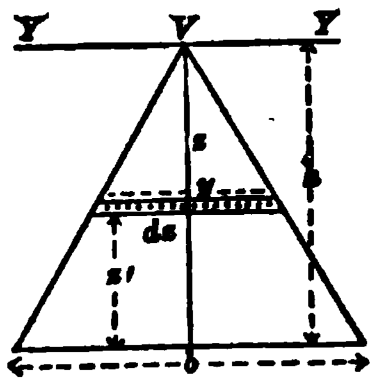


Fig. 141

4. The Moment of Inertia of a triangle about an axis *thru its vertex* parallel to the base. Fig. 141.

The element is ydz, in which

$$Y=\frac{b}{h}\cdot z$$

hence
$$I_v=\frac{b}{h}\int_0^h z^3dz=\frac{bh^3}{4}=A.\frac{h^2}{2}=Ak^2.\ \therefore k=\frac{h}{\sqrt{2}} \tag{28}$$

5. Had we used *the base as the axis* of reference, we should have had (Fig. 141).

$$I_b=\frac{b}{h}\int_0^h (h-z')z'^2dz=\frac{b}{h}\left(\frac{h^4}{3}-\frac{h^4}{4}\right)=\frac{bh^3}{12} \tag{29}$$
$$=A.\frac{h^2}{6}=Ak^2$$

7. The relations of the three values of I for a triangle are thus shown in Fig. 142.

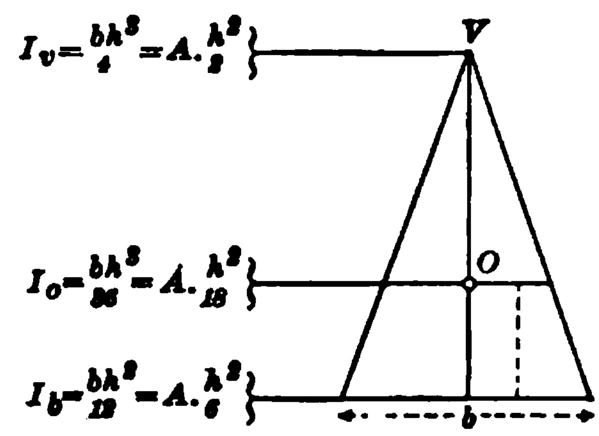


Fig. 142

153. An exterior axis in the plane. The Moment of Inertia of a surface with reference to an axis not passing thru the centroid may be found directly by integration, as has just been shown, but it is often more convenient to make I depend upon I_o for a parallel axis. Thus:

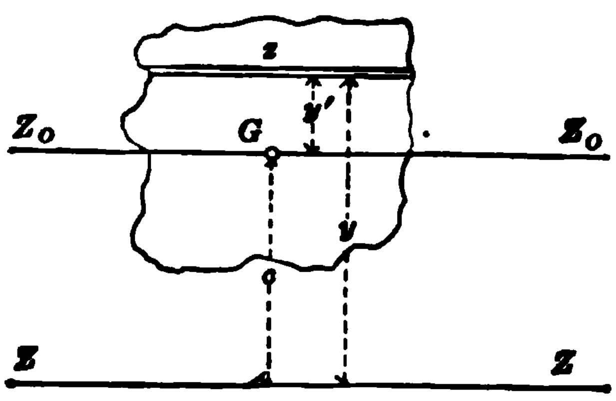


Fig. 143

To find I for the axis ZZ, Fig. 143:

$$I_z=\int y^2zdy,\ \text{but}\ y=c+y'$$

(so that $dy=dy'$).

If Z_oZ_o is parallel to ZZ, and the distance from ZZ to the centroid is c,
Hence
$$I_z=\int (c^2+2cy'+y'^2)zdy'$$
$$=c^2\int zdy'+2c\int y'(zdy')+\int y'^2zdy'$$
$$=c^2A+I_o$$

Since the first integral is plainly the area of the given surface; the second integral is the moment of a surface about its own centroid axis, which must be zero; and the last is merely I_o. Hence the rule:

$$I_z=I_o+Ac^2 \tag{1}$$

The Moment of Inertia of a surface with reference to any axis in its plane is equal to I_o for a parallel axis thru the centroid of the surface, *plus* the product of the given area multiplied by the square of the distance between the two axes.

1. Thus the I_v for the triangle in **152** could have been found from the knowledge of

$$I_o = \frac{bh^3}{36}$$

for

$$I_v = \frac{bh^3}{36} + \frac{bh}{2}\cdot\left(\frac{2}{3}h\right)^2 = bh^3\left(\frac{1}{36}+\frac{8}{36}\right) = \frac{bh^3}{4}$$

2. **Conversely,** knowing I_x, I_o for a parallel axis can be found from the formula

$$I_o = I_x - Ac^2 \qquad (2)$$

3. For a rectangle about its base we have, since $I_o = \dfrac{bh^3}{12}$

$$I_b = \frac{bh^3}{12} + \frac{h^2}{4}\cdot bh = \frac{bh^3}{3}$$

4. For a *semicircle* about Z_oZ_o Fig. 144,

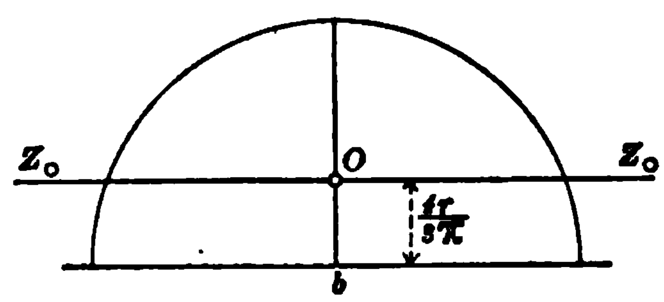


Fig. 144

$$I_o = I_b - A\left(\frac{4r}{3\pi}\right)^2.$$

Hence

$$I_o = \frac{\pi r^4}{8} - \frac{\pi r^2}{2}\cdot\frac{16r^2}{9\pi^2} = \frac{r^4}{72\pi}(9\pi^2 - 64) = A.\left(\frac{9\pi^2-64}{36\pi}\right)r^2 \qquad (3)$$

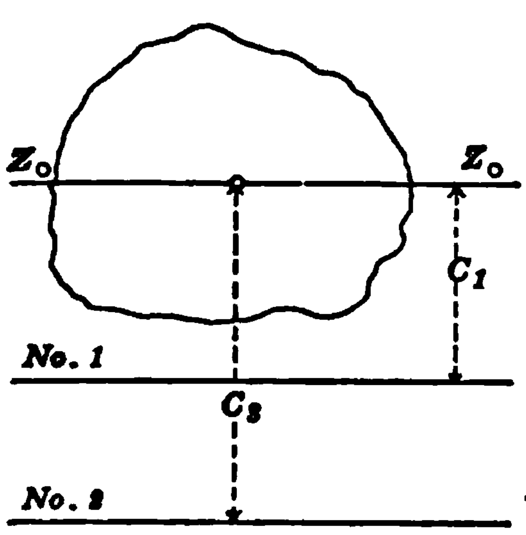


Fig. 145

5. Finally, knowing I and c for any axis, we can find I for any *known parallel co-planar axis.* Suppose we have I_1 and c for No. 1; and we wish to find I_2 for No. 2. Fig. 145.

$$I_o = I_1 - Ac_1^2$$

$$I_2 = I_o + Ac_2^2 \quad I_1 + A(c_2^2 - c_1^2)$$

154. The Moment of Inertia of a composite surface. It is first necessary to find the centroid of the surface, then I_o is found by use of the formula:

$$I_o \text{ for } Z_oZ_o = \Sigma I_o + \Sigma(Ac^2)$$

The addition in practice is algebraic, since both I_o and A may be negative, as will now be illustrated. See Fig. 146.

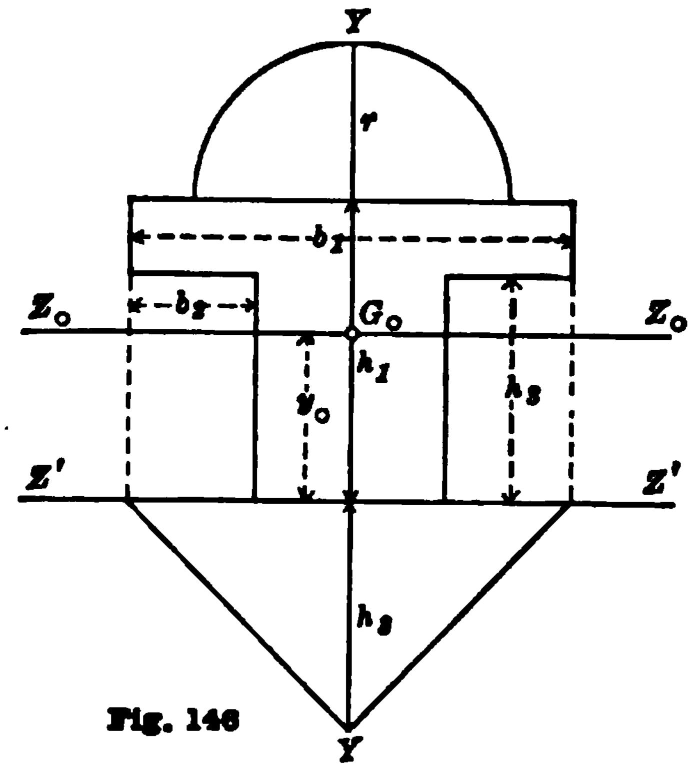


Fig. 146

Example. The line YY is an axis of symmetry, and must contain the centroid G_0. Take any convenient line as a preliminary Z-axis as $Z'Z'$.

The total area is

$$\Sigma A = \frac{\pi r^2}{2} + b_1h_1 - 2b_2h_2 + \frac{1}{2}b_1h_3$$

The total moment with reference to $Z'Z'$ is

$$\Sigma(Ay_0) = \frac{\pi r^2}{2}\left(h_1 + \frac{4r}{3\pi}\right) + \frac{b_1h_1^2}{2} - b_2h_2^2 - \frac{b_1h_3^2}{6}$$

In the case of the two absent rectangles, both areas and moments are negative. In the case of the triangle the area is positive but the moment is negative. Hence

$$y_0 = \frac{\Sigma(AY_0)}{\Sigma A} = \frac{\pi r^2\left(h_1 + \frac{4r}{3\pi}\right) + b_1h_1^2 - 2b_2h_2^2 - \frac{b_1h_3^2}{3}}{\pi r^2 + 2b_1h_1 - 4b_2h_2 + b_1h_3}$$

Having found and located G_0, find the I_0 for the areas separately, with reference to their own centroid axes. Then find ΣAc^2 and add results.

Numerical example—

Let $r=2$, $b_1=6$, $h_1=4$, $h_2=3$, $h_3=3$, $b=7/4$

Find the position of G_0 and the value of I_0.

155. Built up beams, struts and columns or posts are very frequently made of separate steel bars or plates securely riveted together so that the cross-section consists of parts, and the moment of inertia must be found as in the last example.

When a post or column is used to carry a vertical load, its ***stiffness***, whereby it resists lateral bending, is an important element in its strength; and its moment of inertia is needed with reference to more than one axis thru the centroid of a section. For example, in the case of an ellipse (Fig. 140) we have

$$I_y = \int z^2 dA, \text{ and } I_z = \int y^2 dA$$

N. B.—When two Moments of Inertia are found for the same surface, about different rectangular axes, with O at the centroid,

the *larger* will be called I or I_1, and the *smaller* may be called J or I_2. Thus for the ellipse whose major axis is $2a$, and whose minor axis is $2b$,

$$I_1 = \frac{\pi a^3 b}{4} \text{ and } J = \frac{\pi a b^3}{4}.$$

156. Commercial shapes of rolled steel. In determining shapes of rolled structural steel, exact dimensions are given including rounded corners and fillets, as is shown in Fig. 147; but in computing Moments of Inertia for strength and weight they are often ignored or their fillets are assumed to balance the rounded corners, as in Figs. 148-9, which are taken with notation and formulas from the "Carnegie Pocket Book."

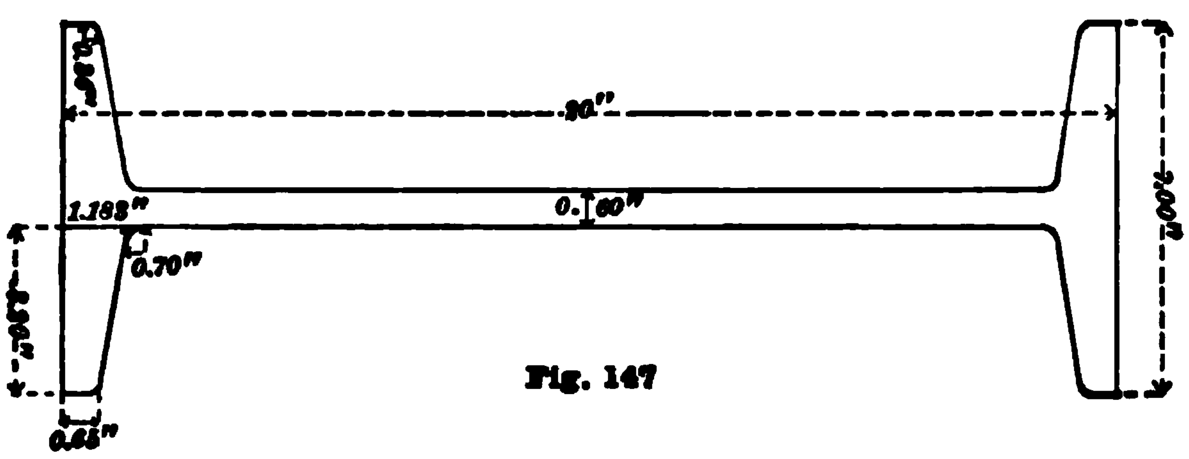


Fig. 147

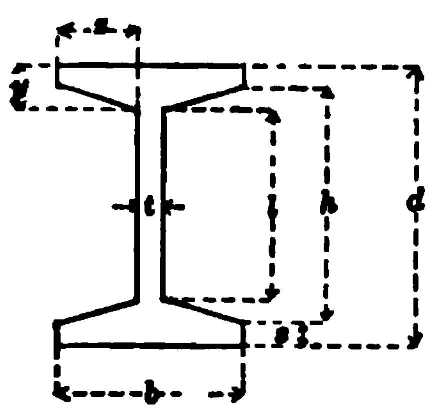


Fig. 148

Let the student check the values of A, I and J.

I = Mom. In. for a neutral axis *parallel* to a *flange*.

J = Mom. In. for a neutral axis *parallel* to a *web*.

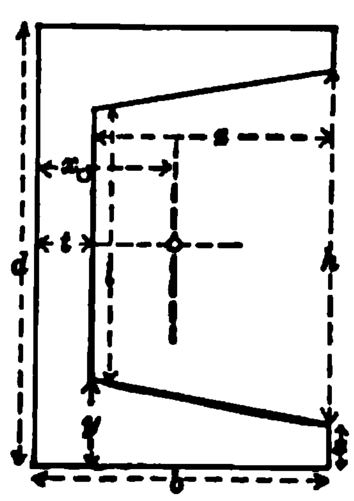


Fig. 149

An I-Beam. Fig. 148.

Area $= A = dt + (s+y)2z$

$$I = \frac{1}{12}\left[bd^3 - \frac{3}{2}(h^4 - l^4)\right].$$

$$J = \frac{1}{12}\left[b^3(d-h) + lt^3 + \frac{1}{24}(b^4 - t^4)\right].$$

A Channel Section. Fig. 149.

Area $= dt + (s+y)z$

$$x_o = \frac{1}{A}\left[b^2 s + \frac{1}{2}ht^2 + \frac{1}{18}(b-t)^2(b+2t)\right]$$

$$I = \frac{1}{12}\left[bd^3 - \frac{3}{4}(h^4 - l^4)\right]$$

$$J = \frac{1}{3}\left[2sb^3 + lt^3 + \frac{1}{12}(b^4 - t^4)\right] - Ax_o^2$$

N. B.—"The flanges of both *I*-beams and standard channels have now a uniform slope of 2 inches to the foot."

Hence $\frac{b-t}{k-l} = 6$ for *I*-beams, and $\frac{b-t}{h-l} = 3$ for channels.

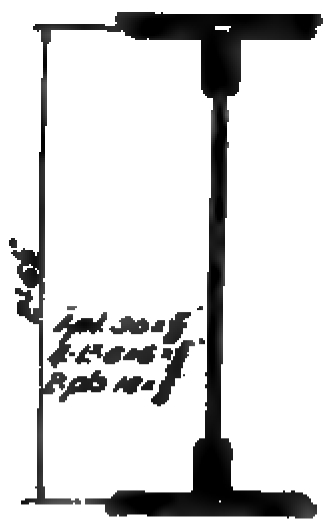


Fig. 150

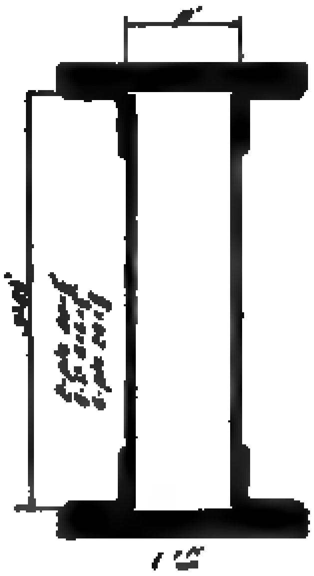


Fig. 152

Fig. 151 consists of: a Web-Plate 60"x⅞"; each Flange has 2 Angles 6"x6"x⅝", and 3 Plates 14"x⅝".

Fig. 152 is a cross-section of a Column of the Chicago & N. W. R'y Office Building, Chicago; 6 Plates 16"x⅝"; 2 Plates 16"x$^{11}/_{16}$"; 2 Plates 12½"x¾"; 4 Angles 6"x6"x¾".

Fig. 151

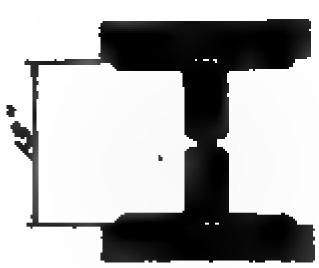


Fig. 153

Fig. 154

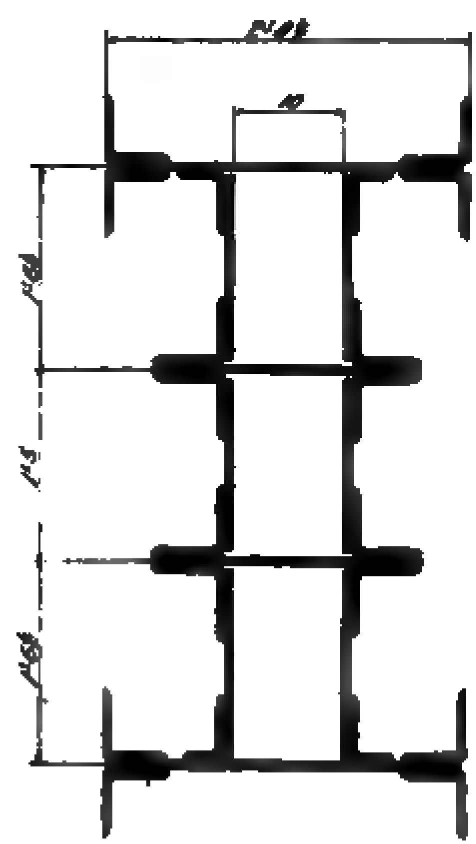


Fig. 155

Fig. 154 is a cross-section of Column 280, Waldorf-Astoria Hotel, New York. It consists of: 10 Plates 32½"x⅝"; 4 Plates 36"x¾"; 4 Angles 6"x4"x$^{11}/_{16}$"; 8 Angles 6"x3½"x⅝".

Fig. 155 is a cross-section of a Column in Chicago Steel Co. Building, Chicago. It consists of: 2 Plates 32"x½"; 2 Plates 28"x⅜"; 4 Plates 18"x$^{7}/_{16}$"; 2 Plates 16"x$^{7}/_{16}$"; 4 Angles 6"x6"x⅝"; 12 Angles 6"x6"x$^{7}/_{16}$"; 4 Angles 6"x4"x½".

Fig. 156, is a cross-section of the lower chord in the Quebec Bridge which failed during the erection.

It consisted of materials as follows:

Outer Ribs: 2 Plates 54"x$1\frac{3}{16}$"
Each 1 Plate 54"x⅞"
1 Plate 87¾"x$1\frac{5}{16}$"
2 Angles 8"x6"x$1\frac{5}{16}$"

Inner Ribs: 3 Plates 54"x$1\frac{3}{16}$"
Each 2 Plates 46"x$1\frac{5}{16}$"
2 Angles 8"x3½"x$1\frac{5}{16}$"

Lattices across the top and bottom of all four ribs.

Fig. 156

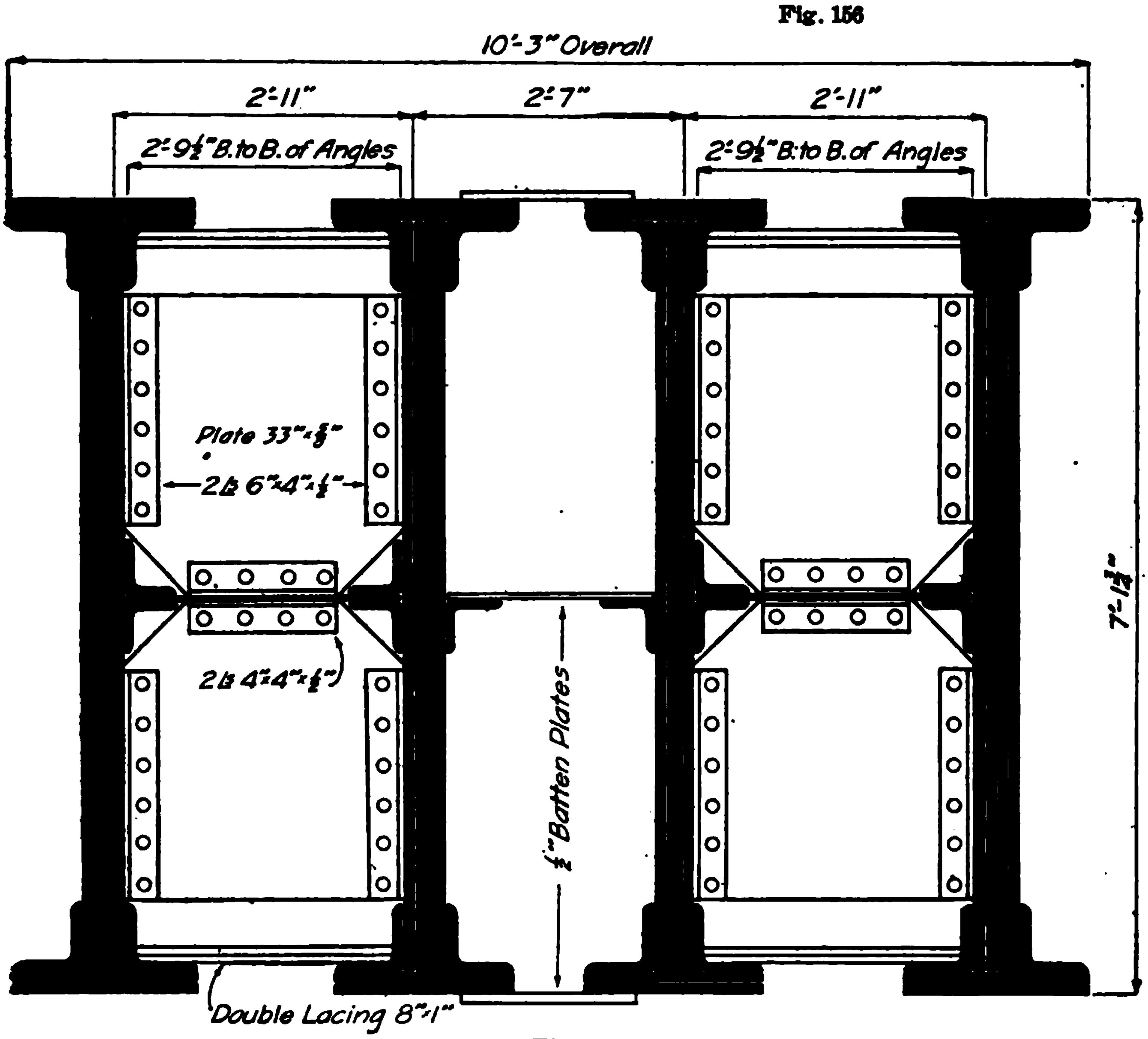


Fig. 157

Fig. 157 is a cross-section of the lower chord of the new Quebec Bridge, as designed for the position of the one that failed. The exquisitely drawn cut, is almost self-explanatory. However, the section requires:—

8 Plates 81.5"x¾"
8 Plates 81.5"x1⅛"
16 Plates 20"x1" for flanges
16 Angles 8"x8"x1" for ribs
10 Angles 6"x6"x⅝" for diaphragm plates
2 Plates 33"x$1\frac{1}{16}$" for diaphragms
4 Plates 33"x⅝" vertical between ribs
16 Angles 6"x4"x½" for vertical plates
8 Angles 4"x4"x½" for diaphragms
4 Lattice Lacings 8"x1"
8 Plates ½" thick horizontal between central ribs

157. A variety of examples showing cross-sections of beams and struts in actual use is given above. The moment of inertia of each one may be found from the dimensions given.

Remarks upon Figs. 156 and 157.

The chord member which failed, *A9L*, had lattice bracing along the edges of the ribs as shown by the broken lines. The purpose of this bracing was to prevent the buckling of the ribs sideways. Prof. Geo. F. Swain, of Harvard, who made a careful study of the design, is of the opinion that failure was primarily due to the inadequacy of that bracing.

CHAPTER X.

Moments of Inertia—(Continued).

The Relation Between Strength and Moment of Inertia.

158. The strength of a beam is measured by the load it can carry with safety. The test of strength may come from a load at the end of a cantilever, or at the center of a beam supported on piers. In either case the bending moment is proportional to the load; hence M, the bending moment, or its equal, the Moment of Resistance, is a measure of its strength. It was found that

$$M = \frac{p_1}{c} I_o$$

in which c is the distance from the neutral axis to the extreme fiber, and p_1 is the greatest intensity allowed. It follows that the strength of a beam is *directly proportional to I*, and *inversely proportional* to c, p_1 being the same.

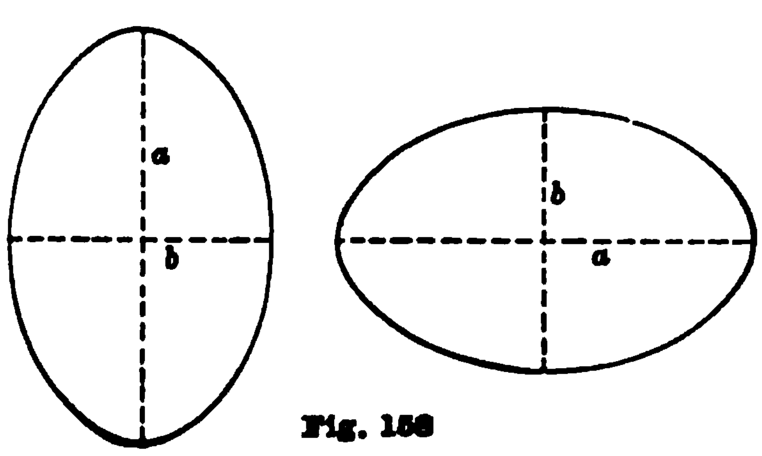


Fig. 158

1. To illustrate, suppose a beam has an elliptical cross-section and that it is loaded first with its major axis vertical, and second with that axis horizontal. Fig. 158.

We have
$$M_1 = \frac{p_1}{a} \cdot \frac{\pi a^3 b}{4} = p_1(\pi ab) \cdot \frac{a}{4}$$

$$M_2 = \frac{p_1}{b} \cdot \frac{\pi ab^3}{4} = p_1(\pi ab) \frac{b}{4}$$

Hence
$$\frac{M_1}{M_2} = \frac{a}{b}$$

or the strengths are to each other as the semi-axes.

2. Again take a rectangular beam. Fig. 159. When it is "edgeways"

$$M_1 = \frac{2p_1}{h} \cdot \frac{bh^3}{12} = p_1(hb) \frac{h}{6}$$

When "flatways"

$$M_2 = \frac{2p_1}{b} \cdot \frac{hb^3}{12} = p_1(hb) \frac{b}{6}$$

so that
$$\frac{M_1}{M_2} = \frac{h}{b}$$

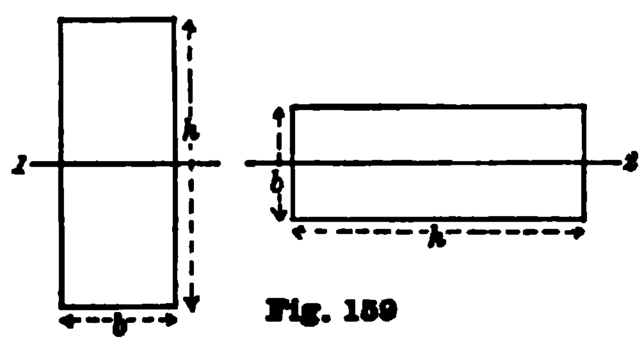


Fig. 159

This means that a floor joist 2″x12″ will safely support six times as much when on edge as when flat. This is abundantly confirmed by personal experience. Proper *lateral* support is assumed.

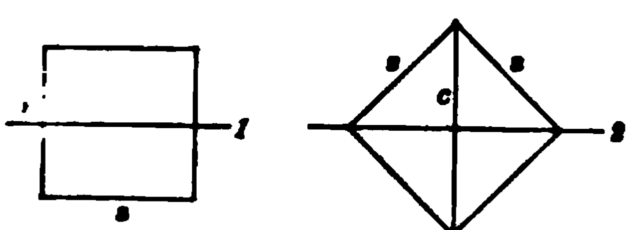


Fig. 160

3. A third case is hardly a matter of experience. Suppose a square beam is placed with sides vertical. Fig. 160. Its strength is

$$M_1 = \frac{2p_1}{s} \cdot \frac{s^4}{12} = p_1(s^2) \frac{s}{6}.$$

If its diagonal is vertical,

$$M_2 = \frac{p_1}{c} \cdot 2 . \frac{2cc^3}{12} = p_1(2c^2) \frac{c}{6}$$

$$\frac{M_1}{M_2} = \frac{s}{c},$$

or the strength of No. 1 is 1.4 times that of No. 2.

The unmathematical reason is plain: In No. 1 the whole extreme *layers* of fiber are utilized for the stress p_1; in No. 2, only a single fiber

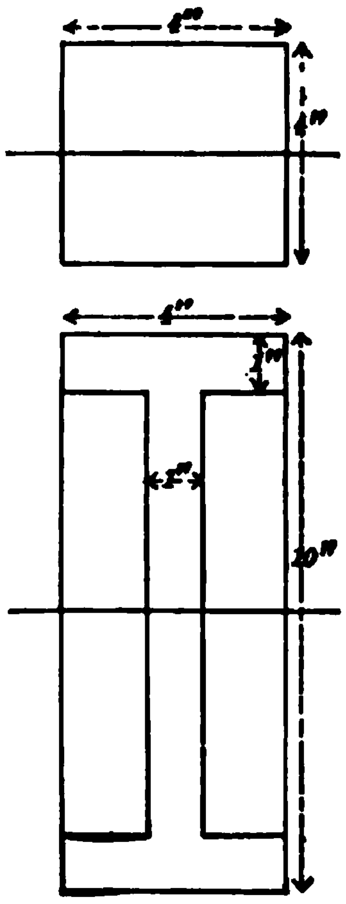


Fig. 161

on the top and the bottom has the stress p_1 while the increased material near the horizontal diagonal adds very little to the strength.

156. The moral derived from the above illustrations is, to put as much of the material as far from the neutral axis as possible, leaving near the neutral axis only enough to preserve the unity and stability of the beam. This will be illustrated by two ideal examples.

1. Compare the strength of a steel bar 4″ square with an I-beam made of the *same amount of steel*, the flanges being 1″x4″, and the web 1″x8″. Fig. 161.

$$M_1 = p_1 \frac{bh^2}{6} = p_1 10\frac{2}{3} \text{ for No. 1.}$$

$$M_2 = p_1 \cdot \frac{1}{5} \cdot \left(\frac{4(10)^3}{12} - \frac{3.(8)^3}{12}\right) = p_1 41\frac{1}{15} \text{ for No. 2.}$$

Hence $$\frac{M_2}{M_1} = \frac{41\frac{1}{15}}{10\frac{2}{3}} = 4 \text{ nearly.}$$

or, **the I-beam with the** *same weight* **of material is almost** *four times* **as strong.**

2. A certain load, W, is to be carried by a cantilever beam of length l. Compare the *weight* of a beam in the shape of a solid cylindrical rod, with the *weight* of a cylindrical tube of the same length, with an exterior diameter n times as great, and *equally strong.*

The expression for the strength of a solid cylinder is

$$M_c = p_1(\pi r^2)\frac{r}{4} = p_1 A_1 \frac{r}{4}$$

If the exterior radius of the tube is nr, and the inner radius x, we have,

$$M_t = \frac{p_1}{nr} \cdot (I_{nr} - I_x) = \frac{p_1}{nr} \cdot \frac{\pi}{4}(n^4r^4 - x^4)$$

Equating the two expressions for M, since they are equally strong, and solving for x^2, we have

$$x^2 = r^2\sqrt{n^4 - n}$$

Hence, the area of the ring, which is the cross-section of the tube, is

$$A_2 = \pi(n^2r^2 - r^2\sqrt{n^4 - n})$$

Since the weights of two beams of the same length are to each other as the areas of their cross-sections, we have.

$$\left.\begin{aligned}\frac{\text{Weight of solid cylinder}}{\text{Weight of tube}}=\frac{A_1}{A_2}&=\frac{\pi r^2}{\pi r^2(n^2-\sqrt{n^4-n})}\\&=\frac{2n}{1}\text{ nearly}\end{aligned}\right\}$$

(since $\sqrt{n^4-n}=n^2-\dfrac{1}{2n}-\dfrac{1}{8n^4}$ etc.), which means that:—

If the tube's exterior diameter is $\left\{\begin{matrix}\textbf{twice}\\\textbf{thrice}\end{matrix}\right\}$ as great as that of the solid cylinder, and ***equally strong,*** **it will weigh only one $\left\{\begin{matrix}\textbf{fourth}\\\textbf{sixth}\end{matrix}\right\}$ as much; and so on:** under *ideal* conditions, which would be quite *unreal* if n be made too large.

The saving of weight while preserving strength, thru the use of tubes, is the secret of light-framing and light-rolling stock. For a discussion of the *stiffness* of hollow shafting; see Chapter XIX.

157. The axis for the maximum value of I_o. The student is now prepared to see that while an unsymmetrical surface may have an indefinite number of values of I for different axes across it, there always must be one greatest I_o, and one least. In engineering practice both the greatest and the least are very important. It is now necessary to show how they are found and valued.

Let O, Fig. 162, be the centroid of a plane surface, and OZ and OY a set of rectangular axes so taken that I_z and I_y can be, and have been, found. Let the larger be I_z. Let OZ_1 and OY_1 be a second set of rectangular axes meeting at O, and determined by the angle θ. Let P mark the position of an element of the surface, d^2A, whose co-ordinates are (z,y) and (z_1, y_1) for the respective axes.

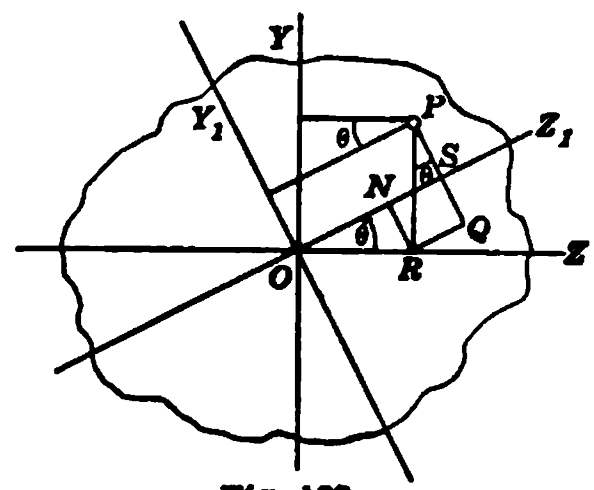


Fig. 162

The quantities $I_z=\int\!\!\int y^2d^2A$ and $I_y=\int\!\!\int z^2d^2A$ are supposed to have been found. The surface element d^2A may be thought of as $dydz$, or $dy'dz'$.

From the figure,

$$\left.\begin{aligned}y_1&=PQ-RN=y\cos\theta-z\sin\theta\\z_1&=RQ+ON=y\sin\theta+z\cos\theta\end{aligned}\right\}$$

Hence $$I_{z1}=\int\int y_1^2 d^2A=\int\int(y^2\cos^2\theta-2yz\sin\theta\cos\theta+z^2\sin^2\theta)d^2A$$

$$I_{z1}=\cos^2\theta I_z-2\sin\theta\cos\theta\int\int yzd^2A+\sin^2\theta I_y.$$

Similarly $$I_{y1}=\sin^2\theta I_z+2\sin\theta\cos\theta\int\int yzd^2A+\cos^2\theta I_y.$$

The integral $\int\int yzd^2A$ is sometimes called (for the sake of a name) the *Product of Inertia*. More conveniently it will be called simply K

$$K=\int\int zyd^2A.$$

When necessary its value will be found by integration. If either OZ or OY is an axis of symmetry, K is zero, since the differential terms in the integral are in pairs which balance.

Therefore, we have

$$I_{z1}=\cos^2\theta I_z-2\sin\theta\cos\theta K+\sin^2\theta I_y \qquad (1)$$

$$I_{y1}=\sin^2\theta I_z+2\sin\theta\cos\theta K+\cos^2\theta I_y \qquad (2)$$

Adding, we have

$$I_{z1}+I_{y1}=I_z+I_y \qquad (3)$$

Knowing I_z and I_y, and having calculated I_{z1}, I_{y1} can be found from the last equation.

1. Take for example the simple case of an ellipse and suppose we want I_{z1}.

We know that $$I_z=\frac{\pi ba^3}{4}, \text{ and } I_y=\frac{\pi ab^3}{4}$$

and by inspection we see that $K=0$.

Let $$\theta=\frac{\pi}{4}$$

Then $$I_{z1}=\frac{I_z+I_y}{2}=I_{y1}=\pi ab\cdot\frac{a^2+b^2}{8}$$

Interesting values follow if $\theta=\frac{\pi}{6}$ and if $\theta=\frac{\pi}{3}$.

Going back to Eq. 1, we see that the value of I_{z1} varies as θ varies, and will have either a maximum or a minimum value when $\frac{dI_{z1}}{d\theta}$ is zero. Hence, differentiating (1) and remembering that I_z, I_y and K are constants, we have

$$\frac{dI_{z1}}{d\theta} = -2\sin\theta\cos\theta\, I_z + 2(\sin^2\theta - \cos^2\theta)K + 2\sin\theta\cos\theta\, I_y$$

or $$\frac{dI_{z1}}{d\theta} = -2\sin\theta\cos\theta\,(I_z - I_y) + 2(\sin^2\theta - \cos^2\theta)K$$

Placing the second member equal to zero so as to find the values of θ for the greatest and least values of I_z, we have

$$(I_z - I_y)\sin 2\theta = -2\cos 2\theta . K$$

$$\tan 2\theta = \frac{-2K}{I_z - I_y} \qquad (4)$$

If $K=0$: $2\theta = 0$ and π;

or $\theta = 0$ and $\frac{\pi}{2}$, hence in the case of the ellipse, the major and minor axes of the ellipse are the axes for minimum and maximum values of I.

158. When K is not zero. The angle 2θ always has two values differing by π; hence θ has two values differing by $\frac{\pi}{2}$, one acute, the other obtuse, so that the axes for greatest and least values of I are at right angles. In all cases, since $I_z > I_y$, the axis for maximum I, makes an angle numerically less than 45° with the original Z-axis. The axes for I(max.) $= I$; and I (min.) $= J$, are called the *Principal Axes of the Surface.*

Since $K=0$ for an axis of symmetry, it follows that that axis and its perpendicular thru the centroid *are the Principal Axes.*

159. The value of K_1 for the axes OZ_1 and OY_1.

Substituting the values of y_1 and z_1 from (157) in the expression

$$K_1 = \int\int y_1 z_1 d^2A, \text{ we have}$$

$$K_1 = \int\int [y^2\sin\theta\cos\theta + zy(\cos^2\theta - \sin^2\theta) - z^2\sin\theta\cos\theta]d^2A$$

$$K_1 = \sin\theta\cos\theta\,(I_z - I_y) + (\cos^2\theta - \sin^2\theta)K$$

Comparing the value of K_1 with the value of $\frac{dI_{z1}}{d\theta}$ we see that

$$\frac{dI_{z_1}}{d\theta} = -2K_1$$

so that, when dI_{z_1} is zero, $K_1 = 0$ and conversely.

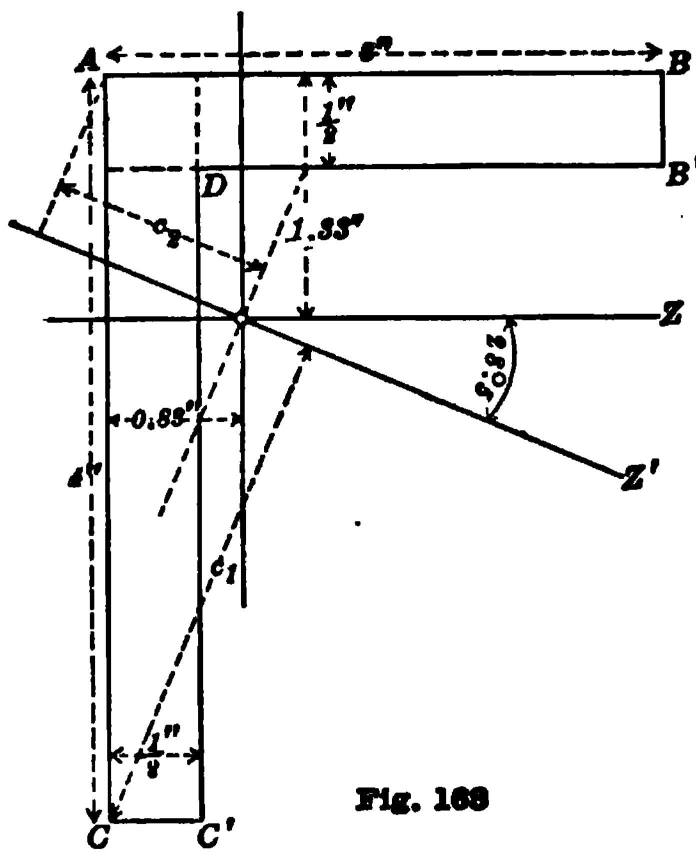


Fig. 163

Hence, the "Product of Inertia" is zero, for the Principal Axes, without regard to symmetry.

159. Angle irons. It is frequently necessary to know the maximum and minimum values of I_o for a cross-section of an "Angle Iron" shown approximately* in Fig. 163. It has no axis of symmetry, and the solution of an illustrative problem will be best with numerical values.

The centroid is found by taking moments of areas about AB and AC. The centroid is distant from AB 1.33″, and from AC, 0.83″ The position of the centroid is shown at o.

160. How to find I_z, I_y and K. They can, of course, be found directly from the figure by means of the formula for parallel axes, but the following is more simple.†

1. Draw two axes, Z' and Z'', parallel to OZ, and thru the centroids of A_1 and A_2 respectively. Fig. 164.

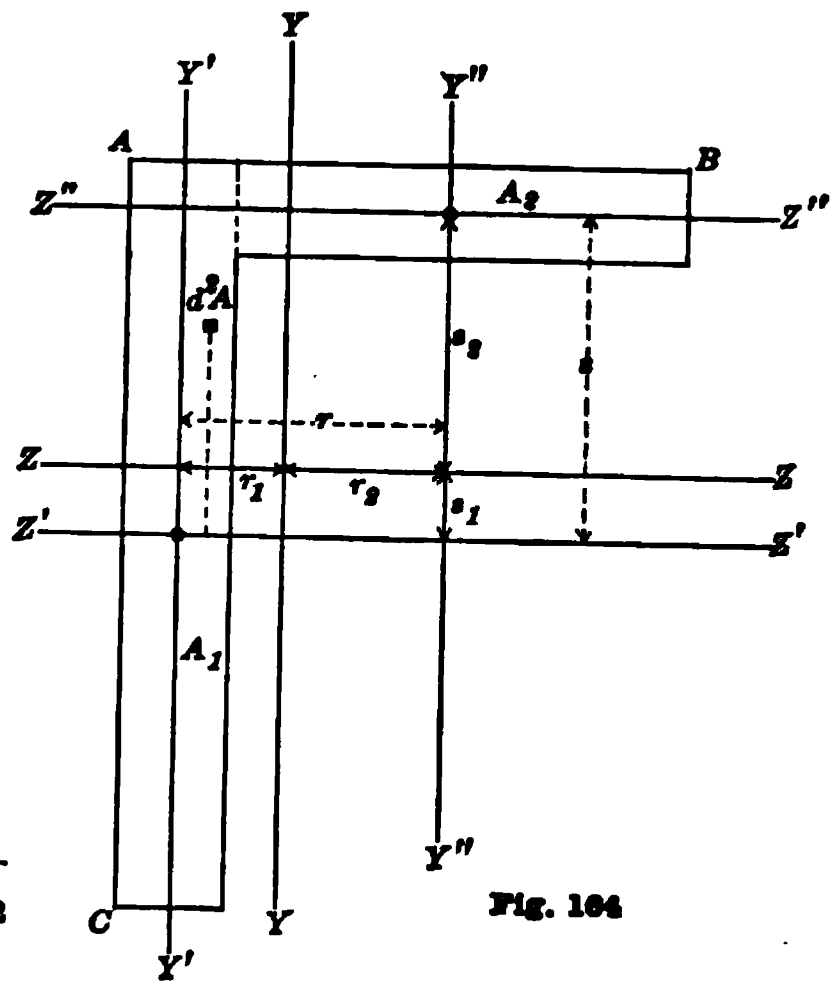


Fig. 164

Then $\quad I_z = I_{z1} + A_1 s_1^2 + I_{z''} + A_2 s_2^2$

But $\quad A_1 s_1 = A_2 s_2$, and $s = s_1 + s_2$,

hence $\quad s_1 = \dfrac{A_2 s}{A_1 + A_2}$, and $s_2 = \dfrac{A_1 s}{A_1 + A_2}$

so that $\quad A_1 s_1^2 + A_2 s_2^2 = \dfrac{A_1 A_2^2 s^2 + A_2 A_1^2 s^2}{(A_1 + A_2)^2} = \dfrac{A_1 A_2 s^2}{A_1 + A_2}$

and
$$I_z = I_{z'} + I_{z''} + \frac{A_1 A_2 s^2}{A_1 + A_2} \tag{1}$$

In the same way it is readily shown after drawing the axes Y' and Y'' that
$$I_y = I_{y'} + I_{y''} + \frac{A_1 A_2 r^2}{A_1 + A_2} \tag{2}$$

---

* In angle irons as actually used, the salient angles B' and C' are rounded, and the re-entrant angle D has a fillet. See Fig. 147.

† This elegant analysis appears to be due to Muller-Breslau.

2. To find $K=\int\int yzd^2A_1$ we take the parts A_1 and A_2 separately so that $K=K'+K''$. We see that

$$y=y'-s_1$$
$$z=z'-r_1$$

and

$$yz=y'z'-r_1y'-s_1z'+s_1r_1$$

hence

$$K'=\int\int y'z'd^2A_1-r_1\int\int y'd^2A_1-s_1\int\int z'd^2A_1+s_1r_1\int\int d^2A_1$$

But the axis $Z'Z'$ is an axis of symmetry of A_1, hence

$$\int\int y'z'd^2A_1=0$$

As the axes z' and y' pass thru the centroid of A_1, we have $\int\int y'd^2A=0$, and $\int\int z'd^2A_1=0$; and since $\int\int d^2A_1=A_1$

we have $K'=s_1r_1A_1$ and similarly $K''=s_2r_2A_2$

But, as before $s_1=\dfrac{A_2s}{A_1+A_2}$, and $s_2=\dfrac{A_1s}{A_1+A_2}$

$$r_1=\frac{A_2r}{A_1+A_2},\ \text{and}\ r_2=\frac{A_1r}{A_1+A_2}$$

hence

$$K=K'+K''=\frac{A_1A_2^2sr}{(A_1+A_2)^2}+\frac{A_1^2A_2sr}{(A_1+A_2)^2}$$

and

$$K=\frac{A_1A_2sr}{A_1+A_2} \qquad (3)$$

3. The numerical values of I_z, I_y and K are now easily found. From Fig. 163 we have $A_1=2$, $A_2=\dfrac{5}{4}$

and from figure 164

$$s=\left(2-\frac{1}{4}\right)=\frac{7}{4}\ \text{and}\ r=\frac{1}{4}+\frac{5}{4}=\frac{3}{2}$$

Hence

$$I_z=\frac{\frac{1}{2}(4)^3+\frac{5}{2}\left(\frac{1}{2}\right)^3}{12}+\frac{2\times\frac{5}{4}\times\frac{49}{16}}{2+\frac{5}{4}}$$

$$I_z=2.69+2.35=5.05$$

$$I_y=\frac{4\left(\frac{1}{2}\right)^3+\frac{1}{2}\left(\frac{5}{2}\right)^3}{12}+\frac{2\times\frac{5}{4}\times\frac{9}{4}}{2+\frac{5}{4}}$$

$$I_y = 0.692 + 1.731 = 2.423$$

$$K = \frac{2 \times \frac{5}{4} \times \frac{21}{8}}{2 + \frac{5}{4}} = 2.02$$

4. It will be noted that all these quantities, tho they refer to axes meeting in the centroid of the surface, do not involve a knowledge of where that centroid is. The quantities s and r are the co-ordinates of the centroid of A_2 with reference to the axis meeting at the centroid of A_1. When, however, we calculate the angle θ for a principal axis, and proceed to draw that axis, the centroid of the entire surface must be used as found above.

The angle θ is determined by formula

$$\tan 2\theta = \frac{-2K}{I_z - I_y} = \frac{-4.04}{5.06 - 2.42} = -1.53$$

$$2\theta = -56° \ \ 50'$$

$$\theta = -28° \ \ 25'$$

Having the numerical values of I_z, I_y, K, θ and the co-ordinates of the centroid of the entire surface, we can plot the center, draw the principal axes and compute the values of $I(max.)$ and $(J\ min.)$.

5. The computation is somewhat simplified by combining equations (1) and (2) **157**, with the equation

$$\tan 2\theta = \frac{-2K}{I_z - I_y}$$

and we get

$$I = I_z - K \tan \theta^* \tag{4}$$

Combining this last equation with $I + J = I_z + I_y$, we get

$$J = I_y + K \tan \theta \tag{5}$$

---

* Eq. 4, 157, gives

$$I_y = I_z + \frac{2K}{\tan 2\theta} = I_z + \frac{2(\cos^2\theta - \sin^2\theta)K}{2 \sin\theta \cos\theta}$$

Substituting in (1) we get

$$I = I_z - K\left(2 \sin\theta \cos\theta - \frac{\sin\theta\,(\cos^2\theta - \sin\theta^2}{\cos\theta}\right)$$

$$= I_z - K\left(\frac{2 \sin\theta \cos^2\theta - \sin\theta \cos^2\theta + \sin^3\theta}{\cos\theta}\right) = I_z - K \tan\theta$$

Since $K=2.02$ and $\tan(-28° 25')=-0.541$, we have

$$I=6.145 \qquad k_1^2=1.88$$
$$J=1.33 \qquad k_2^2=0.41$$

since $A_1+A_2=\frac{13}{4}$, and $k^2=\frac{I}{A_1+A_2}$, k being the radius of gyration.

161. When the angle-iron which we have been studying is used as a beam, the axis for I should be horizontal, and the distance to the extreme fiber should be computed or measured from an accurate drawing. See Fig. 163. Then the greatest Moment of Resistance the bar is safely capable of is

$$M_{max}=\frac{p_1}{c_1}\cdot A\times k_1^2=\frac{p_1}{c_1}\cdot I \qquad (1)$$

However, if an angle-iron is used as a strut of post, it is liable to bend by buckling, and as the buckling is always in the plane of *least resistance*, it is only the *least* Moment of Resistance which is to be counted upon.

$$M_{min}=\frac{p_1}{c_2}\cdot Ak_2^2=\frac{p_1}{c_2}J \qquad (2)$$

Steel "angles" may be rolled of various dimensions, with some allowances for round corners, and a fillet, but, in fact, only certain standard sizes are in the market. A general formula for k_1 and k_2 is hardly worth while here.

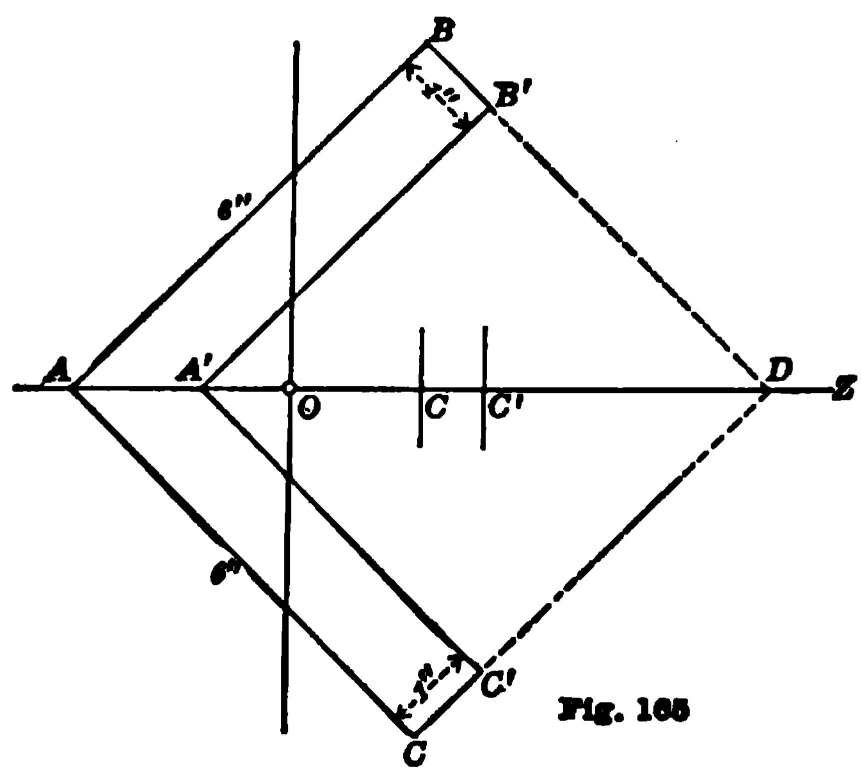


Fig. 165

162. When the arms of the angle iron are equal, as in Fig. 165, there is an axis of symmetry, AD. Consequently, $K=0$, and AD and the perpendicular thru the centroid of the surface are the principal axes. The Moment of Inertia, I_z, is readily found by subtracting the moment of inertia of the triangle $A'B'D$, with reference to the axis OZ, from the moment of inertia of the triangle ABD, and doubling the result. The moment of inertia, I_y, is found by subtracting the moment of inertia of the small square from the moment of inertia of the large square, after finding the centroid of the surface.

The distance $\overline{OC}$ is found from the equation

$$(\overline{OC})[(6)^2-(5)^2]=(5)^2\sqrt{\tfrac{1}{2}}=(5)^2(\overline{CC'})$$

Using the dimensions shown in the figure, find I_z and I_y.

168. A *Z*-bar. A *Z*-bar usually has equal flanges, and tho the cross-section has no axis of symmetry, its centroid is readily seen. With the axis and dimensions as shown in the figure (Fig. 166), find the Principal Axes and the max. and min. values of $M=aI$.

$$I_x=\frac{cb^3}{12}+2\left(\frac{ac^3}{12}+ac\times\left(\frac{b-c}{2}\right)^2\right)$$

$$I_x=\frac{cb^3+2ac^3}{12}+\frac{1}{2}ac\,(b-c)^2$$

$$I_y=\frac{bc^3+2a^3c}{12}+\frac{ac}{2}\,(a+c)^2$$

$$K=\frac{1}{2}ac(b-c)(a+c).\quad \text{See (3) } \mathbf{160.}^*$$

Hence
$$k_x^2=\frac{I_x}{A},\ \text{and } k_y^2=\frac{I_y}{A}$$

Now, as
$$\tan 2\theta=-\frac{2K}{I_x-I_y}$$

the angle θ, which is a negative angle, is readily found, as are the principal moments of inertia:

$$I_1=I_x-K\tan\theta$$
$$I_2=I_y+K\tan\theta.$$

Example.

Let $a=\frac{11}{4},\ b=6,\ c=\frac{3}{4}$, inches, so that $A=A_1+A_2+A_3=8.625$

Then $$I_x=\frac{1}{12}\left(\frac{3}{4}\times 216+2\times\frac{11}{4}\times\frac{27}{64}\right)+\frac{1}{2}\left(\frac{11}{4}\times\frac{3}{4}\times\frac{441}{16}\right)=42.12$$

$$I_y=\frac{1}{12}\left(6\times\frac{27}{64}+2\times\frac{1331}{64}\times\frac{3}{4}\right)+\frac{1}{2}\times\frac{11}{4}\times\frac{3}{4}\times\frac{49}{4}=15.44$$

$$K=\frac{1}{2}\times\frac{11}{4}\times\frac{3}{4}\times\frac{21}{4}\times\frac{7}{2}=18.95.$$

---

* $K=\dfrac{A_1A_2sr}{A_1+A_2}$. $A_1=A_2=ac,\ s=b-c,\ r=a+c.$

Hence $$k_z^2=\frac{42.12}{8.625}=4.88, \text{ so that } k_z=2.21$$

and $$k_y^2=\frac{15.44}{8.625}=1.79, \text{ and } k_y=1.34$$

$$\tan 2\theta=-1.42$$

$$2\theta=-54°51'$$

$$\theta=-27°25'$$

$$\tan\theta=-0.519$$

$$K\tan\theta=-9.835$$

$$I_1=42.12+9.835=51.96$$

$$I_2=15.44-9.835=5.60$$

so that $$k_1^2 \max=\frac{51.96}{8.625}=6.02$$

and $$k_2^2 \min=\frac{5.60}{8.625}=0.649$$

Hence $$k_1=2.45, \text{ and } k_2=0.81.$$

164. The relation of M to I. In computing the strength (moment of resistance) of a beam, it must not be forgotton that it is *not enough* to know the moments of inertia. Suppose we have, as is the case of the Z-bar, the max. I_1 and the min. I_2. The ratio of strength is

$$\frac{M_1}{M_2}=\frac{\frac{p_1}{c_1}I_1}{\frac{p_1}{c_2}\cdot I_2}=\frac{c_2I_1}{c_1I_2}. \tag{22}$$

This shows that the strength varies *directly* as the I, and *inversely* as the extreme fiber distance. For the Z-bar

$$\frac{M_1}{M_2}=\frac{I_1}{I_2}\cdot\frac{P'N'}{PN}.=\frac{51.96}{5.60}\cdot\frac{P'N'}{PN}.$$

Now, from Fig. 166, $PN=\frac{b}{2}\cos\theta+\left(a+\frac{c}{2}\right)\sin\theta=4.10$

$$P'N'=\frac{b}{2}\sin\theta+\frac{c}{2}\cos\theta=\cos\theta=1.71$$

Hence

$$\frac{M_1}{M_2} = \frac{51.96}{5.60} \cdot \frac{1.71}{4.10} = \frac{88.85}{22.96} = 3.9 \text{ nearly.}$$

165. *Z*-bars are used as purlins in the framing of roofs, the flanges taking the slope of the roof. *Z*-bars are used in the construction of columns and struts, especially where it is necessary to have ready access to all surfaces for the purpose of painting. The method of using them is shown by Fig. 167, which gives a general horizontal section.

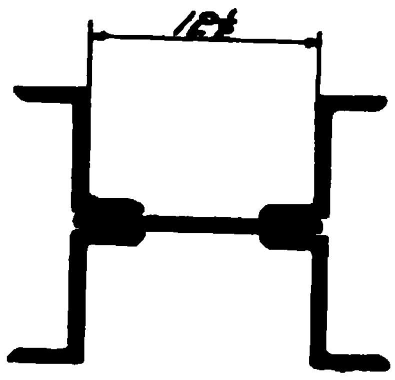


Fig. 167

The combination of 4 *Z*-bars, united by an interior plate, makes an open column whose principal moments of resistance are readily calculated, as the axes are parallel to those already used. The central connecting plate has usually the same thickness as the *Z*-bars.

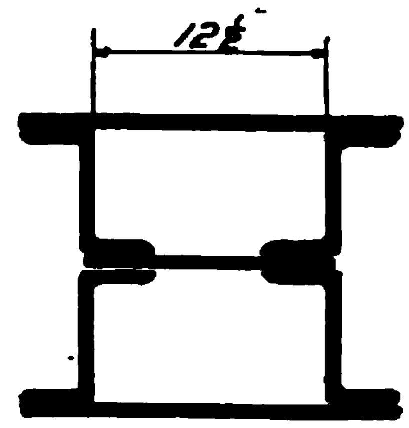


Fig. 168

In large columns, two wide thick plates are riveted to the outer flanges of the *Z*-bars, as shown in the drawing, Fig. 168. There is, however, a slight lack of economy in the use of material as compared with some other forms.

CHAPTER XI.

Elementary Graphical Statics.

166. Direct stresses in the members of a frame.

1. The frames considered in this chapter are ideal; they are assumed to be rigid; all members lie in a vertical plane; all joints are by means of pins which are central to the connected bars; and the only stresses (unless otherwise stated) taken into account are the *direct stresses* due to concentrated imposed loads. All loads are applied at pins, and their vectors are *external* to the frames; that is, a load on a lower pin is represented by an arrow *below* the pin; if on an upper pin the arrow is *above* the pin. Every pin is in equilibrium under the action of a system of balanced forces, and in every case the stress diagram will show for the balanced forces a static polygon.

2. A bar in tension will be called a "tie"; one in compression will be called a "strut." It should go without saying that a *tie pulls* on *both* pins, and a *strut pushes* on *both* pins. All drawings of frames and diagrams should be executed on a drawing board, with good instruments and on a scale at least twice that used in this book. Great

pains must be taken in drawing parallel lines. In laying off load vectors and in measuring stress vectors, the same scale must be used, but the scale itself is entirely arbitrary, tons, lbs., kilos, poundals or dynes.

167. Review static polygons. At this point the student will do well to review (**70**) and make sure that he can draw the static polygon for a set of co-planar forces, whose lines of action meet at a point and balance, provided all directions are given and all magnitudes but two are known.

It is also necessary to recall the fact that the arrows on the lines of a static polygon show the directions in which the members meeting at a pin act upon it, so as to balance. For example, suppose the bars 1, 2, 3, 4, 5, Fig. 169, act upon a pin at P, and their actions are accurately shown by the force polygon which closes, as the arrows follow each other around the *Area P.* These arrows show that Nos. 1, 3 and 5 must be *struts*; and that Nos. 2 and 4 are *ties*. The student will do well to study the figure 169 carefully so as to see the *system* followed.

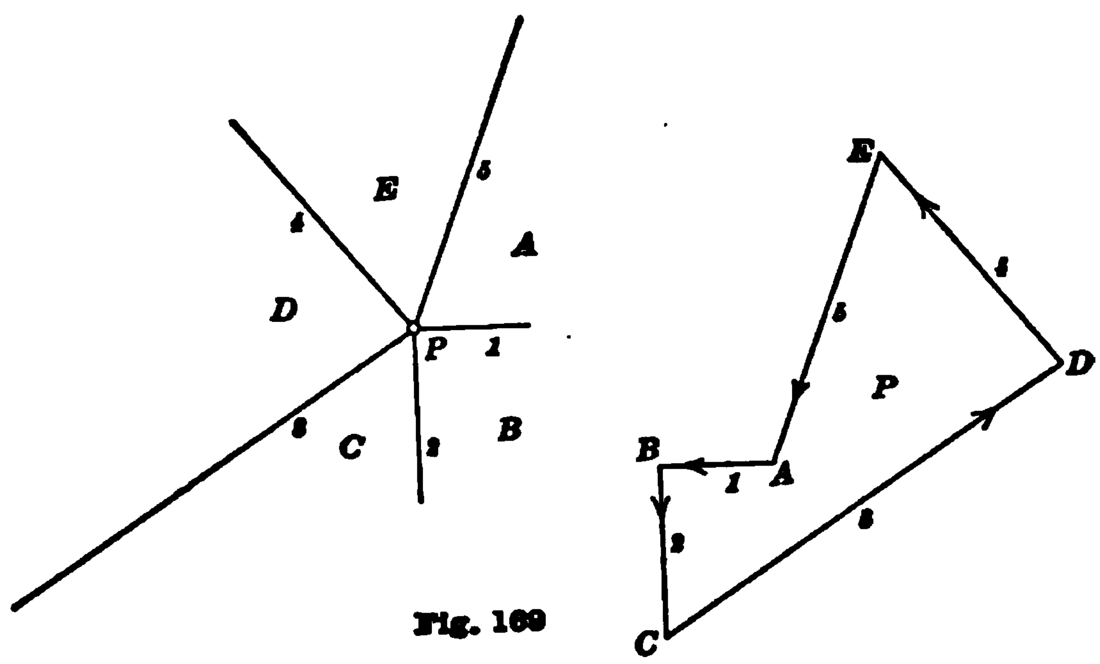


Fig. 169

168. Given an irregular cantilever frame supporting at different pins six equal loads. Fig. 170.

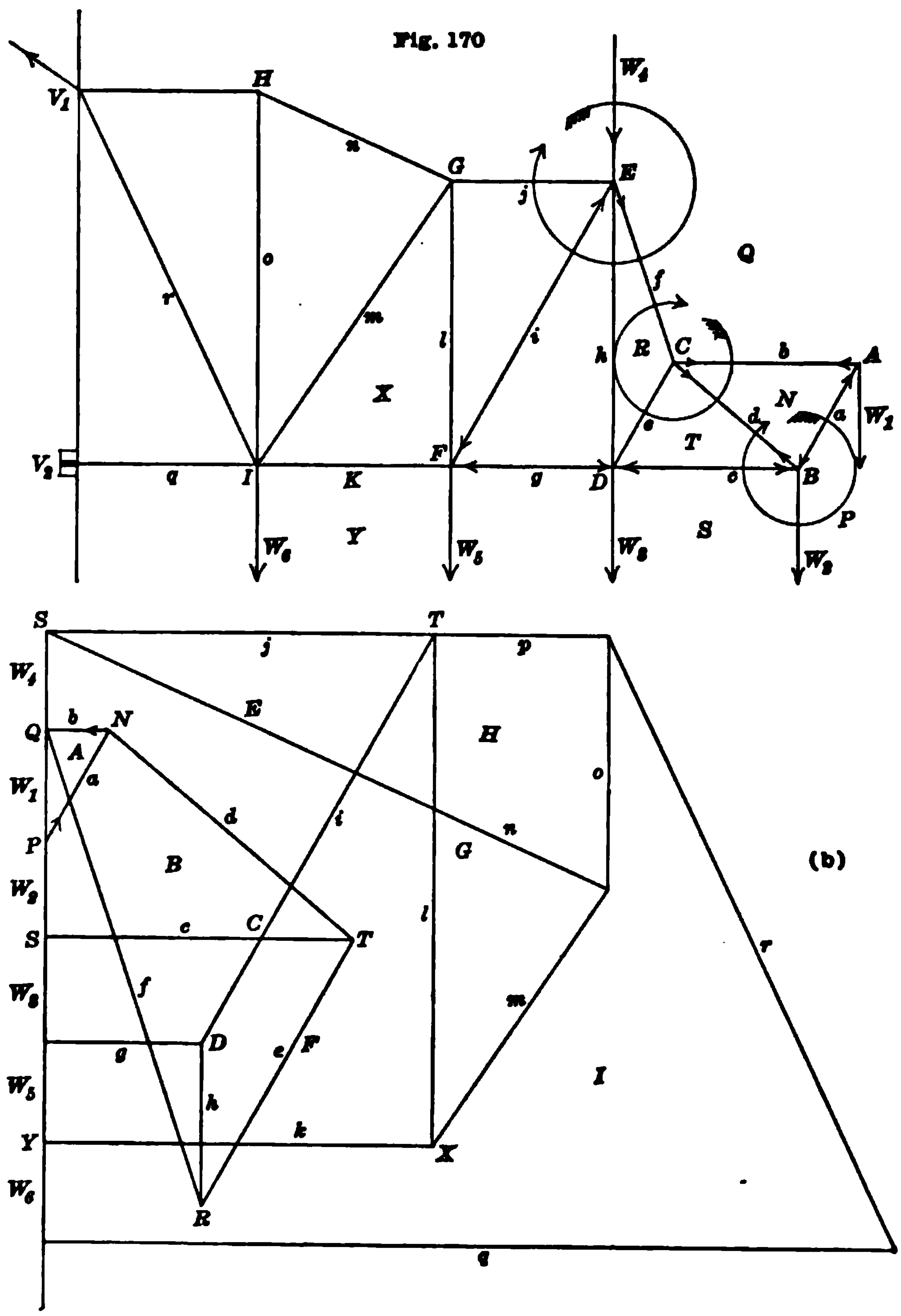
1. We begin with the pin A, because that is the only one whose static polygon (triangle) can at first be drawn. The forces are the weight W_1, and the actions of a and b; Fig. (b) shows the static triangle, QPN, whose area is also lettered A. Taking the forces in *clockwise* order round the pin, *i. e.*, W_1, a, b, the triangle shows that a acts *up against* the *pin*, and is therefore a strut; and that b *acts to the left, from the pin,* being therefore a tie.

2. The next pin is B acted upon by four forces: a already found to be a strut acting *down* on B; a known load, W_2, and the bars c and d. Taking the known forces in clockwise order, and using the stress line, NP, for a as drawn, we add $W_2 = PS$; then a line, ST, parallel to C, and get back to our starting point, N, at the upper end of a, by a line TN parallel to d. We thus have for the pin B the static polygon: $a - W_2 - c - d$, or $NPST - N$. The polygon should be clearly identified

no matter what other lines are in the neighborhood. It is seen that c is a *strut*, and d is a *tie*.

3. We next consider the pin C, acted upon by two known, and two unknown forces. Starting at Q, we trace $b-d$, and close back to Q by $e-f$. The reader must clearly identify the polygon $b-d-e-f$, or $QNTR-Q$, thinking of the *directions* of the actions as he goes around: e is *down*, and f is *up*. All the bars acting on the pin C are ties.

Fig. 170



4. Five forces act on D, of which three are known. Beginning with e at the point R, we retrace $e-c$, lay down W_3, draw g and h, and we have the polygon D. Remember that every static polygon must close.

5. In drawing polygon for E, we must begin with W_4, since the *known forces must be taken first in clockwise order*; and as f must *follow* W_4, the latter must be laid off *above* W_1; hence we have $W_4-f-h-i-j$, the last two being determined by required directions, and the necessity of returning to the point S.

6. The polygon for F is readily drawn, beginning and ending with the point T.

7. The four-sided polygon for G: $j-l-m-n$, being a quadrilateral with an internal intersection, is readily seen in Fig. (b).

8. The reader should re-draw the figure from the *frame* alone, identify every polygon, and note the nature of the stress in every bar. If he puts an additional load at G, he will find that the stress lines for p and j will not overlap.

169. A careful study of the above exposition will bring to the reader's notice several important relations between Fig. 170 and Fig. 170*b*.

1. We followed a *clockwise* order in considering the forces acting on a pin; as for example, the pin I; we had four known forces; so we took them thus: $o-m-k-W_6-q-r$: right-handed; in identifying the *polygon*, we took the stresses in the same order, but we went about the *area* I in a left-handed direction.

2. Not only does every pin or point of intersection in the frame have an *area* in Fig. (*b*), but every terminal meeting *point* in (*b*) has an *area* in Fig. 170. For example, take the point X, in (*b*) where the stress lines of l, m, k meet; in Fig. 170 the *bars* l, m, k surround the area X.

3. Again, we see that at the point Q, in Fig. (*b*), the force lines of W_4, b, f, and W_1 meet; now, turning to Fig. 170 we see that the *bars* f, b, and the indefinitely extended load lines, W_1 and W_4, bound, or shut off, the indefinite area represented by Q in Fig. 170. Similarly, the area Y, Fig. 170, corresponds to the meeting of the boundary lines W_5, k and W_6.

4. We note that the bar k, in Fig. 170, is the line separating the areas X and Y (like a division fence); it is evident that the bar k might be read: "the bar XY" (like the Mo.-Kansas line); and the stress in that bar as XY in Fig. *b*.

5. It will be seen that the external forces, *i. e.*, the loads W_4, W_1, W_2, ... W_6, all fall in the same vertical line in (*b*), and in regular order as one passes around the frame externally in a clockwise manner. The line of loads is called, "The Load Line."

6. If all the areas, within and without the given loaded frame, were suitably lettered, any pin could be identified by naming the areas around it; thus the pin A could be read: "the pin QPN," which is the way we read the static triangle of the forces at that pin.

The last two suggestions will be adopted hereafter in lettering frames and stress diagrams.

170. Method of lettering. A single example will show its advantages.

Fig. 171 shows a second example of an irregular cantilever frame with a definite load at each pin. All areas are lettered both within and without, and the order of procedure at every pin is indicated by a curved arrow. The stress diagram is shown in Fig. *b*. The

reader will readily see in Fig. 171 the bars CD and AH; and in (b) the magnitude of the stress in CD and again in AH. If he wishes to know the nature of the *stress* in CD, he must think of its action on *one* of the *pins*. If he thinks of the *upper* pin, he will read that pin (that is the areas around it) as $G-C-D-E-F$; now he will read the stress polygon also as $G-C-D-E-F$, and note that he read $C-D$ *downwards*; hence the action *on the upper pin is down*; so CD is a *tie*. Had he thought of, and read, the lower pin, he would have (mentally) said $D-C$, which reads *up* on the stress; so it pulls at the lower end as well. In the same way he finds that ED is a *strut*.

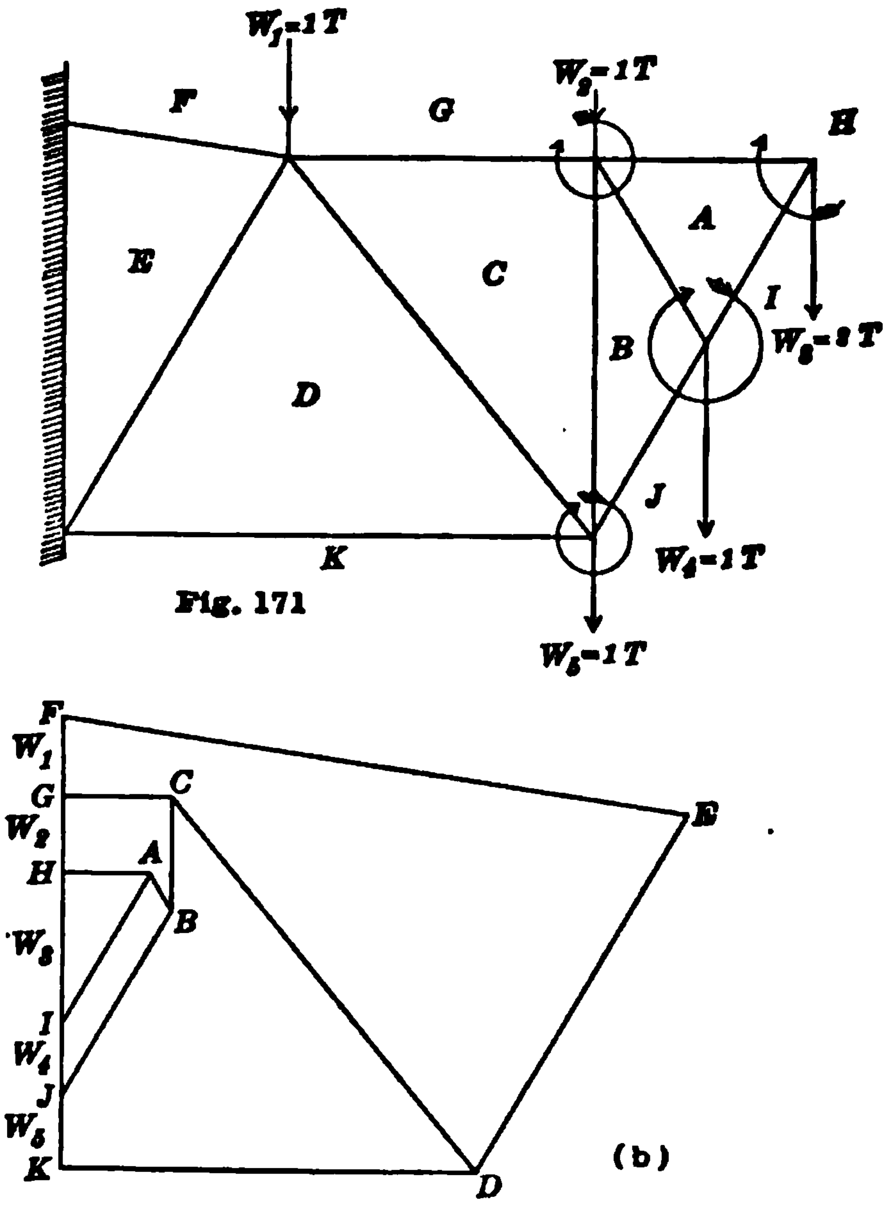


Fig. 171

171. Cantilever truss. Draw a stress diagram and scale off the stresses for the bars shown in Fig. 172.

172. Roof Trusses.

Ex. 1. *Plain Roof Truss,* with equal loads on all upper pins. Fig. 173. By symmetry the supports are equal. Directions: Draw first the Load Line, lay off the loads and supports to scale and letter them from K round to X. Next draw the static triangles, XAR and XKJ. Next, the static polygons $QRAB$ and $BAXC$ on the right, and $JKLI$ and $XJIH$ on the left. Finally, draw for the remaining upper pins. The symmetry of the figure will be an aid to accuracy. The many sided polygon for the central lower pin will be found already drawn.

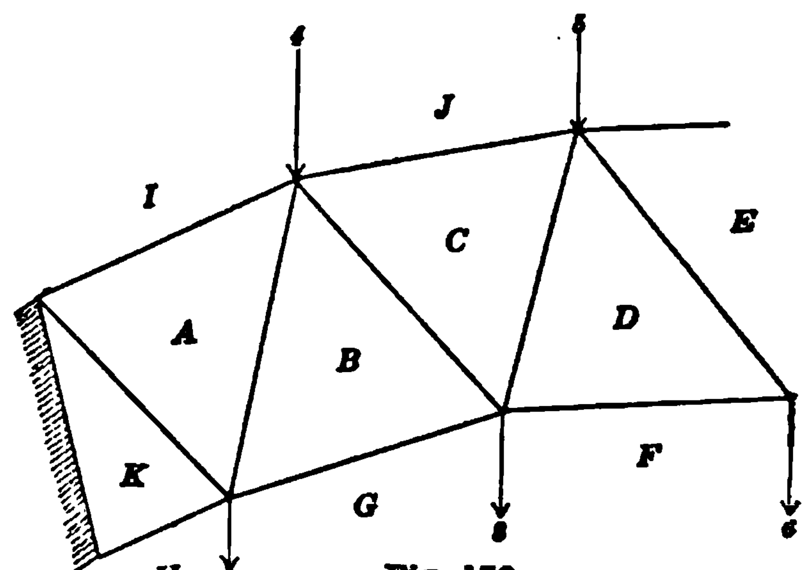


Fig. 172

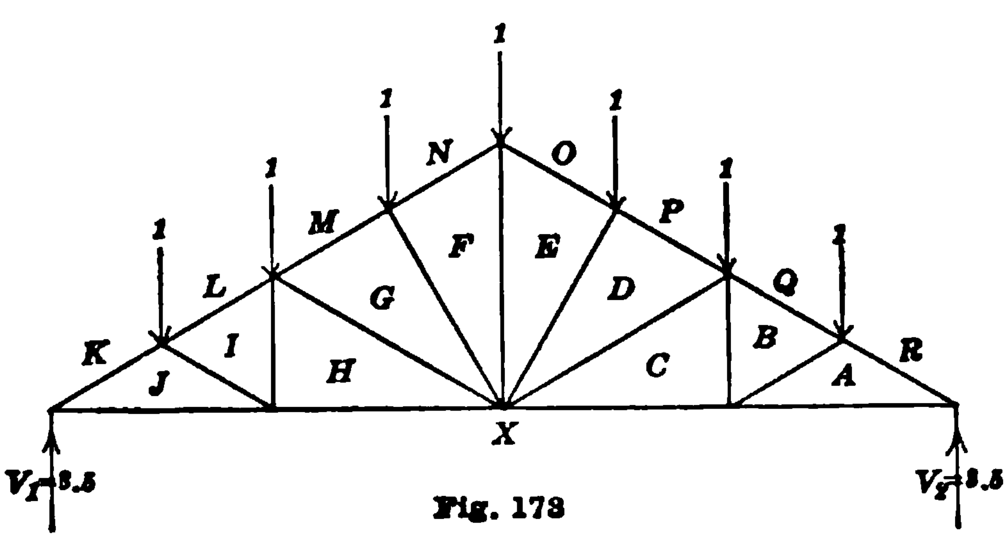


Fig. 173

The drawing will be finished when the character of every member is shown by the letter *s* (strut) or *t* (tie).

Ex. 2. *A Hip Roof.* Fig. 174. The points A and D in (b) coincide. Point out the polygon for the pin near E in the figure, 174 (b).

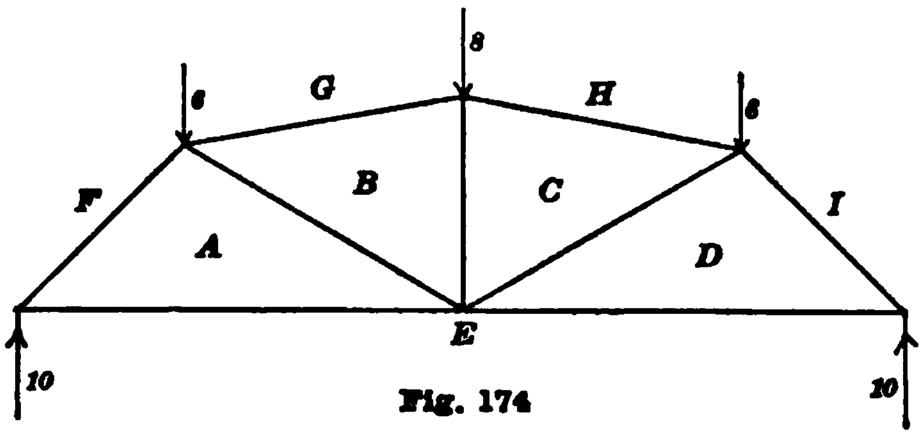


Fig. 174

Ex. 3. Invent a roof truss having only triangular areas in the frame, and place on it symmetrical loads. Letter and draw the stress-diagram, and scale it.

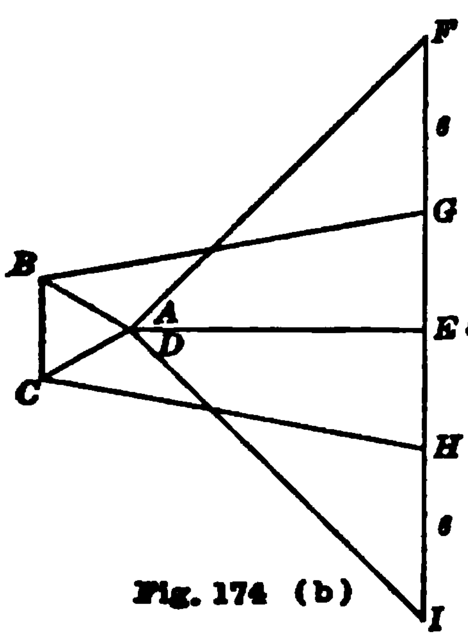


Fig. 174 (b)

173. Unstable frames. Before giving more problems, a word must be said about unstable frames.

1. Suppose a frame with pin joints is hinged at X and Z and loaded as shown in Fig. 175. [The reader of what follows *must draw the stress-diagram as he reads*, or he will not appreciate the thought.] To find the stresses on the assumption of stability, we lay off the loads W_1, W_2 and W_3; then begin with the pin CDA. We can next go to either the upper pin or the lower as in each case there are only two unknowns. Going to the upper, we say: $CABE-C$ and draw AB and BE. Now, going to the lower pin, we say: $BADF-B$; and the only unknown is FB; but, if it be drawn in the direction given, the *polygon will not close*; hence the frame is not *statical*, and *cannot support a load.*

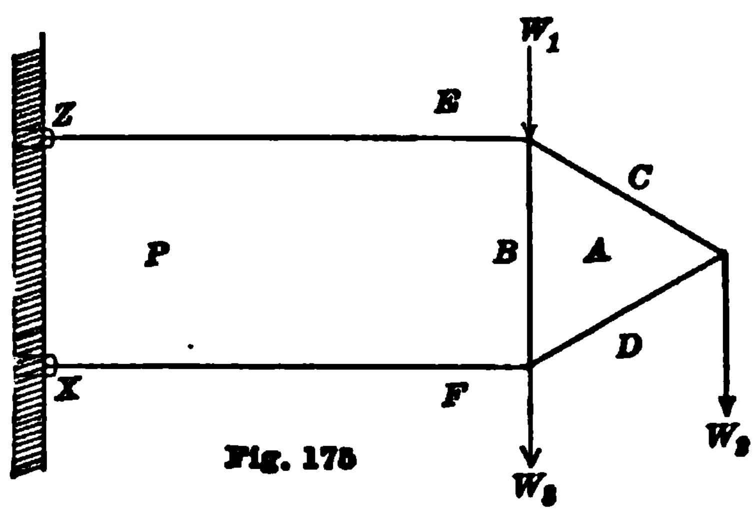


Fig. 175

If a diagonal bar from Z is put in, it will divide the area, and the polygon will close with a stress line BP, the length of which stress line shows *how much* a diagonal *tie* was needed. Equally well (and sometimes better) the other diagonal could be put in as a *strut.* It is thus seen that a four-sided area in a frame is *unstable* with pin joints. Even when the joints are nailed, screwed, glued, bolted or pinioned, they lack the stability which a diagonal strut or tie readily gives. A diagonal wire with its ends well secured is better than much nailing and glueing. Notice the weakness of screen frames, tables and benches when lateral forces act, and the sagging of doors and gates without diagonals when heavily loaded.

2. A rectangular frame without a diagonal is made rigid only by a large expenditure of material and labor on the joints. The economy and safety gained by the use of struts or metallic ties are worthy of more consideration by carpenters and cabinet-makers. Note the stiffness of aeroplanes. A four-sided frame often may profitably have two diagonal ties. A tie rod, wire or bar is a very simple affair, requiring little material and little workmanship as compared with a strut.

174. Queries to be answered by drawings.

1. If only a diagonal tie is put in, would the tension in it be greater or less than the compression in the strut, had the other diagonal been used?

2. Suppose the diagonals are of different lengths, the area B not being rectangular? Answer as above.

3. Suppose *both* diagonals are put in, which would take the stress? This is rather "indeterminate"; generally the tie or the one with the closest fittings. This query is answered later under **Deformations,** if the fittings are perfect and independent, and all bars are elastic, with known dimensions, and the strut does not buckle.

4. It was seen a few lines above that the diagonal inclined *downward towards* the support was a strut, while that inclined *upward towards* the support was a tie. When a vertical frame is subject to horizontal forces (in its plane) which may act either way, two diagonal ties suffice for both contingencies, and are vastly simpler than a single bar built to act as either tie or a strut as may be required.

Fig. 176 shows a rectangular screen frame, exposed to the action of horizontal forces tending to rack the joints.

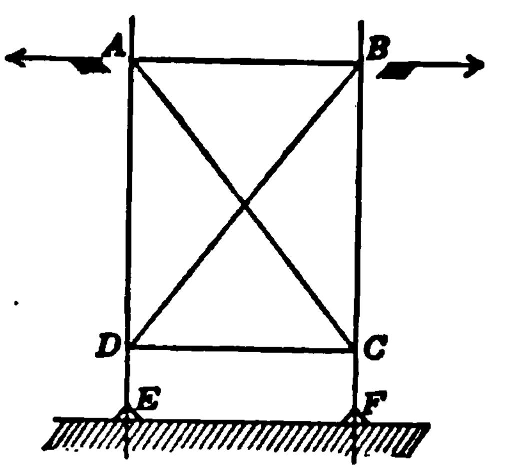


Fig. 176

When the force at B is acting, the tie, BD, supports the frame; when the force at A is acting, AC is in tension. In either case, if the feet are blocked, the action upon the parts DE and CF causes them to bend, and develop stresses which will be discussed later on.

175. Roof trusses under symmetrical loads.

1. By means of "purlins" (girders which cross from rafter to rafter), the loads on large roofs are brought to the trusses at the joints, or at supported points which *might* be joints. *All loaded points,* therefore, may be supposed to be joints with pins.

2. Draw the stress diagram for the loaded roof shown in Fig. 177, and mark the character of each member. Though the rafters may be continuous from eaves to ridge, they may be supposed to have

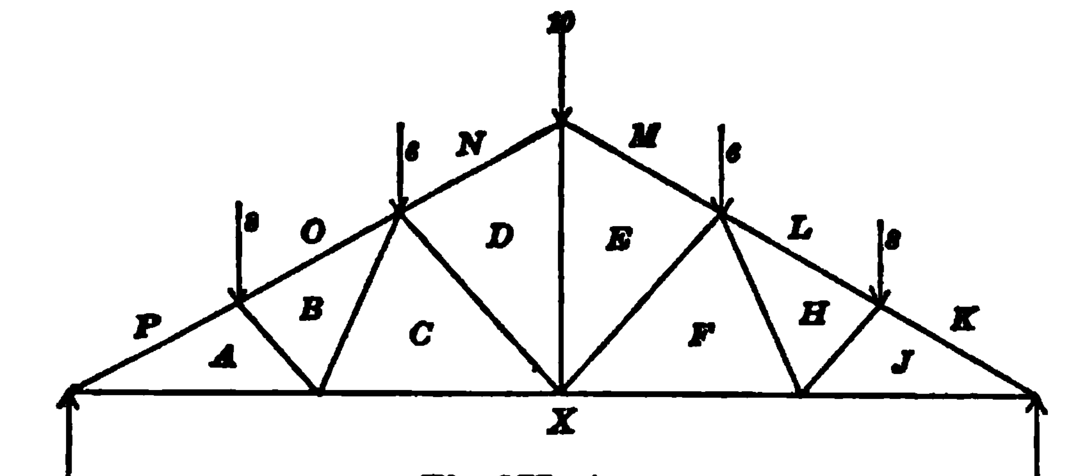


Fig. 177 (a)

pins where the loads are applied. The stress in JK is found to be greater than in EM.

3. Figures 178 and 179 are possible designs of a roof truss, which the reader may load, letter and draw.

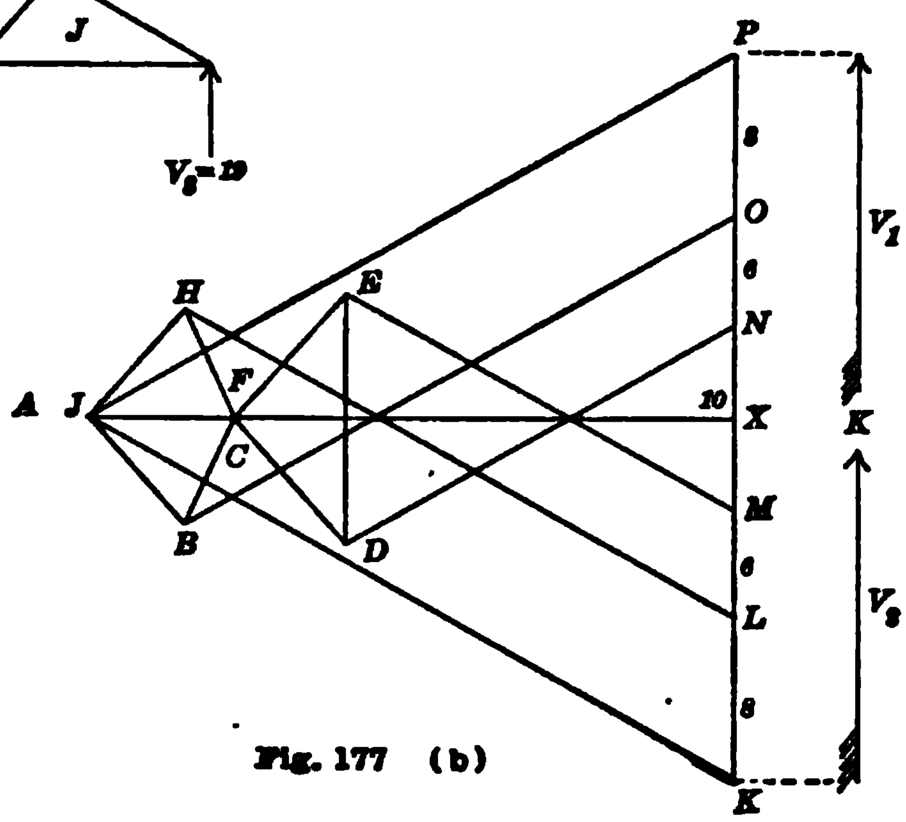


Fig. 177 (b)

176. Loads both above and below. A roof truss is often required to carry loads suspended from the lower pins. In that case the closed polygon of loads and supports (external forces) is a vertical line, some parts of which are used four times.

For example, take the simple truss, Fig. 180, in which the lower ties

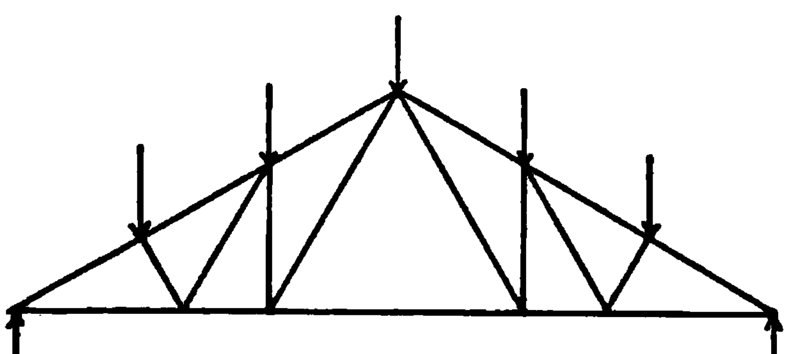


Fig. 178

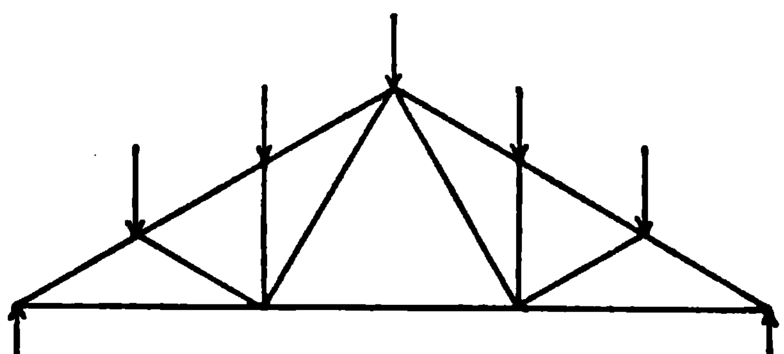


Fig. 179

(called the lower chord) are arranged to give more head-room below as shown in the sketch. The "load line" (the external forces taken in order around the truss, beginning at F) should be drawn first. The succession of forces is shown by lines not quite vertical on the right of the proper load line. Fig. 180b, on next page.

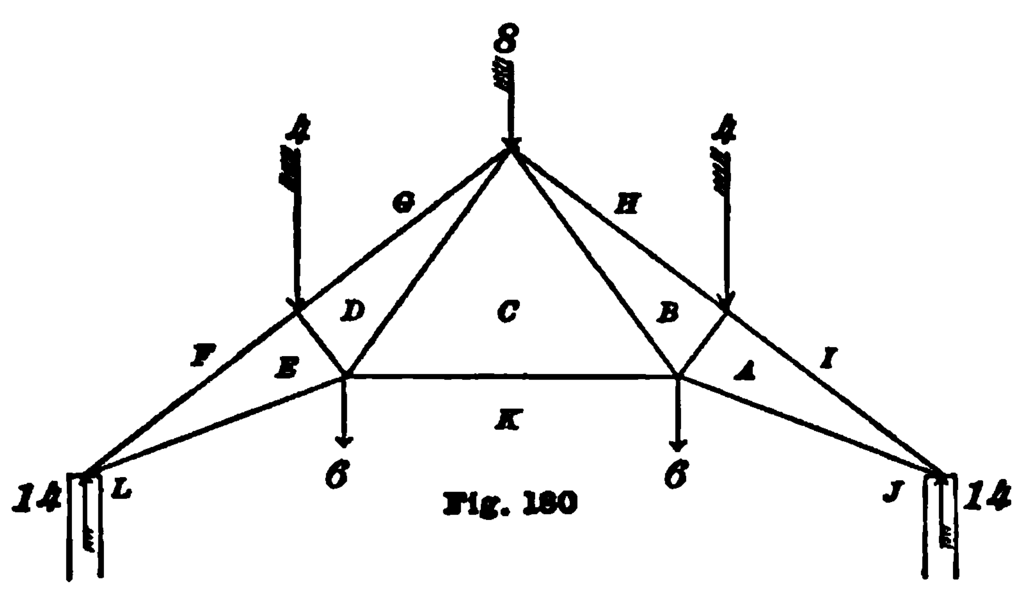


Fig. 180

177. A Fink truss. It often happens that a frame, or truss, has its long struts trussed (to prevent bending or sagging) by what may be called secondary trusses, thereby creating what to a novice seems an impossible case; this is well illustrated in what is

called the "Fink Truss." Fig. 181. It is seen that the rafters, after having been bisected, are again sub-divided by small secondary trusses, tho they are not wholly independent. For the sake of simplicity, uniform loads are given. There is no difficulty with three pins at either end. On going to a fourth pin, one finds both roads blocked, each pin having three unknowns. The difficulty may be overcome in different ways, but the simplest will be for the student to calculate the stress in XG by the method of moments. The whole trussed rafter to the right support is loaded with $30T$ (centered at the middle pin), and is held in equilibrium by the *support*, by XG, and by an action at the upper end, Z. Taking the upper end, Z, as an axis of moments, we have the moment equation, calling the span s,

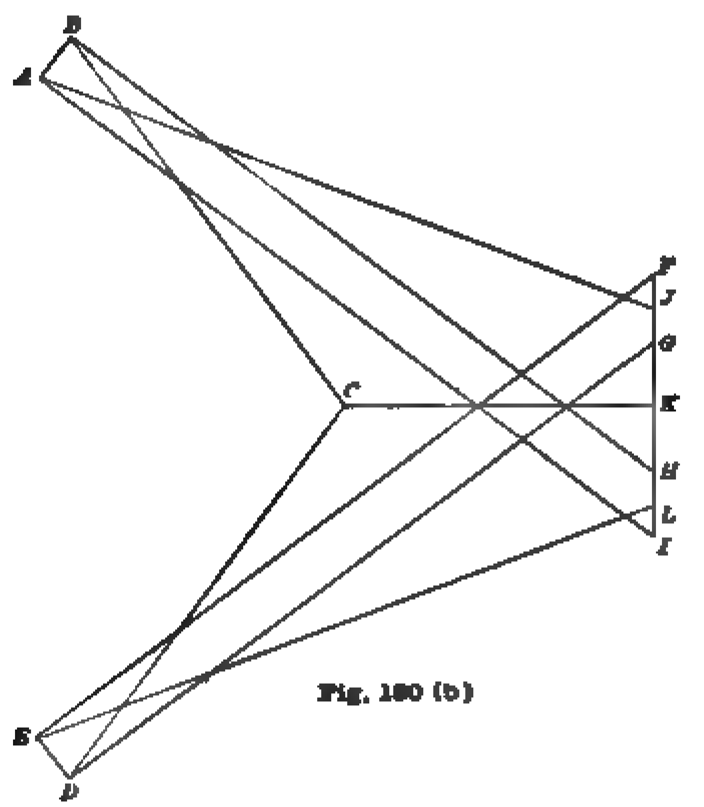


Fig. 180 (b)

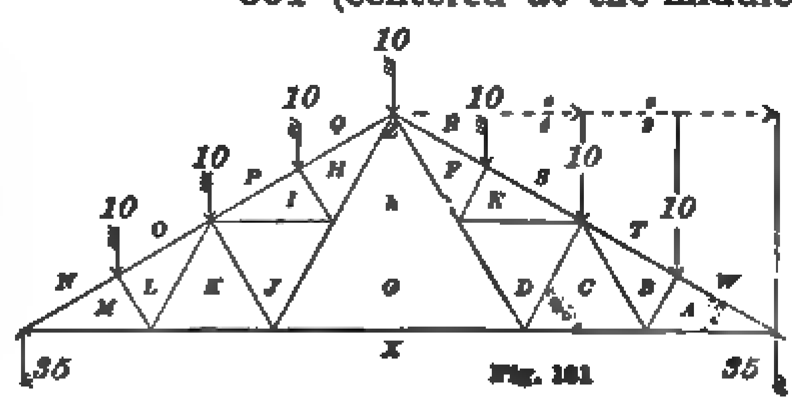


Fig. 181

$$M_z = \frac{30s}{4} - \frac{35s}{2} + XG.h = 0.$$

If the height $h = \dfrac{s}{2\sqrt{3}}$,

we find that

$$XG = 20\sqrt{3} = 34.64,$$

which is laid off to scale from X, Fig. 181(b) determining the point G on the diagram. Now take the pin $DCXG$. The rest of the problem is easy.

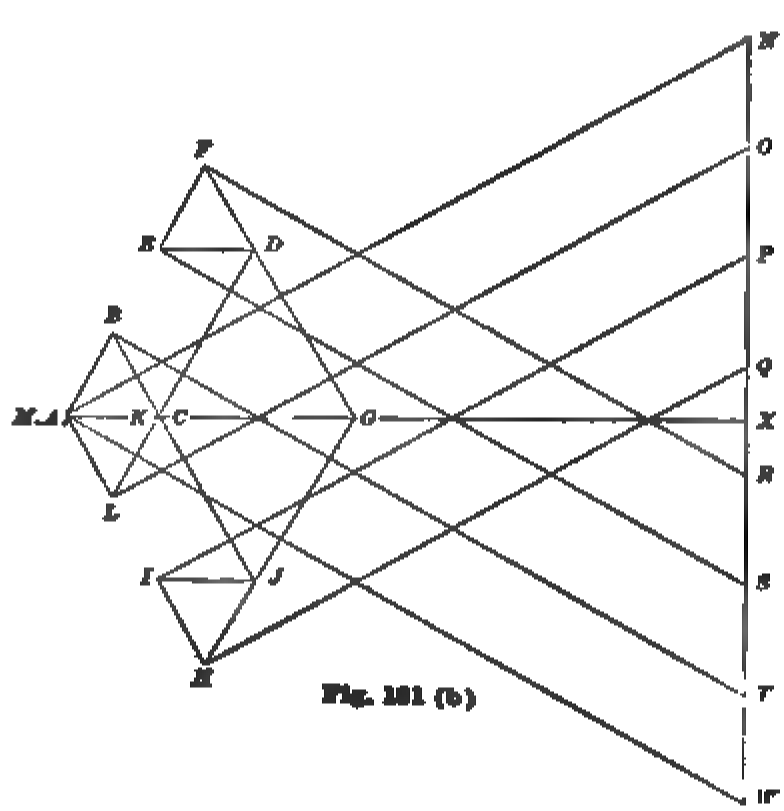


Fig. 181 (b)

178. Unsymmetrical loads. 1. Vertical loads

directly over supports do not affect the stresses in the members of a truss, hence they do not appear in the diagram. Pin loads are rarely equal, and it is found that when the interior members (the "bracing") are most severely stressed, the loads are not symmetrical. In such cases the supports are unequal and should be calculated (most readily by the method of moments), as already illustrated in Chapters III and V. See Fig. 182.

To find the support at one end, take a moment axis at the other end. In getting moments, use any convenient unit of length; the absolute length of a truss is of no importance.*

2. Ex. Let Z be the axis for finding V_2. The upper joints divide the span into eight equal parts; the lower joints into three equal parts.

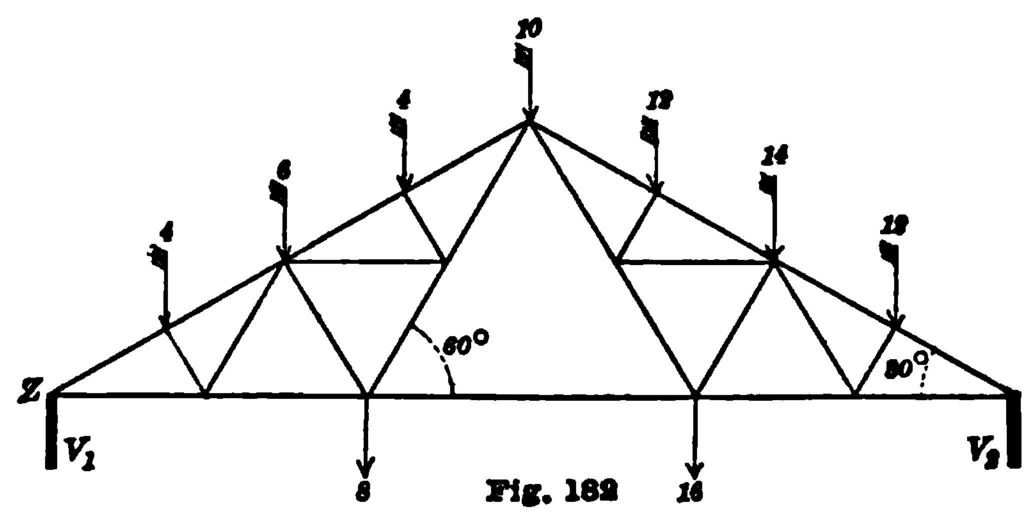


Fig. 182

Hence, let the span be 24 units; then we have for finding V_2

$$24V_2 = 12+36+36+120+180+252+252+256+64$$
$$V_2 = 50\tfrac{1}{3}, \text{ and consequently } V_1 = 35\tfrac{2}{3}.$$

The stress diagram will be found to be unsymmetrical.

3. There are other ways of finding V_1 and V_2, chiefly by drawing, but none are more simple or more readily used; even the above solution may be made more simple (and the student should simplify every mathematical and mechanical process) by noting that $12=4+8$; $14=6+8$, and $16=8+8$, so that the loading is equivalent to a symmetrical load which gives $27T$ on each wall, a load of $24T$ on the center pin of the right-hand rafter, and a load of $8T$ on the right-hand lower pin. The right-hand support, V_2, carries $\frac{3}{4}$ of the $24T$, and $\frac{2}{3}$ of the $8T$. So that the whole support on the right is

$$V_2 = 27+18+5\tfrac{1}{3} = 50\tfrac{1}{3}$$

all of which is seen without putting pencil to paper. (The student should go over this paragraph till every step is clear.)

179. Inclined loads, wind pressures. When load lines are not vertical, the supporting forces, one or both, are not vertical, and the external forces acting on a truss form an *open* static polygon. Some-

---

* Given the concentrated loads, the mere size of a truss has nothing to do with the direct stresses in its members, tho it has much to do with the way in which the members shall be given sufficient strength and stiffness.

times the stresses due to vertical loads and those due to inclined loads (the wind) are found separately and added. The following illustration takes them all together.

1. Steel trusses are generally anchored firmly to the windward wall (on the side which is exposed to the highest winds), while the free ends are either resting on rollers, or placed in grooves or shoes which admit of an easy horizontal motion. The end is left free so that the truss may expand or contract as the temperature rises or falls, or lengthen and shorten again when loads are put on and taken off, without moving the walls. Accordingly the supporting force at the free end is vertical, or nearly so.

It is usually assumed that wind pressure is normal to the surface of action, and that it is uniformly distributed. Neither assumption is very near the truth.* The wind pressure per square foot is a very uncertain quantity. It depends chiefly on the velocity of the wind and the slope of the roof; and somewhat on the extent of surface. Its action is concentrated at the joints as is the "dead" load.

2. *To find the stresses in the members of the roof truss shown in Fig.* 183, *which carries dead loads, and wind pressures.* The support, V_2, is found by getting the moment of the wind pressure about an axis at Z, the fixed end, and adding the support due to the symmetrical dead load. [The drawings are on the opposite page.]

The support due to the dead load is 12 tons.

The total wind pressure is 36, and its "center" is at the middle of the rafter. If r be the length of the rafter, the wind's moment is $18r$. Let the inclination of rafter be $30° = \beta$.

Then the support, V_2' due to the wind, at the wall on the left, has a moment arm, $2r \cos\beta$; hence we have the equality of moments

$$2V_2'r \cos\beta = 18r$$

$$V_2' = \frac{9}{\cos\beta} = 6\sqrt{3} = 10.39$$

and $$V_2 = 12 + V_2' = 22.39$$

Now, the static polygon for external forces can be drawn, beginning with XJ, which is 22.39. Since the polygon *must close*, the last line, SX, shows the magnitude and direction of the support V_1. The builder must see that this support is furnished. The effect of the wind is plainly shown in the stresses of rafters and bracing on the windward side, and the chords XA and XC.

* See a paper on Wind Pressure by Prof. F. E. Nipher, Proceedings of St. Louis Academy of Science.

180. Supports modified by wind pressures. The student who has read Chapter V will recall that the resultant of vertical and inclined loads could have been found graphically by means of the *Chain Polygon*, and that the components of the force balancing it, *i.e.*, V_1 and V_2, could also have been found by the method there explained. However it is doubtful if there is a simpler solution than that just given, but the student should not assume that there is no other method.

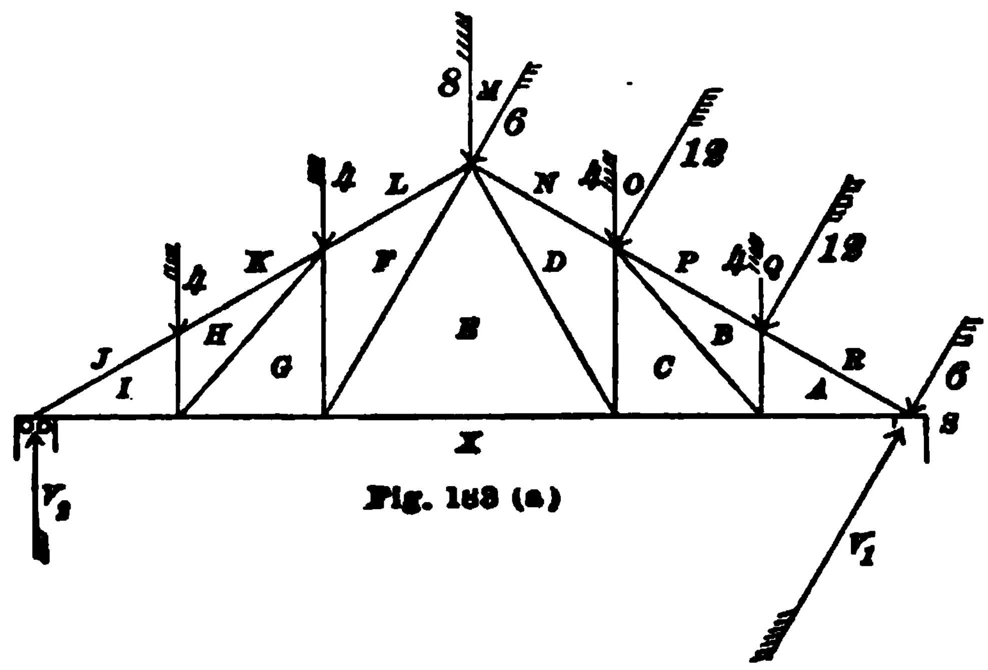


Fig. 183 (a)

Ex. Let the student add wind pressures to the load on the truss shown in Fig. 177, assuming that the windward support is anchored, while the other rests on rollers.

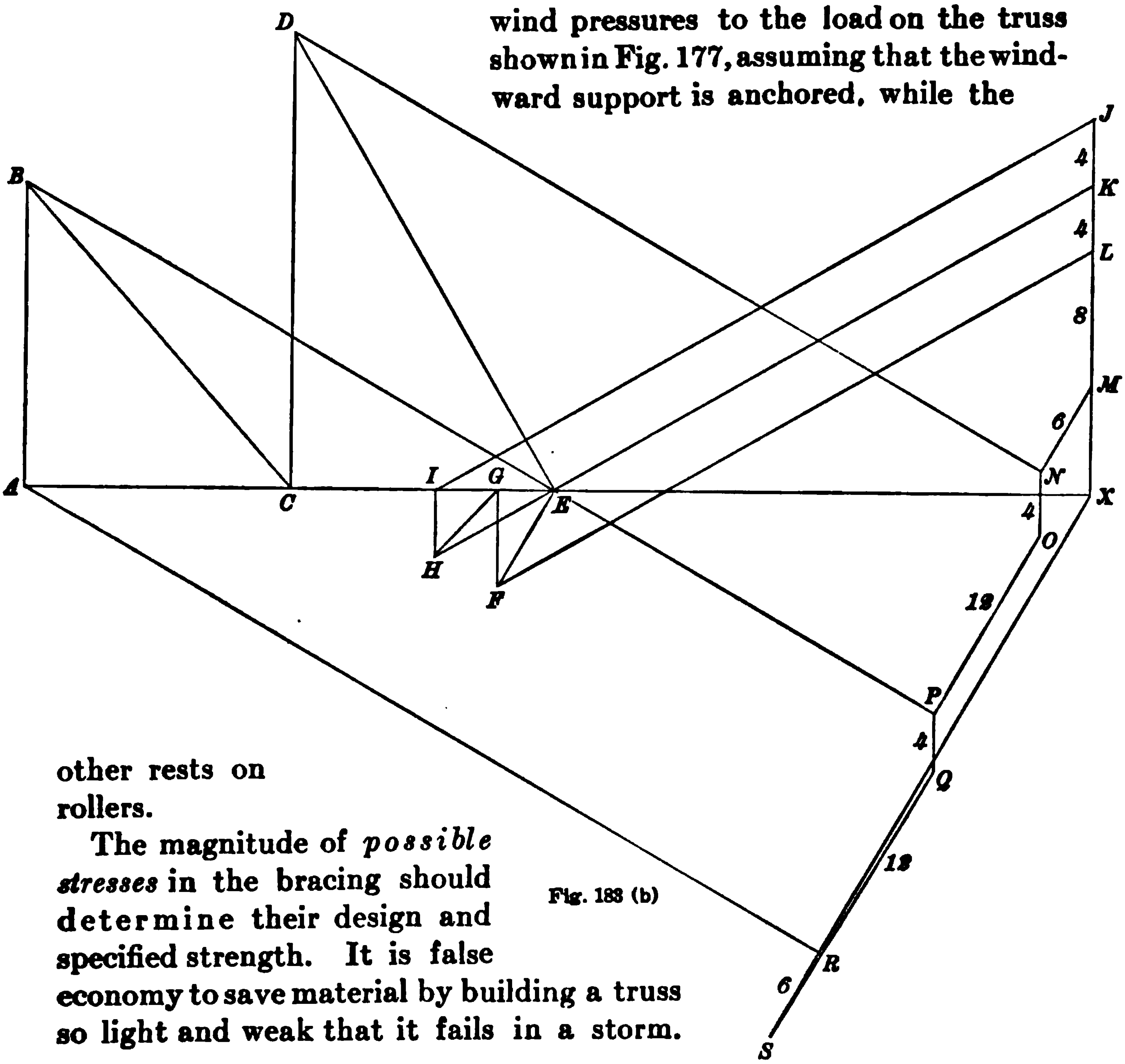


Fig. 183 (b)

The magnitude of *possible stresses* in the bracing should determine their design and specified strength. It is false economy to save material by building a truss so light and weak that it fails in a storm.

Bridge Trusses.

181. The two trusses of a bridge are connected by cross-girders which usually rest upon the pin joints of the trusses. These girders carry stringers, and the stringers carry the roadway; hence the loads are applied to the trusses at the joints. The weight of a member itself is divided equally between its ends. Bridges may have "thru" or "trellis" spans for moving loads to pass between the trusses, or "deck" spans when the roadway and its immediate supports are above the trusses. The stresses in members are readily found graphically, the results may be checked by other methods.

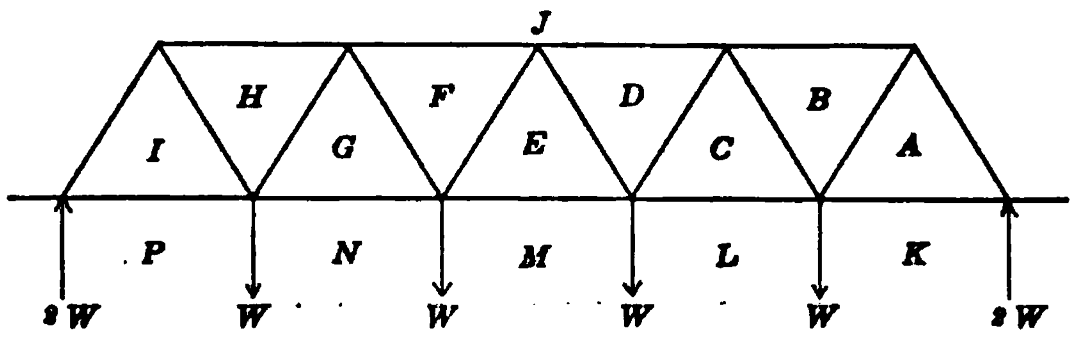


Fig. 184

182. A "Warren Girder," or triangular truss. Fig. 184. The triangles are equilateral, and only loads on the lower pins are considered. The load line is, in order,

$$+JK-KL-LM-MN-NP+PJ.$$

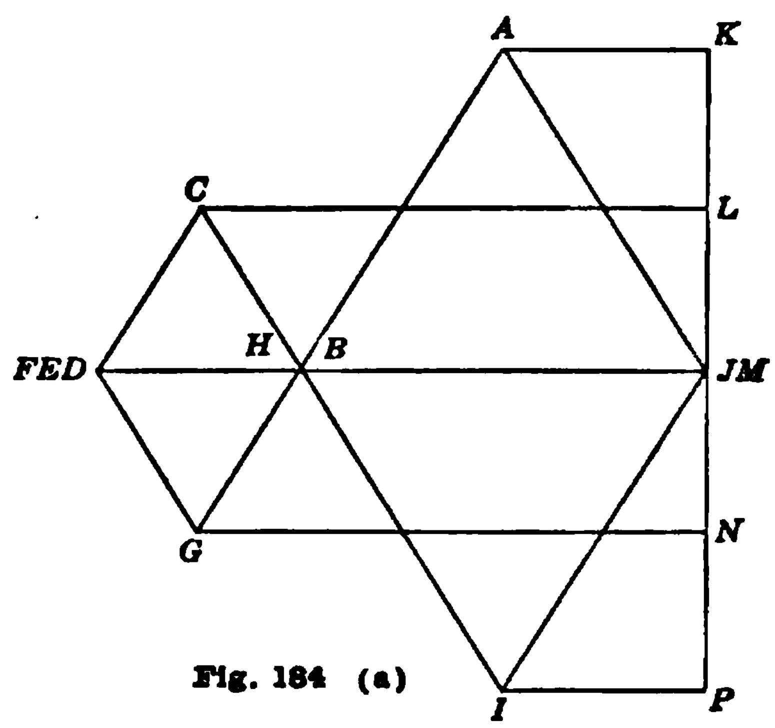


Fig. 184 (a)

The fact that the letters D, E and F fall together shows that, under a symmertical load, the bars DE and EF have no stress. This is thus explained: the stress in FE (tension), caused by the load MN, is exactly cancelled by the compression caused by the equal load LM. Were the loads NM and ML unequal, both FE and ED would be under stress.

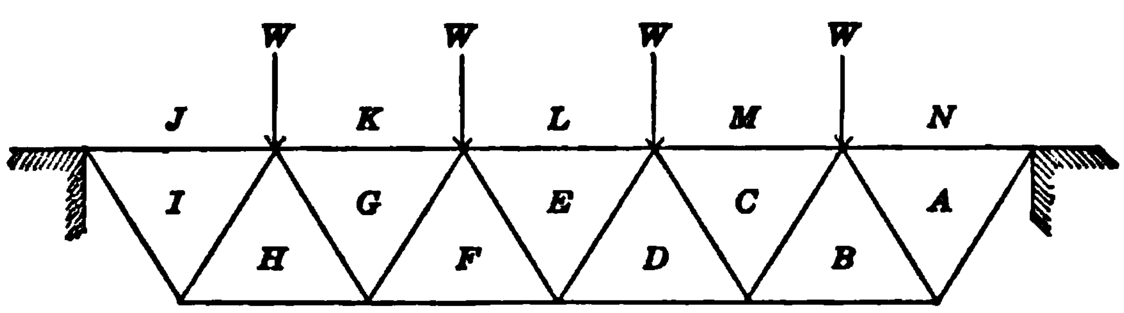


Fig. 185

2. **A deck span.** The supports are each $2W$, and the load line begins $+PJ$. Fig. 185. With this ideal loading the bars DE and EF are idle, but such would not be the case with irregular loading. Had the bridge been one panel longer, there would have been no idle member. Make a drawing to confirm this.

3. Draw a stress diagram for a thru span, and one for a deck span, with eight or ten panels under uniform loads. Mark struts and ties, and take note of laws of increase and decrease of stress in members.

183. Combination trusses. When panel lengths are too great for roadway stringers, two Warren girders are sometimes combined, as shown below. One rests on masonry, as in Fig. 184, and the other hangs on piers of steel or masonry as in Figure 185. Such a combination is known as a **Lattice Girder.**

2. Draw a diagram for the suspended truss with its *six* loads, and a separate diagram for the other with its *five* loads. Fig. 186.

Scale off all the stresses for the different members, and *add the stresses in the chords* wherever they have been used in both trusses. Finally write the stresses found on the *members themselves*, as represented in the drawing.

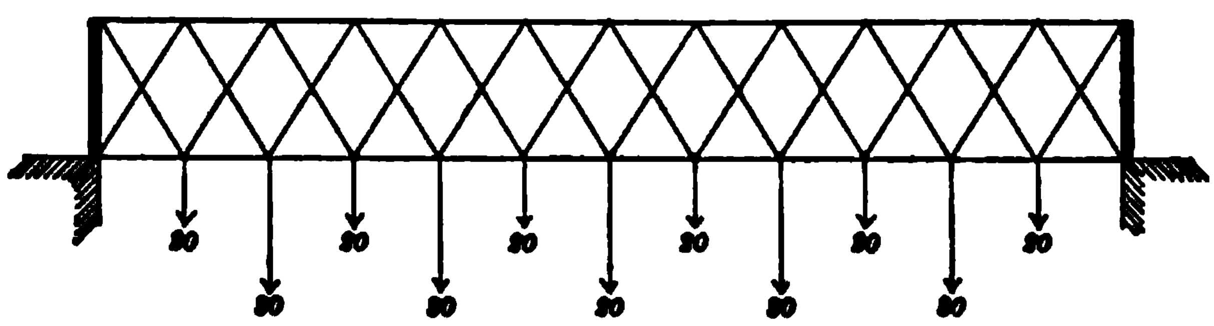


Fig. 186

It will be seen that the end panels of both chords are used but once. Let the component girders be drawn on good paper with an actual span of 12 inches.

3. Lattice girders are often met with which have three or four component trusses. In such cases some of the end members are inclined at special angles. See Fig. 187.

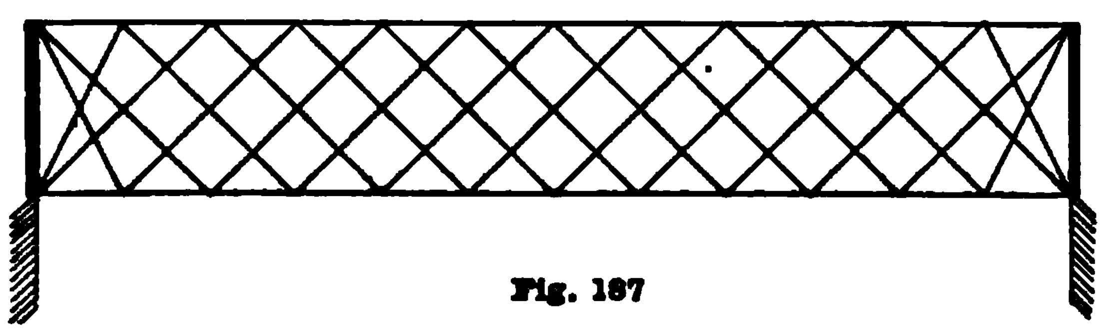


Fig. 187

184. Two typical bridge trusses are shown in their simplest form in order that the student may see their characteristic features, and by personal experience may learn the beauty and simplicity of the graphical analysis when one has good drawing instruments at hand.

1. **The Pratt Truss,** shown in Fig. 188.

2. The "Double Pratt" has short vertical struts and long diagonal ties. The two trusses have the same chords and end struts, but separate diagonals. The load on the vertical rod, near the end, is divided, usually, half and half. See Fig. 189.

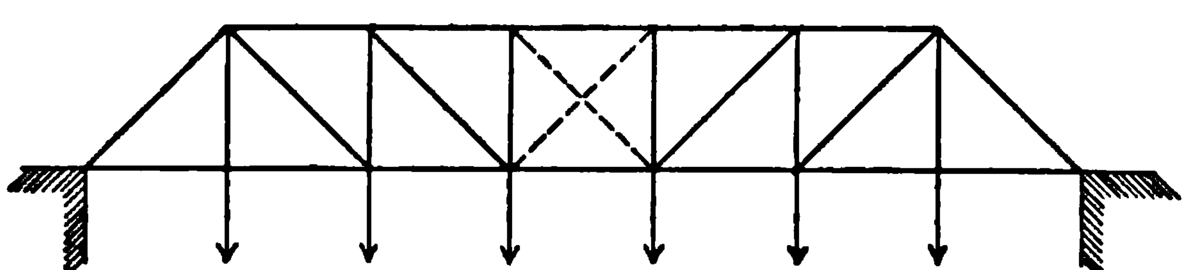


Fig. 188

Draw the diagrams separately, measure stresses and where the members are superposed, add the stresses.

185. A Howe truss. If a load is perfectly symmetrical (and it rarely is), the dotted members, which are both ties, will not be needed; with a moving load both are needed, but not both at once. The student will note the long diagonal *struts* in the "Howe," and the long diagonal *ties* in the "Pratt." The "Howe" went out of use when steel replaced wood in bridge construction. Fig. 190.

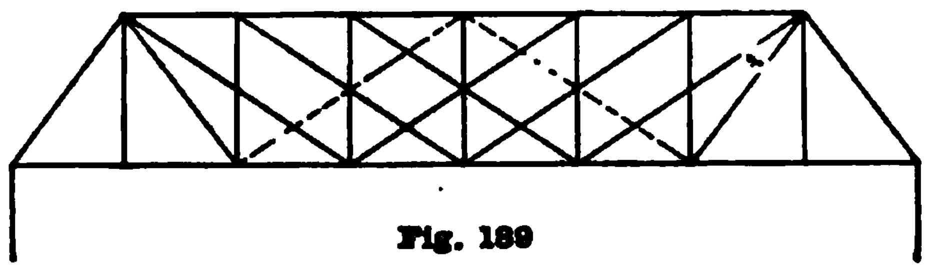


Fig. 189
A DOUBLE PRATT

For a full discussion of "Framed structures" which the above few pages are intended to lead up to, the student is referred to Building Construction and Bridge Engineering.

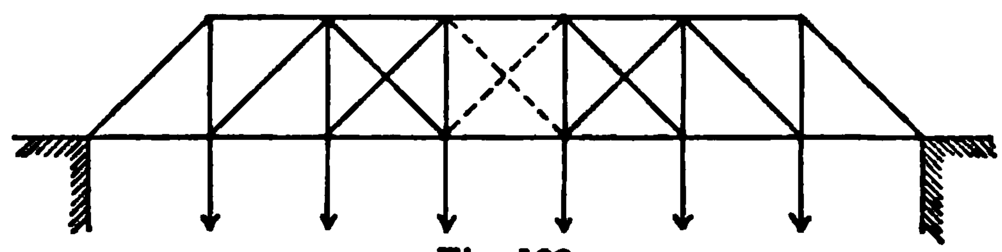


Fig. 190
A HOWE TRUSS

Calculations base on stress diagrams. In all the problems of the present chapter, the magnitude of stresses in frame members was determined graphically, and the degree of accuracy depended upon the quality of the instruments and the skill of the draftsman in laying off and measuring lines and angles. But it is easily seen that the diagrams, even when made "free-hand," lend themselves to trigonometric calculation. An examination of any one of the diagrams will suffice to discover how an unknown length can be computed, as all angles are known.

1. For example, let the student calculate the stresses in several members of the Fink truss, Fig. 181, and also of the Pratt Truss, Fig. 188, and compare results with his measurements of his stress-diagrams. He is quite likely to find defects in his instruments and faulty drawing.

2. Let the student make a stress diagram of the cantilever truss, Fig. 42.

186. The endless variety of designs. There is no limit to the number of frames for roofs and bridges. "King Post" and "Queen Post" roof trusses offer no difficulty, but no unusual design should be built without a careful drawing which shows both the *nature* and the *magnitude* of the stress in each piece. The writer once saw a roof truss for a church actually built with a tie (an eye-bar), where there should have been a strut. The builder's "horse-sense" was inferior to a stress diagram. A drawing of thetruss follows. Fig. 191.

The reader may assume any reasonable loading, and proceed with the drawing, using one of the members, *FG*; and note the stress in it. Then make a second diagram with the other diagonal, *FG*. See which would alone be a tie, and which alone would be a strut. The reason why the truss does not fall is, that the rafter is continuous to the ridge and very strong, and its *stiffness* saves it.

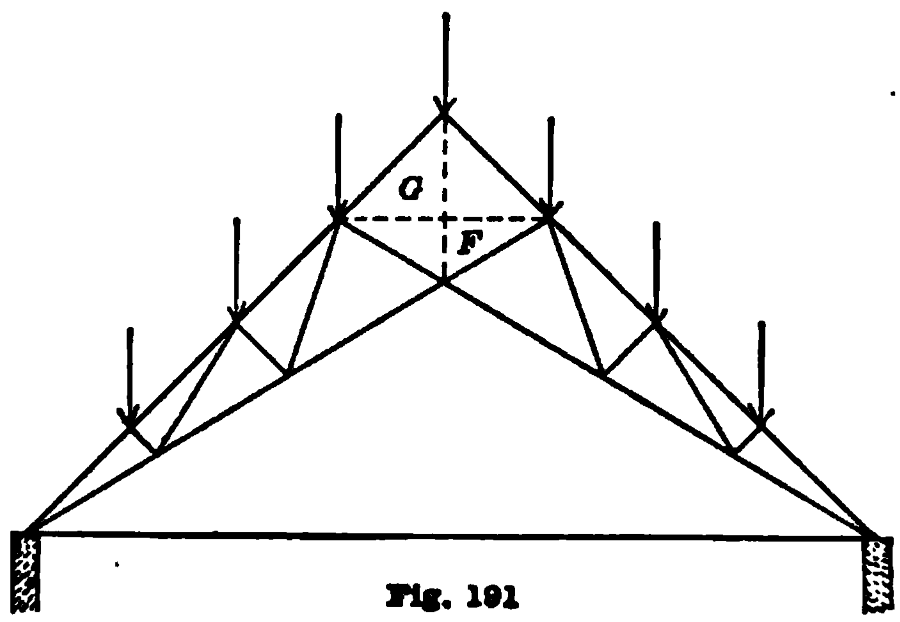


Fig. 191

CHAPTER XII.

Internal Stress.

Composition and Resolution of Stresses.

187. In Chapter IX it was shown that at every internal point of a loaded beam there was stress normal to a cross-section plane, and a shearing stress in the same plane.

The definition that "force is an action between two bodies," is to apply to two contiguous parts of a body.

Let *A*, *B* and *C* be parts of a horizontal loaded beam. Both *A* and *C* and the load Δw are acting on *B*, and holding it in equilibrium. There are actions at every point of the planes which bound *B*, and those actions vary from point to point.

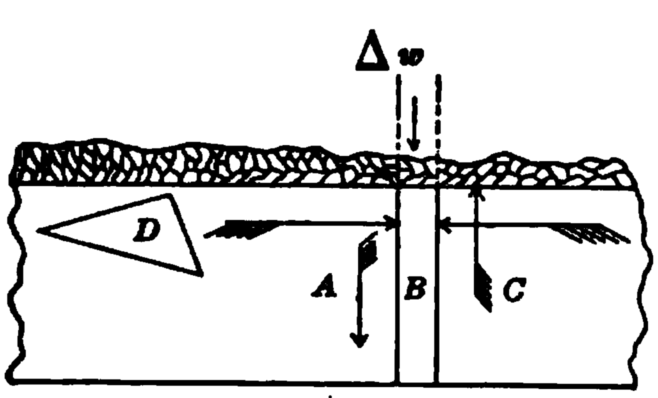


Fig. 192

But that is not all. If a prism like that at *D* be formed by three general planes cutting the same rectangular loaded beam, there is stress (actions) at every point of every lateral face. The adjacent material is pressing or pulling *D*, and is dragging or tending to drag it up or down; and yet, as *D* is at rest, we know that all such actions actually balance. The prism may be very small, infinitely small, yet have three sides or faces, and the stresses on those sides are normal, oblique and tangential. It is the purpose now to study the stresses on *every plane* that can pass thru a given point in a loaded beam, particularly planes which are perpendicular to the plane of external forces (loads and supports).

188. Simple stress and its components. Fig. 193 represents a bar in tension but under no other stress. If the *cross-sectional* area S is uniform, the "*Normal Intensity*" is

$$p_x = T/S$$

T

upon every unit of surface of a right section.

If an oblique section, AC, is made, the stress is oblique, and as the section has the area $S_1 = S \sec\theta$, we have

$$p = \frac{T}{S \sec\theta} = p_x \cos\theta$$

If now p be resolved into its normal and tangential components, we have

$$p_n = p\cos\theta = p_x \cos^2\theta$$

$$p_t = p\sin\theta = p_x \sin\theta\cos\theta$$

T

Fig. 193

If a second plane HK be taken so that its inclination to a right section is $\phi = \frac{\pi}{2} - \theta$, we shall have

$$p_n' = p_x \cos^2\phi = p_x \sin^2\theta$$

$$p_t' = p_x \sin\phi\cos\phi = p_x \cos\theta\sin\theta$$

Hence $\qquad p_n + p_n' = p_x$, and $p_t = p_t'$ if $\phi + \theta = \pi/2$.

This is particularly important, and is stated as follows: **If two oblique planes having their normals in the same axial plane are at right angles, the tangential components of the oblique stresses are equal.** A graphical representation of magnitudes of the stresses on an oblique plane is readily shown: The value of p_n is found by projecting p_x *twice* thru the angle θ. Fig. 194.

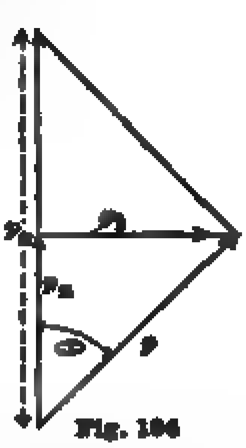


Fig. 194

189. Compound stress. More often than not, material is subjected to, or sustains, more than one simple stress. When the simple stresses are known, the stress on any given plane is readily found if all the normals lie in the same plane. This may be illustrated by *finding the stress on a third plane when the simple stresses on two planes are given.*

1. Take the case of **the shell of a steam boiler** or a cylindrical gas **tank.** Let the radius of the cylinder be r, the thickness of the shell be c, and the gauge pressure of the steam or gas per square inch be q in *excess of the external air-pressure.* The total direct longitudinal

tension is $q\pi r^2$ due to the pressures upon the ends of the cylinder, which must be sustained by the shell at every cross-section, whose area is $A = 2\pi rc$.

$$\text{Hence } p_y = \frac{q\pi r^2}{2\pi rc} = \frac{qr}{2c}$$

if the direct longitudinal tension is p_y.

2. In addition to this longitudinal tension, there is a "hoop tension" which is found by considering the tension in a hoop one inch wide around the cylinder. The resultant internal pressure on one half of the hoop is the same as the pressure would be on a *diametral* strip one inch wide; hence $2r \times q = 2T$, where T is the tension at every cross-section of the hoop. See Fig. 195.

$$\text{Hence } T = rq.$$

Now, the cross-section of the hoop is $c \times 1$, so that

$$p_x = T/c = rq/c$$

This shows that the *intensity of the hoop tension is twice* the intensity of the longitudinal tension.*

3. Unless otherwise stated as to the drawings which follow, the planes thru OX and OY are thought of as perpendicular to the plane of the paper, and intersecting in an axis OZ (not shown); and all oblique planes of action are parallel to OZ.

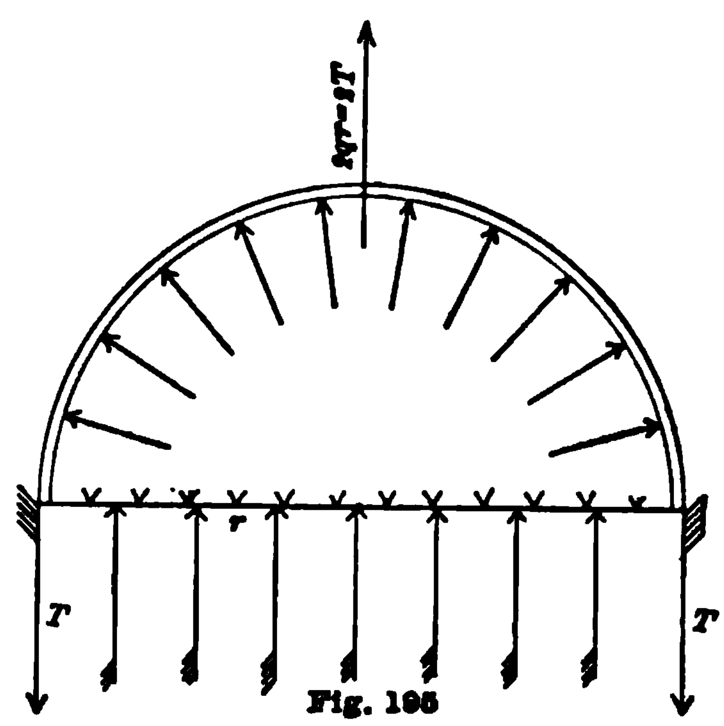


Fig. 195

4. It is now required to find the stress on the plane, or at the section AB, which makes an angle θ with OY. Fig. 196.

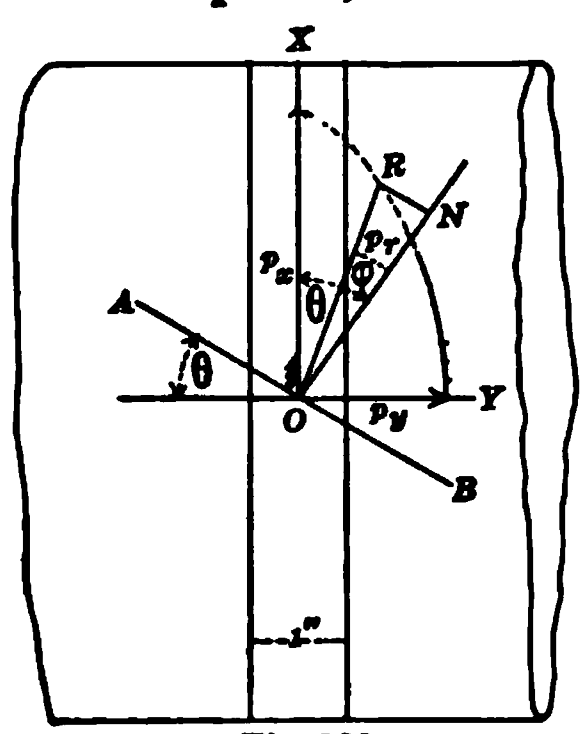


Fig. 196

Let ON be normal to the section. By the formulas just derived, resolving p_x into components on AB.

$$p_n' = p_x \cos^2\theta, \qquad p_t' = p_x \sin\theta\cos\theta;$$

Resolving p_y,

$$p_n'' = p_y \sin^2\theta, \qquad p_t'' = p_y \sin\theta\cos\theta$$

* The observing student will recall the fact that there are two rows of rivets along a longitudinal seam of a steam boiler, but only one row on a cross-sectional seam.

The magnitudes of p_n', p_n'', p_t', p_t'', etc., are found by projection in Fig. 197.

$$OX=p_x;\ OY=p_y;\ \overline{OC}=OC'=p_n';\ OD=OD'=p_n''.$$

$$ON=p_n=p_n'+p_n'';$$

$$NR=p_t'-p_t''=p_t$$

$$OR=p_r$$

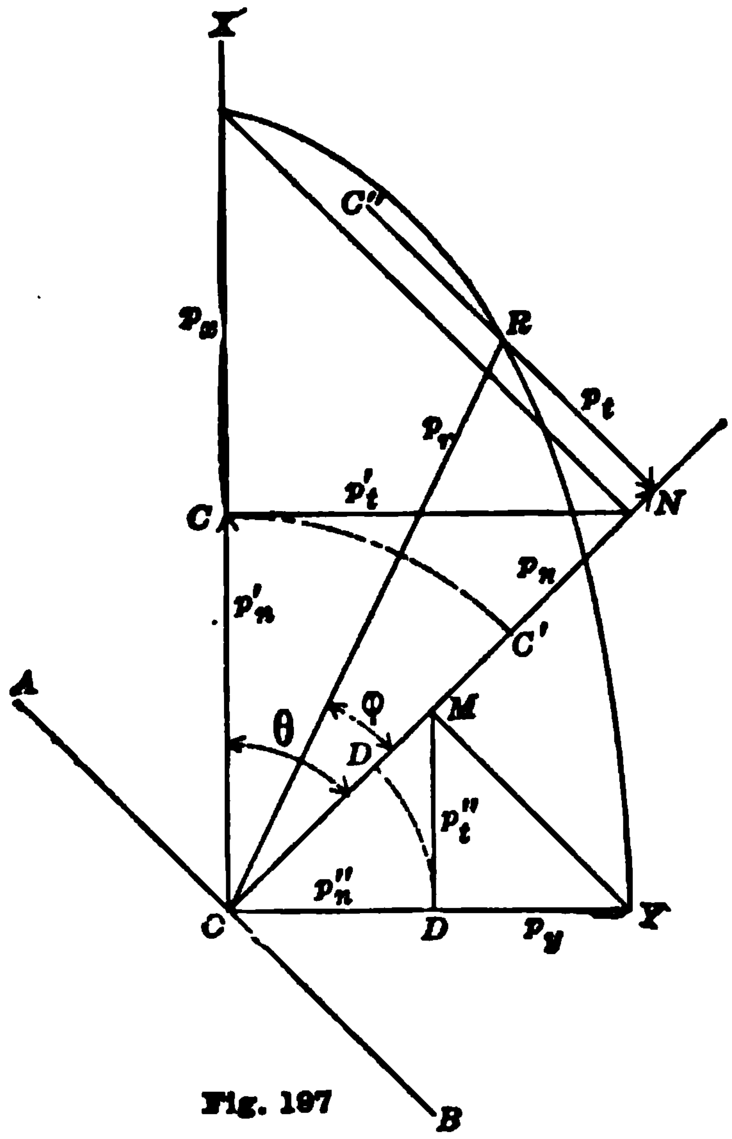


Fig. 197

From the figure it is seen that p_n' and p_n'' have the same sign, while p_t' p_t'' have opposite signs.

Hence we have the final equations

$$\left.\begin{aligned} p_n=p_n'+p_n''&=p_x\cos^2\theta+p_y\sin^2\theta \\ p_t=p_t'-p_t''&=(p_x-p_y)\sin\theta\cos\theta \end{aligned}\right\}$$

If we lay off p_n on ON, and p_t at right angles to ON on the *side* of the *larger* of p_t' and p_t'', then $p_r=OR$ **is the stress on the plane** AB **per unit of surface.** It is thus seen that the stress p_r is oblique, and that the obliquity, that is, the angle with the normal, is ϕ, and that

$$\tan\phi=\frac{(p_x-p_y)\sin\theta\cos\theta}{p_x\cos^2\theta+p_y\sin^2\theta}.$$

Substituting for p_x and p_y the values qr/c and $qr/2c$, we have

$$p_n=\frac{qr}{2c}(1+\cos^2\theta);\ p_t=\frac{qr}{2c}\sin\theta\cos\theta$$

$$\tan\phi=\frac{\sin\theta\cos\theta}{1+\cos^2\theta}$$

190. When the stresses on OX and OY are normal and known, it is much simpler to use the static triangle. Suppose we wish to find the stress on a general plane which is parallel to the inter-section of the planes OX and OY, and which makes an angle θ with OY. Fig. 198.

1. ON is normal to AB, hence $XON=\theta$. Take an elementary prism, whose faces are parallel to the planes AB, OX and OY, whose altitude is c, and whose bases have the edges $bk=dx$, $ak=dy$, $ab=ds$. It is evident that $dy=ds\cos\theta$, and $dx=ds\sin\theta$.

Now the face of the prism, $ak \times c$, has the stress p_x on every unit of surface; hence the stress on that face, which we will call $\overline{OE}$, and lay off on OX, is

$$\overline{OE} = cp_x \cos\theta\, ds$$

Similarly, the stress on the face, kb, is found to be

$$OD = cp_y \sin\theta\, ds$$

The resultant of these two stresses is

$$OR = cds\ (p_x^2 \cos^2\theta + p_y^2 \sin^2\theta)^{\frac{1}{2}}$$

If the intensity of stress on the third face is represented by p_r, the total stress on that face is $p_r cds$, but that stress must balance the resultant OR. Hence the *static triangle*, OER, and

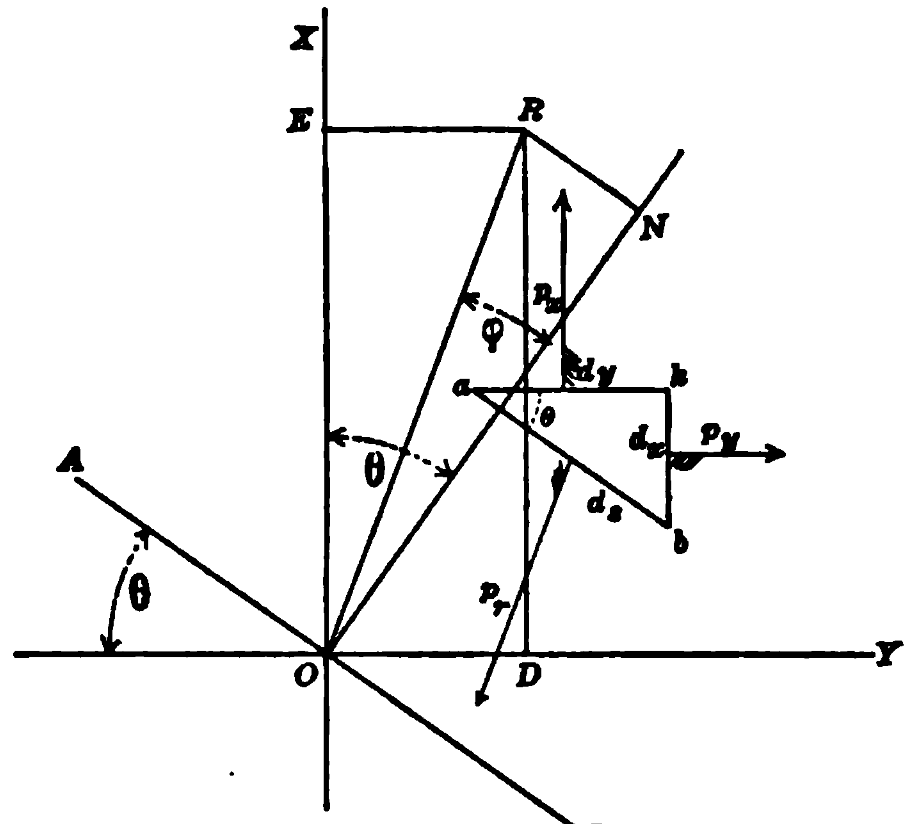


Fig. 198

$$RO = p_r cds,$$

$$p_r = (p_x^2 \cos^2\theta + p_y^2 \sin^2\theta)^{\frac{1}{2}}$$

This formula could easily have been found by reducing $p_r = (p_n^2 + p_t^2)^{\frac{1}{2}}$ in the former solution, **188**.

2. It will be noted that the height of our prism, and the hypotenuse ds have disappeared; only *unit* stresses remain. Hence, we will use only unit stresses, and in the place of $\overline{OE} = cp_x \cos\theta\, ds$, we will use $OF = p_x \cos\theta$. Fig. 199.

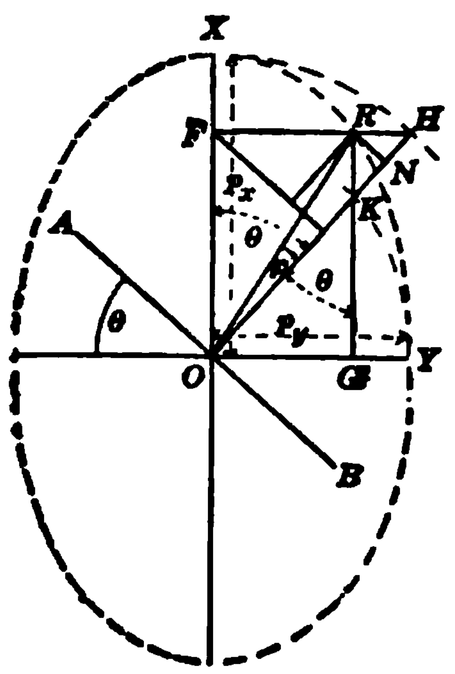


Fig. 199

If we lay off on OX $p_x \cos\theta = OF$, and on OY, $p_y \sin\theta = OG$, and draw the diagonal, **that diagonal, if drawn towards O, and read RO, will represent in *length* and *direction* the unit stress which *balances* the *stresses* p_x and p_y.** This statement is readily grasped if the student bears in mind that the stress of adjacent material on ds *must balance* the stresses on the faces dx and dy, since the prism is in equilibrium.

191. The ellipse of stress. Fig. 199. Let AB be the plane on which the stress is to be found. Draw the normal, ON, and lay off on it *both* $p_x = OH$ and $p_y = OK$, each to scale, and project them upon their respective axes, thus getting the co-ordinates of R and the magnitude of p_r to scale.

In the case of the steam boiler under internal pressure, it was found that $p_x/p_y=2$, but that relation was special; in general they may have any ratio. Whenever p_x and p_y are normal, and are the *only stresses* on their respective planes, which are perpendicular to each other, they are called the **principal stresses,** and their planes of action are called the **principal planes,** and the magnitude of p_r on any oblique plane lies between the magnitudes of p_x and p_y. From this construction it is evident that the locus of R as the plane AB changes its position, is an ellipse, whose semi-major axis is p_x, and whose semi-minor axis is p_y. If we call $OF=x$ and $OG=y$, we have $x=p_x\cos\theta$, and $y=p_y\sin\theta$; eliminating θ, we have

$$\frac{x^2}{p_x^{\,2}}+\frac{y^2}{p_y^{\,2}}=1.$$

The figure is known as the *Ellipse of Stress.*

The "*obliquity*" of the stress is shown by ϕ, and the components of p_r, ON and NR, are

$$ON=p_n,\ \text{and}\ NR=p_t$$

Their analytic values have already been found by projection;

$$ON=p_n=p_x\cos^2\theta+p_y\sin^2\theta,\ \text{and}\ NR=p_t=(p_x-p_y)\cos\theta\sin\theta.$$

and $$\tan\phi=\frac{(p_x-p_y)\sin\theta\cos\theta}{p_x\cos^2\theta+p_y\sin^2\theta}$$

192. Caution. Unless great care is exercised, the young student will overlook an important matter in thinking about finding the component stresses on an oblique plane, when p_x and p_y are given as the stresses on two rectangular planes.

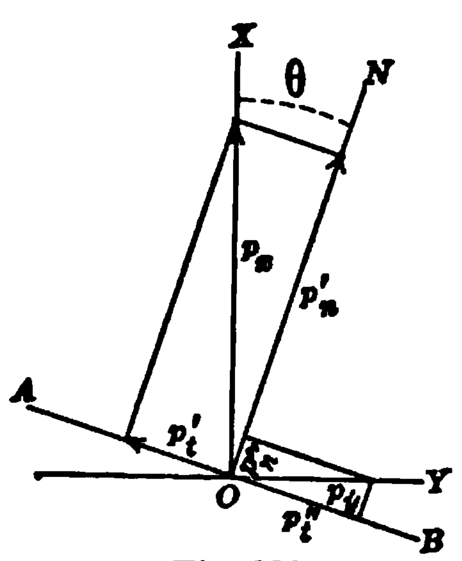


Fig. 200

The temptation is to resolve p_x into p_n' and p_t', and p_y into p_n'' and p_t'' as shown in Fig. 200, which is **all wrong.**

It is *not* the unit stress, p_x, on the plane OY, which is to be resolved into its components; it is *the unit stress on the plane AB which is to be resolved.* That stress is not p_x, it is $p_x\cos\theta$, and *that* is **resolved** into $p_x\cos^2\theta=p_n'$, and $p_x\cos\theta\sin\theta=p_t'$.

193. 1. The practical value of a problem like that of a boiler shell lies in the fact that it shows at a glance that when for any reason an oblique section in the shell is made, and the adjacent plates are connected by butt-straps, bolts or stays **the direction of the bolts or**

stays should ***not*** **be normal to the section,** but should be parallel to the direction of stress, that is, making the angle ϕ with the normal. See Fig. 201.

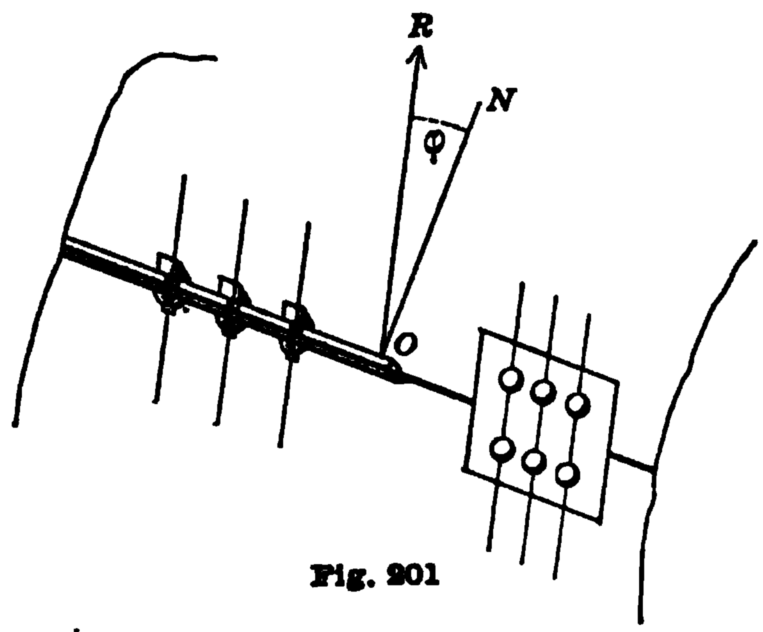


Fig. 201

194. Given stresses of different kinds. In the above example of the shell of a steam boiler or tank of compressed gas, both the longitudinal and hoop stresses were tensile. In the shell of a *steel water tower* which carries a comparatively heavy structure on its top, the vertical stress is compressive and the horizontal stress is tensile.

1. Let R in inches be the mean radius of the bottom ring of the tower; let the ring have a width of one inch, and a thickness of c inches. If W be the entire weight of the empty tower, the intensity of the vertical stress, which we will call p_x, in pounds per square inch is

$$p_x = \frac{W}{2\pi Rc}.$$

Let h be the height in feet of the column of water when the tower is full. The hydrostatic pressure per square foot (Chapter IV.) is wh, and the pressure per square inch is

$$q = \frac{wh}{144},$$

in which w is the weight of a cubic foot of water.

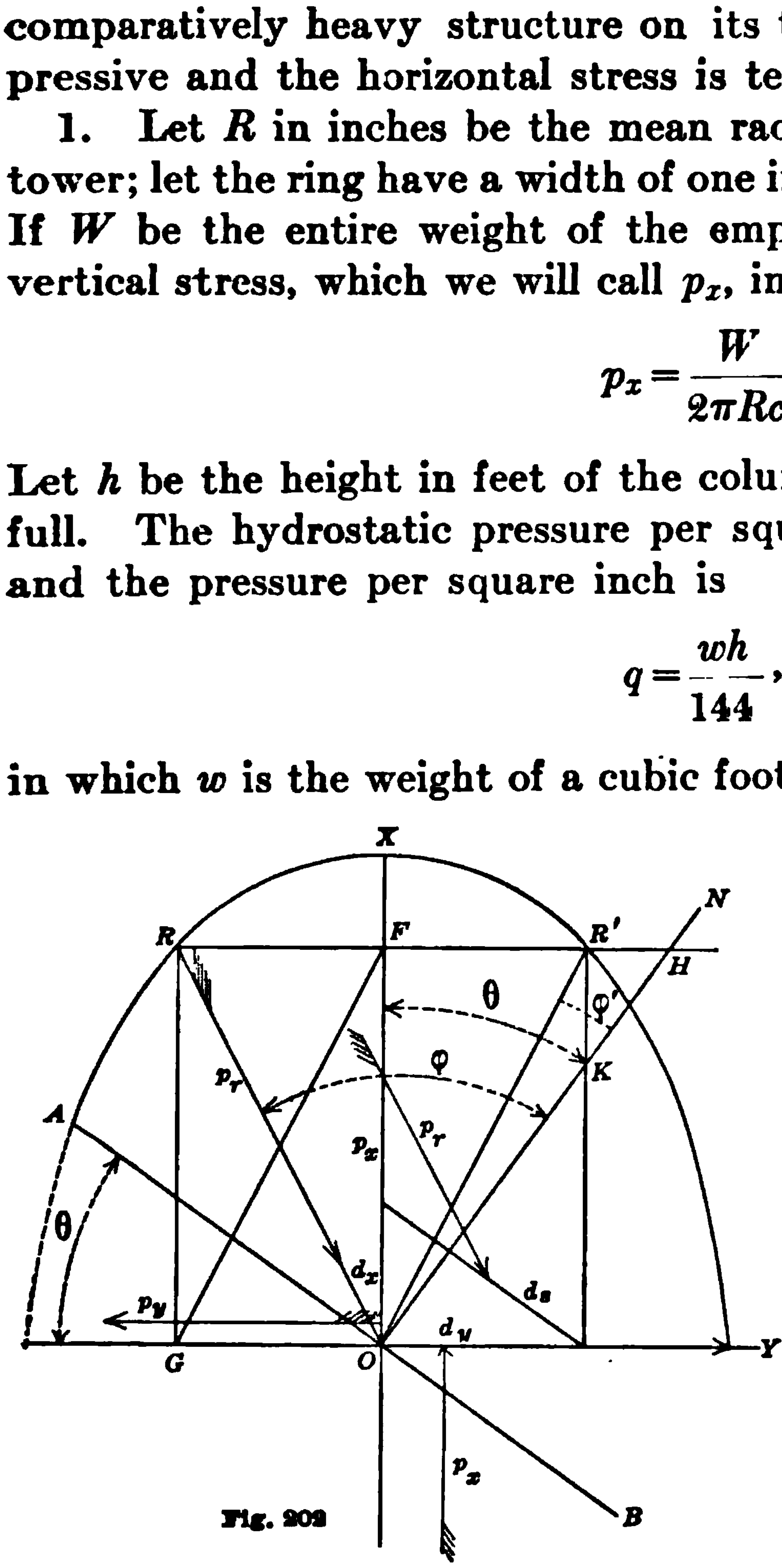


Fig. 202

Consequently, the tension in the ring is

$$T = qR = \frac{whR}{144}$$

and the *intensity* of the tension per square inch is

$$p_y = \frac{T}{c} = \frac{whR}{144c}.$$

2. It will be assumed, for the sake of better comparison of this problem with the last, that p_x is numerically greater than p_y. The elementary triangle can be, as shown in Fig. 202, with faces dy, dx and ds.

Take the vertical compressive stress as positive acting up against dy; and the horizontal stress as negative pulling on dx towards the left. The action on ds must balance. We lay off $OF = p_x \cos\theta$, as before, and $OG = p_y \sin\theta$, since p_y is negative. The resultant unit stress on AB has the same magnitude it had when p_y was positive, but its direction is different, and the "obliquity" is the large angle $NOR = \phi = 2\theta - \phi'$, R being a symmetrical point on the other side of OX, and ϕ' what the obliquity *would have been* had p_y been positive. The student must not fail to see clearly that the stress on the plane AB or ds, and *every parallel plane*, is in the *direction RO*. The point R is still on the circumference of the ellipse, whose semi-axes are p_x and p_y.

195. Equal principal stresses of the same kind. 1. If $p_x = p_y$, the ellipse becomes a circle, and it is evident both from the formulas and from the construction that $p_t = 0$, and $p_r = p_n$. This is the case of *fluid pressure* which is known to be equal on all planes at a point, and therefore in all directions, and always normal to confining surfaces.

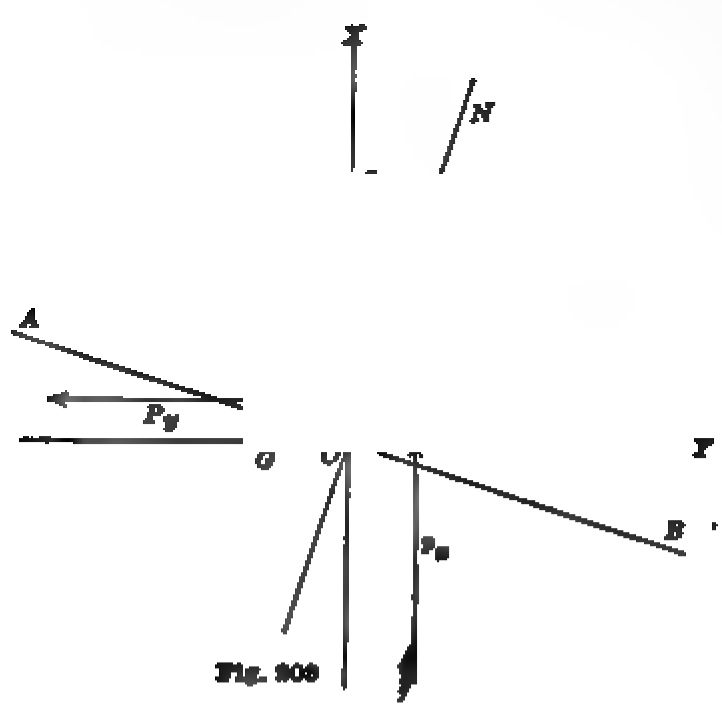


Fig. 208

2. *Equal stresses of opposite kinds.* In this case, since $p_x = -p_y$, Fig. 208,

$$OG : OF : OR = dx : dy : ds$$

inasmuch as the Δ's FOG and dx, dy, ds are similar, and FG is perpendicular to ds, and therefore parallel to ON. Hence the angle ROX = the angle θ, which establishes the important facts that, **when p_x and p_y are equal and of opposite signs, we have**

$$p_r = p_x, \text{ and } \phi = 2\theta$$

3. The two conclusions just reached are briefly stated as follows:

(*a*) If $p_x = p_y$, then $p_r = p_x$ and $\phi = 0$.

(*b*) If $p_x = -p_y$, then $p_r = p_x$ and $\phi = 2\theta$;

they led Prof. Rankine (or some one before him) to adopt them as the basis for graphical solutions of special problems of stress.

196. Stress on rectangular planes. Assume from a mass of material in a state of internal stress, an elementary rectangular prism,

whose edges are perpendicular to the direction of all stresses. Let *ABCD*, Fig. 204, be the base of such a prism in a *region of uniform stress*. The actual *size* is of no consequence, so it may be thought of as infinitely small.

Resolve the resultant stress on the parallel faces *AB* and *CD*, into a normal stress of intensity, p_n, and a tangential stress of intensity p_t, the latter parallel to the lines *AB* and *CD*. In like manner resolve the resultant stress on the parallel planes *AD* and *BC*, using R instead of p.

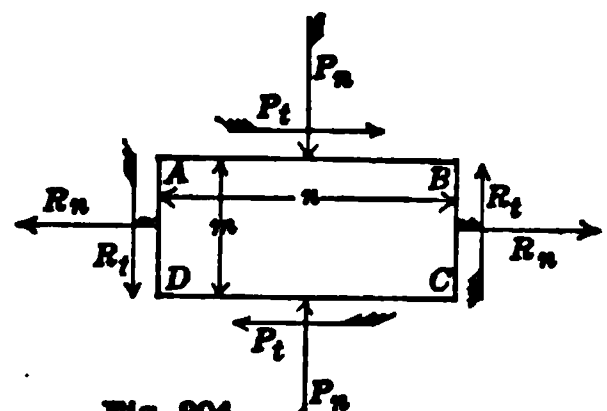


Fig. 204

The normal actions on opposite faces directly balance each other. Hence the two couples formed by the remaining stresses, which are purely tangential, must balance, since the prism is in a state of equilibrium; that is

$$mp_t \times face\ AB = nR_t \times face\ BC$$

Hence $\quad p_t = R_t$, since $m \times face\ AB = n \times face\ BC. = \mathrm{V}$ (volume).

This extends the Theorem of **188** to the case which takes in the resultant of several simple stresses, providing the two sets of parallel planes are perpendicular to each other.

Conclusion. If the stresses on two sets of parallel planes of a body be tangential to those planes, and in directions perpendicular to the intersections of those planes, those stresses must be of equal intensity.

197. Uniform state of stress. In the case of the cylindrical shell of a steam boiler, and in the narrow ring at the base of a water tower, the state of stress was uniform thruout. In the case of a disk acted upon by three tension straps in the same plane, Fig. 205, we can consider the state of stress as uniform only for the region in the vicinity of C, the area where their lines of action meet.

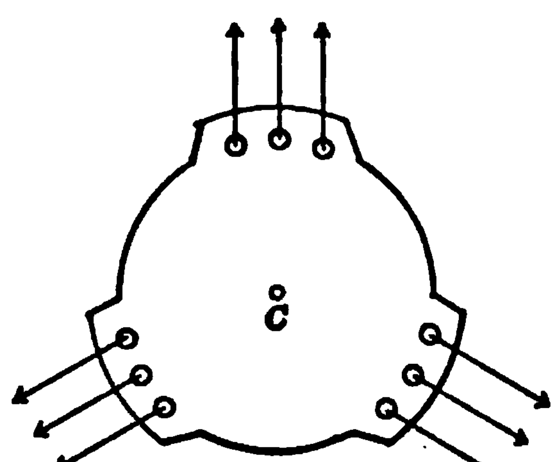


Fig. 205

It is useful to reflect upon the stresses acting upon the faces of different elements, in the region of uniform stress, when we know the stresses on two planes which are at right angles to each other, and to the plane of stresses.

For example, let Fig. 206 show enlarged areas in a region of uniform stress.

Group 1. The triangles I, II, III, IV represent right prisms with bases in the face of the same disk, with the stresses on rectangular faces drawn *consistently*.

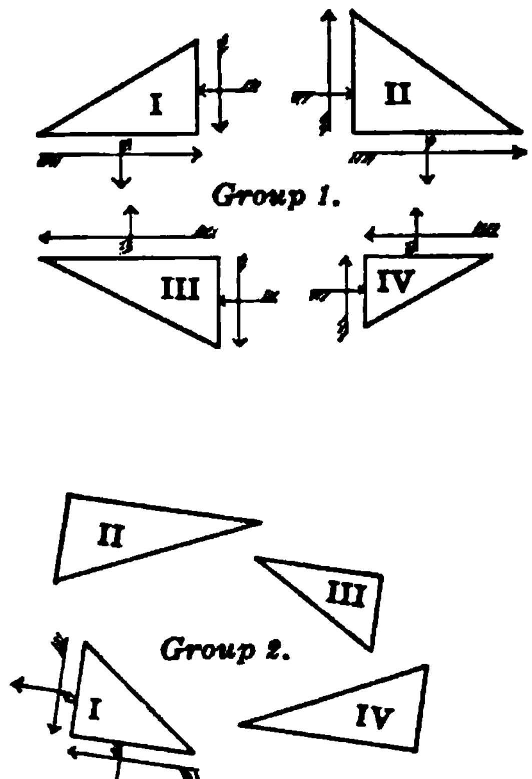


Fig. 206

Group 2. Draw the arrows for II, III, IV consistently with those for prism I of group 2.

198. General problem of stress. 1. Given the stresses on two rectangular planes, XZ and YZ, parallel to the plane XY, to find the stress on a general plane, AB, parallel to OZ. Fig. 207.

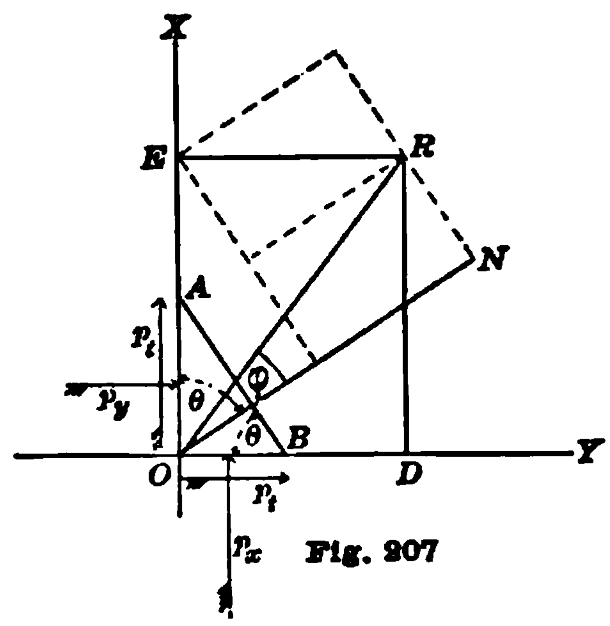


Fig. 207

Assume the triangular prism, OAB. The stresses on OA and OB are given. The prism, whose base is OAB, is in equilibrium; hence the stress on AB must balance the stresses on OA and OB. Let the altitude of the prism be unity.

Let the known unit stress on OA be resolved into its normal and tangential components, p_y and p_t. Similarly let the stress on OB be resolved into p_x and p_t. Note that p_t on $OA = p_t$ on OB.

The stress in the direction of OX is

$$p_x OB + p_t OA = AB\,(p_x \cos\theta + p_t \sin\theta) = OE. \tag{1}$$

The stress in the direction OY is

$$p_y OA + p_t OB = AB\,(p_y \sin\theta + p_t \cos\theta) = OD \tag{2}$$

The resultant OR is the total stress, which must be balanced by $p_r.AB$: that is

$$AB.p_r = AB\left[(p_x \cos\theta + p_t \sin\theta)^2 + (p_y \sin\theta + p_t \cos\theta)^2\right]^{\frac{1}{2}}$$

$$p_r = \left(p_x^2 \cos^2\theta + p_y^2 \sin^2\theta + p_t^2 + 2p_t \sin\theta\cos\theta\,(p_x + p_y)\right)^{\frac{1}{2}} \tag{3}$$

As the length AB disappears, we may consider it unity; hence, that factor will be omitted in projecting OE and ER upon ON and upon NR. From Fig. 207, the projection of OR = the sum of the projections of OE and ER: hence

$$p_n = ON = p_x \cos^2\theta + p_t \sin\theta\cos\theta + p_y \sin^2\theta + p_t \sin\theta\cos\theta$$

$$p_n = ON = p_x \cos^2\theta + p_y \sin^2\theta + 2p_t \sin\theta\cos\theta \tag{4}$$

and

$$p_t = NR = p_x \sin\theta\cos\theta + p_t \sin^2\theta - (p_y \sin\theta\cos\theta + p_t \cos^2\theta)$$

$$p_t = (p_x - p_y)\sin\theta\cos\theta + p_t(\sin^2\theta - \cos^2\theta) \tag{5}$$

The obliquity, $NOR = \phi$, is given by the value

$$\tan \phi = p_t/p_n \qquad (6)$$

PRINCIPAL STRESSES.

2. From Eq. 3, it is seen that the value of p_r varies with θ from $\sqrt{(p_x^2+p_t^2)}$ to $\sqrt{(p_y^2+p_t^2)}$ in each quadrant as the normal ON swings about O; hence there must be maximum and minimum values of p_r, in which cases $d(p_r)^2/d\theta$ must be zero. Differentiating the value of p_r^2, we get

$$d(p_r)^2/d\theta = (p_y^2 - p_x^2)\, 2 \sin\theta \cos\theta + 2p_t (\cos^2\theta - \sin^2\theta)\,(p_x + p_y)$$

Placing the above equal to zero, and solving for θ, we get

$$(p_x - p_y) \sin 2\theta = 2p_t \cos 2\theta$$

$$\tan 2\theta = \frac{2p_t}{p_x - p_y} \qquad (1)$$

If the expression for $d(p_r)^2/d\theta$, after dividing by (p_x+p_y), is compared with the value of p_t, Eq. (5), it will be seen that, **when $d(p_r^2)$ is zero, p_t is zero, and accordingly $p_r = p_n$.**

3. This leads to the general conclusion that as 2θ has two values, 180° apart, the values of θ are 90° apart, and four values satisfy the equation for p_n. If p_t is positive (as assumed in the figure) the values of 2θ are in the 1st and 3rd quadrants. Since $\sin\theta \cos\theta$ is positive, p_n is a *maximum*.

If 2θ is in the 2nd and 4th quadrants, $\sin\theta\cos\theta$ is negative, and p_n is a minimum. As $\tan 2\theta$ is always a possible quantity, it follows that θ is always a real angle, and hence it follows further, that **in every region of uniform stress there are two planes, 90° apart, on which the stress is wholly normal; and that on one the stress is the greatest, and on the other the least, of all the stresses on planes like AB passing thru the axis OZ.**

Those two planes are called the **Principal Planes,** and the Stresses on them are the **Principal Stresses.**

4. In our further discussion of internal stress, the principal planes will be taken as the co-ordinate planes, and p_x will be taken as the maximum stress. In the illustrative problems of the boiler shell and the steel water-tower, the stresses p_x and p_y were principal stresses.

199. Propositions relating to the ellipse of stress. The principal stresses at a point may be resolved into the sum and the difference

of two component stresses, as shown by the following identical equations:

$$\left.\begin{aligned} p_x &= \frac{p_x+p_y}{2} + \frac{p_x-p_y}{2} \\ p_y &= \frac{p_x+p_y}{2} - \frac{p_x-p_y}{2} \end{aligned}\right\} \qquad (1)$$

If we take the first component of each, we have two principal stresses equal and of the *same* kind. When we take the second components together $\left(+\frac{p_x-p_y}{2}; \text{ and } -\frac{p_x-p_y}{2}\right)$, we have equal principal stresses of *opposite* kinds.

1. The stress on any plane is readily constructed as follows: Using the first component in both p_x and p_y, we apply the rule for *equal stresses of the same kind,* and lay off $\frac{p_x+p_y}{2} = OM$ **on the normal.** Fig. 208. Then taking the other component we have the case of *equal stresses* of *opposite* kinds, and we lay off $MR = \frac{p_x-p_y}{2}$ making $NMR = 2\theta$; and we have $OR = p_r$, the stress required on the plane AB. One half of the complete parallelogram is shown with broken lines, since it is unnecessary to draw it in full.

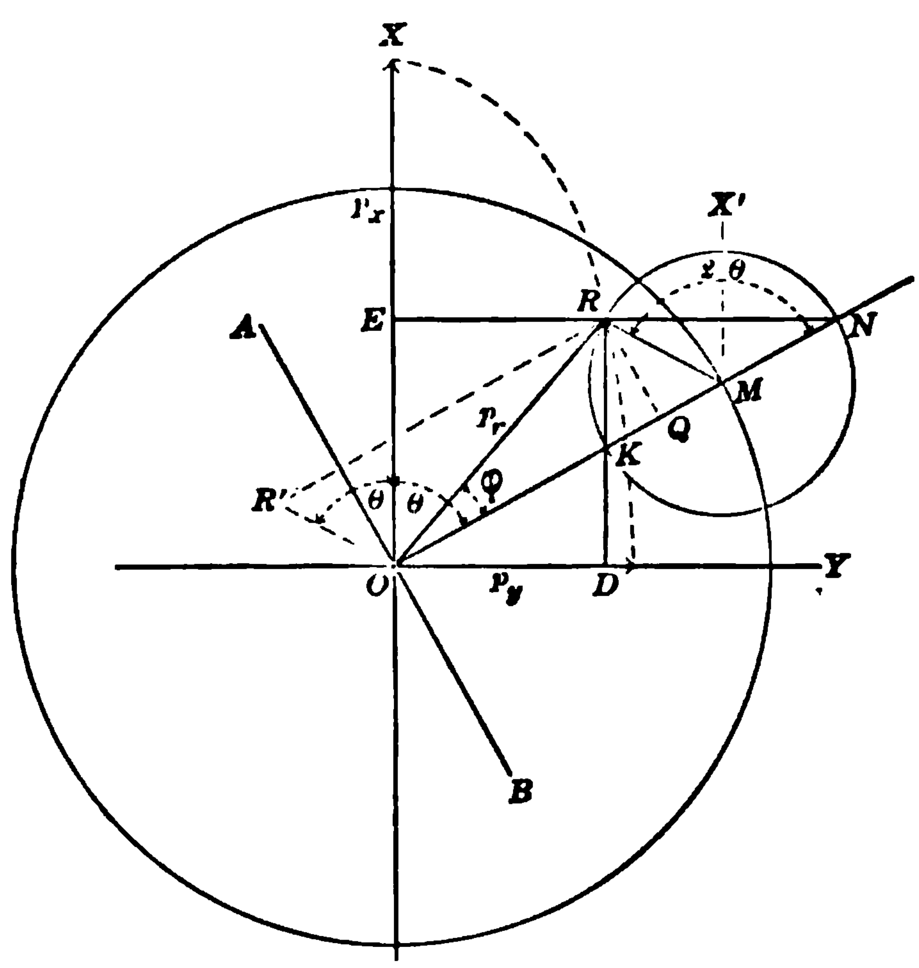


Fig. 208

The algebraic value of OR is readily found as

$$OR^2 = OM^2 + MR^2 + 2OM.MR \cos 2\theta.$$

$$p_r = \sqrt{(p_x^2\cos^2\theta + p_y^2\sin^2\theta)} \qquad (2)$$

which was found in **190.** Also the obliquity is found from the same triangle,

$$\sin\phi = \frac{p_x-p_y}{2p_r} \cdot \sin 2\theta. \qquad (3)$$

2. Equally simple is the following: ON is the normal to the plane AB. OX and OY are the principal planes. The principal stress, p_x, on the plane OY, is laid off on the normal ON. The principal stress,

p_y, on the plane OX, is also laid off on the normal to OK. M is the middle point of KN, and $OM=(p_x+p_y)/2$, and the radius of the small circle is $KM=(p_x-p_y)/2$.

It is then evident that $OE=p_x\cos\theta$, that $OD=p_y\sin\theta$, and that NE and KD intersect in R on the ellipse whose semi-axes are p_x and p_y.

3. The construction employed above for finding p_r affords easy illustrations of several special cases. The point M is always on the circumference of a circle whose center is O, and whose radius is $(p_x+p_y)/2$; R is always on the circumference of a circle whose center is M, and whose radius is $(p_x-p_y)/2$. When $\theta=0$, OMR is a straight line, and $p_r=p_x$. As ON turns from OX to the right, MR turns *equally* to the left. When $\theta=90°$, OMR doubles back upon itself, and $OR=p_r=p_y$. By projecting OR, or its components, OE and ER, upon ON, and then upon a perpendicular to ON, we have

$$OQ=p_n=p_x\cos^2\theta+p_y\sin^2\theta \tag{4}$$

Also

$$RQ=p_t=MR\sin 2\theta=(p_x-p_y)\sin\theta\cos\theta \tag{5}$$

200. Special cases. CASE 1. To find the plane upon which the stress has the *greatest obliquity*, when the principal stresses are of the *same kind*. Referring to Fig. 209, it is at once seen that the line OR must be tangent to the circle about M, giving ϕ its maximum value; so that

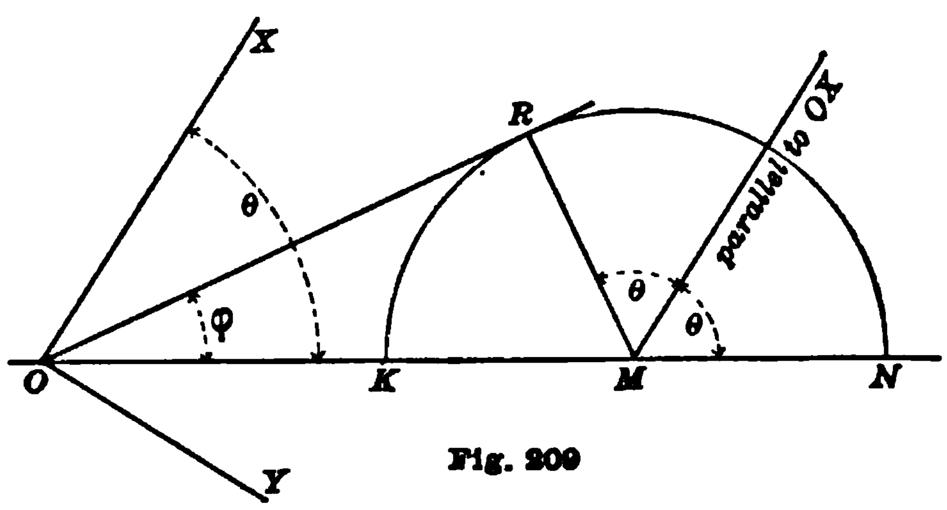


Fig. 209

$$\sin\phi=\frac{MR}{OM}=\frac{p_x-p_y}{p_x+p_y}=-\cos 2\theta \tag{1}$$

If $p_x=2p_y$ (as in a boiler shell), the maximum obliquity is

$$\sin\phi=1/3;\quad \phi=19°\ 20';$$

and its plane makes the angle, $\theta=54°\ 40'$, with the axis of the shell.

Also, $\qquad p_r^2=OK.ON,$ *i.e.* $\qquad p_r=\sqrt{p_x p_y}$

that is, the magnitude of the most oblique stress in the shell of a boiler is a *mean proportional* between the *greatest* and the *least* stresses.

Since the angle $RMN=2\theta$, the angle θ, which determines the plane of most oblique stress, is found graphically and by Eq. (1).

CASE 2. To find the plane on which the stress is most oblique, the

value of that obliquity, and the magnitude of the stress, **when the principal stresses are of *opposite* kinds.**

When p_y is negative, we have

$$(p_x - p_y) > (p_x + p_y), \text{ and } MR > OM,$$

but R is still on the ellipse, whose semi-axis are p_x and p_y, and on the other side of OX from ON. See Fig. 210 for a general case when p_y is negative. No tangent from O to the circle MR is possible, but the *maximum obliquity* is obviously $\phi = 90°$, and the stress is wholly tangential.

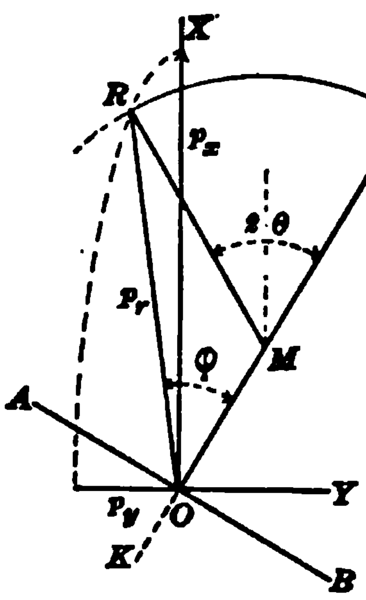


Fig. 210

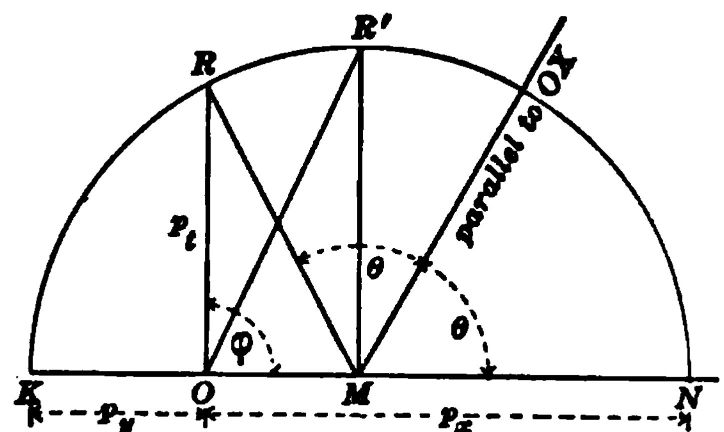


Fig. 211

Fig. 211 shows the construction and gives the angle θ, thereby determining the plane of greatest obliquity.

The formulas of **199** still apply. In this case see Fig. 211.

$$OM = \frac{p_x + p_y}{2}, \qquad MR = \frac{p_x - p_y}{2}$$

$$OK = p_y, \qquad ON = p_x$$

$$\cos 2\theta = -\frac{p_x + p_y}{p_x - p_y}$$

and
$$p_r^2 = OK.ON = p_t^2$$

$$p_r = \sqrt{p_x(-p_y)}$$

Example, If $p_x = 8000$, $p_y = -6000$, $\cos 2\theta = -\dfrac{2000}{14000} = -\dfrac{1}{7}$, $\theta = 49°7'$

$$p_r = \sqrt{48000000} = 4000\sqrt{3} = p_t$$

CASE 3. Find the plane on which the tangential component of p_r is a maximum, when the principal stresses are both positive. A perpendicular dropt from R upon ON is itself a measure of p_t, and its maximum value is evidently $MR = (p_x - p_y)/2$. Tho not shown in any of the above figures, it can be drawn in 209, and it will be shown that $\theta = 45°$.

Figure 210 would show the construction and the plane when p_y is negative, if 2θ were made 90°.*

201. Before taking up the next problem, the student should see that for every position of ON, and of the plane AB, on which the

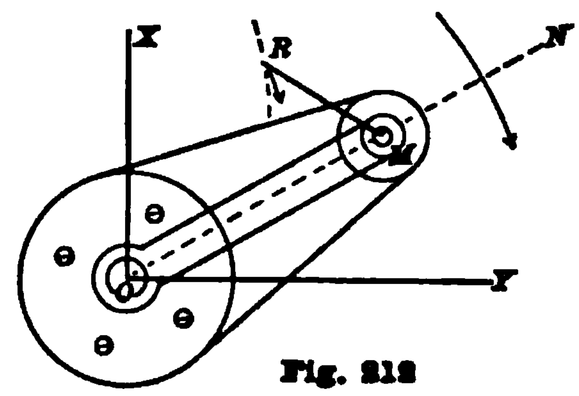


Fig. 212

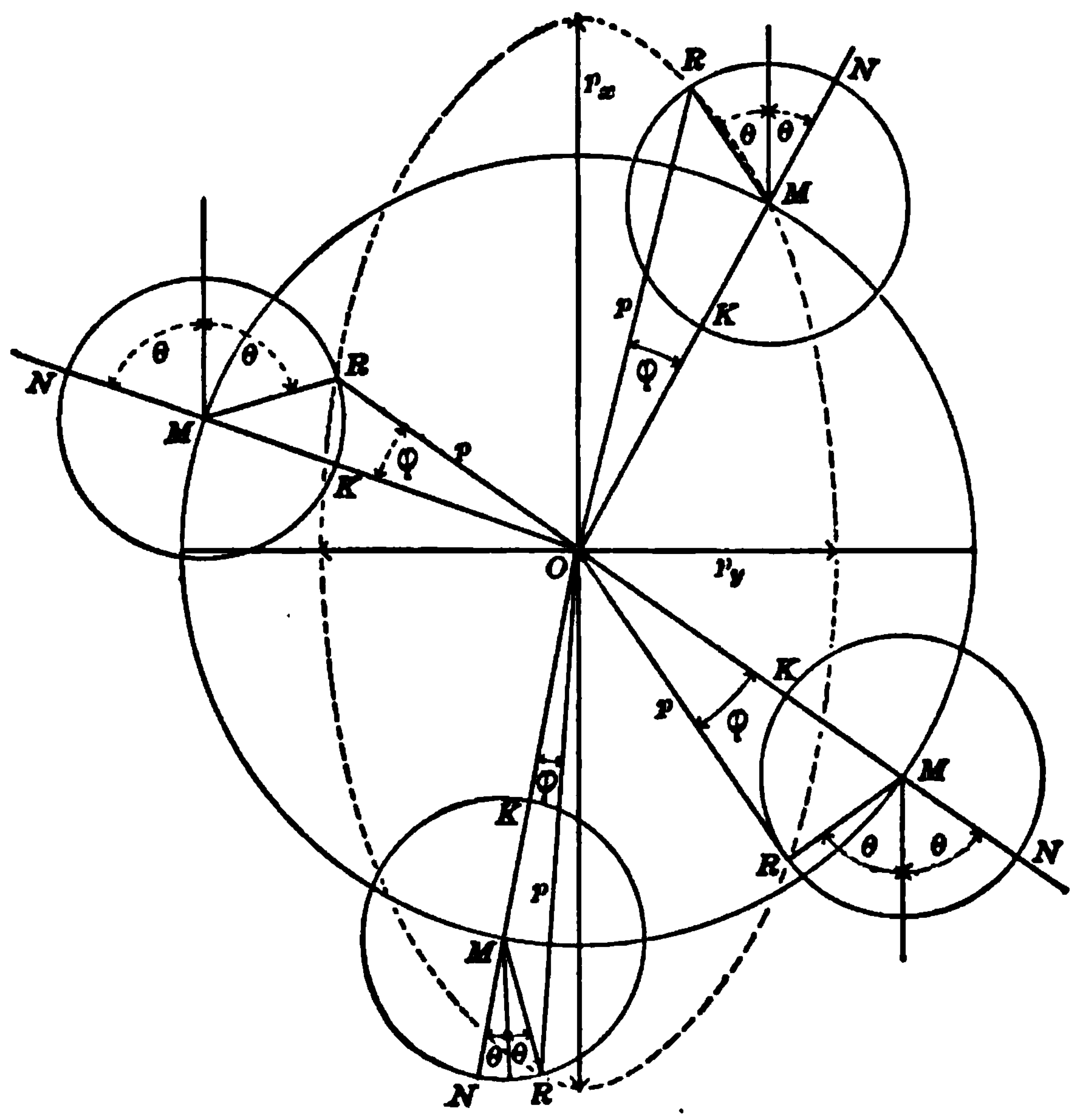


Fig. 213

stress is sought, there is a figure O,M,R,N,K, showing p_r, θ, and ϕ. The angle θ can be measured from either end of the major axis. OR always lies between ON and OX, when p_x and p_y have the same sign. Only two adjacent quadrants of the ellipse are necessary, since every p_r is a semi-diameter of the ellipse, and the obliquities on each side of the plane AB are the same. See Fig. 213.

202. Stresses on two given planes being fully known, to find the principal planes and stresses. *The intensities, kinds and obliquities of any two stresses whose planes of action are perpendicular to the plane of their directions, being given, it is required to find the principal stresses and the axes of stress.*

CASE 1. *When the given stresses are of the same kind.* Since the given stresses are on the given planes, the angle between the normals, NON', is a known angle.

---

* If a sprocket-wheel with any convenient radius be *fixed* in the plane of the axis, OX and OY, with its center at O; and a second wheel in the same plane, with a radius *half as long*, be mounted at M on an axis which is carried by an arm OM revolving in the plane XY about O; and if a chain connect the two wheels as an open belt, every point connected with the moving wheel will describe an ellipse as the arm OM turns about O. See Fig. 212. The mental or actual picture of such a piece of apparatus is helpful in the above discussion.

1. *Graphic solution.* Assume point O as the center of the ellipse and draw the normals ON and ON'. Lay off the given obliquities, ϕ and ϕ', and draw to scale $OR=p$ and $OR'=p'$.

Now, we know that in each normal there is a point M, so placed that $OM=\dfrac{p_x+p_y}{2}$, and that the distance MR for one figure must be the same as MR' for the other. To find M, we pick up one group, say NOR, and lay it so that O' falls on O, and N' on N, with OR' on the same side with OR. Fig. 214. Since M is on ON and equally distant from R and R', we bisect the line RR' by the perpendicular CM, and M is found. If now we draw a circle with M as center and MR as radius, we have $ON=p_x$ and $OK=p_y$, and $\theta=\frac{1}{2}NMR$ and $\theta'=\frac{1}{2}NMR'$. The angle NON' must be equal to the sum or difference of the two θ's; that is, if the normals are in separate quadrants, $NON'=\theta+\theta'=NMX+R'MX'$. If in the same quadrant, $NON'=\theta'-\theta$.

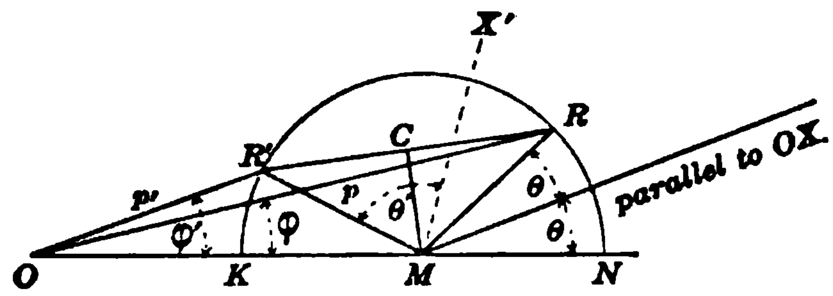


Fig. 214

These last equations serve as a check upon the integrity of the data.

2. *Trigonometric solution.* Since $MR=MR'$,

$$p^2+OM^2-2pOM\cos\phi=p'^2+\overline{OM}^2-2p'\,OM\cos\phi'$$

hence $$OM=\frac{p^2-p'^2}{2(p\cos\phi-p'\cos\phi')}=\frac{p_x+p_y}{2}$$

and $$MR=\frac{p_x-p_y}{2}=\sqrt{\left[\frac{(p_x+p_y)^2}{4}+p^2-(p_x+p_y)\;p\cos\phi\right]}$$

$$\cos 2\,\theta=\frac{2p\cos\phi-(p_x+p_y)}{p_x-p_y}$$

$$\cos 2\theta'=\frac{2p'\cos\phi'-p_x-p_y}{p_x-p_y}$$

Both the graphic and the trigonometric solutions are much simplified in certain special cases.

CASE 2. *When the given stresses have opposite signs.*

Let the numerically greater be p, and the other (which is essentially negative), be p'. The construction differs from Fig. 214 since p' must be drawn negative, and when the two normals are superposed with the obliquities laid off on the same side, the p and p' will appear

on opposite sides, as in Fig. 215. M is found as before, and the circle with MR as radius shows $ON = p_x$, and $OK = p_y$, which is negative. The axes of the ellipse can now be laid off on Fig. 215, and the ellipse drawn thru R.

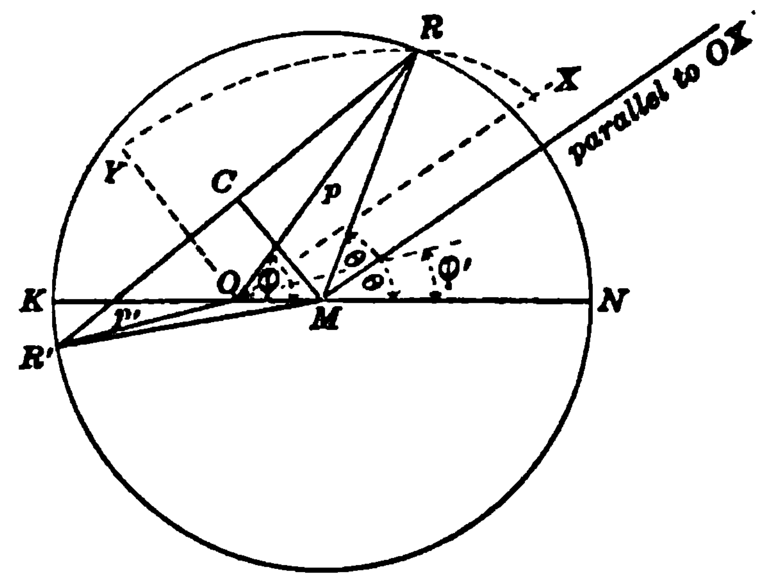


Fig. 215

203. Special cases. There are some special cases of practical value, which it is worth while to consider.

SPECIAL CASE 1. When the given stresses are of the same kind and in planes *perpendicular* to each other. It follows that p_t and p_t' are equal, and the figure when the normals are superposed, takes the form of Fig. 216. Resolve p into p_n and p_t; and p' into p_n' and p_t'.

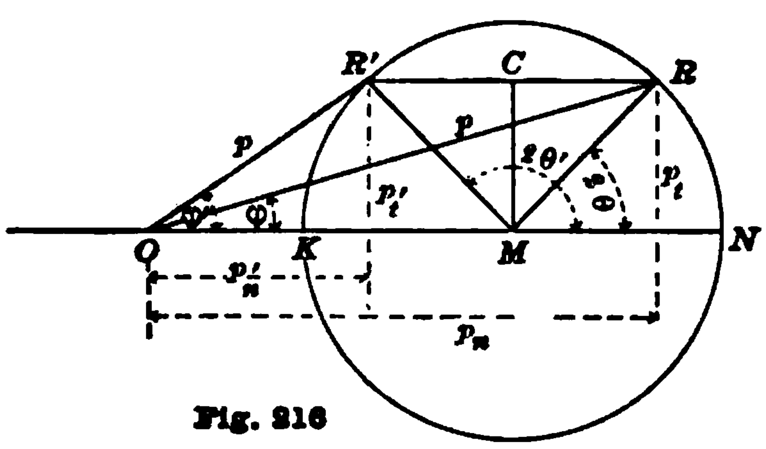


Fig. 216

$$p_x = ON = OM + MR$$

$$p_x = \frac{p_n + p_n'}{2} + \sqrt{\frac{(p_n - p_n')^2}{4} + p_t^2}$$

$$p_y = \frac{p_n + p_n'}{2} - \sqrt{\frac{(p_n - p_n')^2}{4} + p_t^2}$$

$$2\theta = RMN, \qquad 2\theta' = R'MN,$$

$$\theta + \theta' = 90°$$

$$\tan 2\theta = \frac{2\,p_t}{p_n - p_n'}$$

These formulas are used in finding the *maximum stresses* in *beams* and *shafts*.

204. Conjugate stresses. When material is subjected to two independent co-planar oblique stresses in such a way that the *planes* upon which one acts are parallel to the *direction* of the other stress, the stresses are called "conjugate."

Fig. 217 represents a portion of a steel plate, a part of which is subjected to conjugate stresses. The stress P acts upon the planes AA, which lie in the direction of the stress F, which acts upon planes BB, which lie in the direction of the first stress, P.

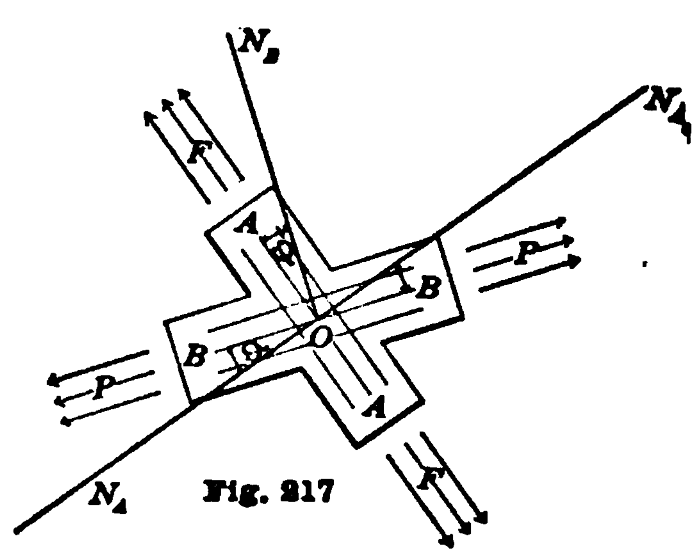


Fig. 217

The two normals, ON_A and ON_B, are drawn, as are the *obliquities* N_AOP and N_BOF, which are seen to be equal, as each is equal to the angle between the normals, less a right angle.

When the stresses P and F are of the same kind, the axis of *maximum* stress lies between them. If one of the given stresses is negative, the axis of *minimum* stress lies between the normals as drawn. It may be remarked that every stress on a plane AB (see Fig. 208) has its conjugate stress on *another plane $A'B'$ parallel to OR*; and the lines p and p' are conjugate semi-diameters of the ellipse of stress.

This property of conjugate stresses introduces a novelty in the graphical solution. The maximum stresses are found as follows, graphically; and by formulas.

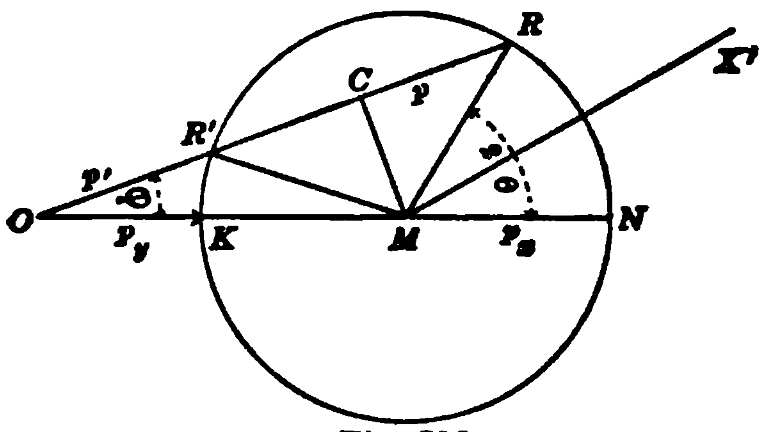


Fig. 218

1. **When the stresses are of the same kind,** since the obliquities are equal, not only do the superposed normals coincide, but p and p' partially coincide. Fig. 218.

$$OM = \frac{p_x + p_y}{2} = \frac{1}{2}(p+p')\sec\phi$$

$$MR = \frac{p_x - p_y}{2} = \sqrt{\frac{(p_x+p_y)^2}{4} - pp'} = \sqrt{\frac{(p+p')^2}{4\cos^2\phi} - pp'}$$

since $(p_x - p_y)^2 = (p_x + p_y)^2 - 4p_xp_y$,

and by geometry, $pp' = p_xp_y$ from the figure.

2. **When the stresses are of different kinds.** Fig. 219.

$OR = p$, $OR' = p'$ (negative)

$ON = p_x$ positive

$OK = p_y$ negative.

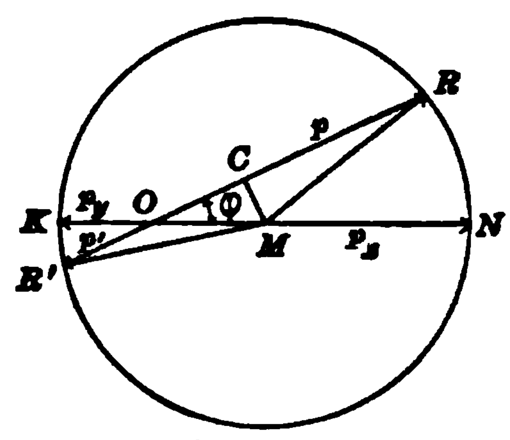


Fig. 219

$$OC = \tfrac{1}{2}(OR - OR') = \tfrac{1}{2}(p_x + p_y)\cos\phi.$$

$$OM = \frac{p_x + p_y}{2} = \frac{p+p'}{2\cos\phi}$$

$$MR = \frac{p_x - p_y}{2} = \sqrt{\frac{(p+p')^2}{4\cos^2\phi} - pp'}$$

p_y and p' being negative.

205. The stresses being thrusts, and the greatest obliquity the material will bear being given, it is required to find the ratio between two conjugate pressures whose common obliquity is known.

Think of a granular material like sand, clay, or dry powder. Common observation tells us that when piled conically, such material will rest indefinitely with a certain maximum slope. That slope varies with different materials and with different conditions of the same material. When the slope is a maximum, it is unstable, and if disturbed it takes a smaller slope. See reference to the sloping banks of the Panama Canal, later on.

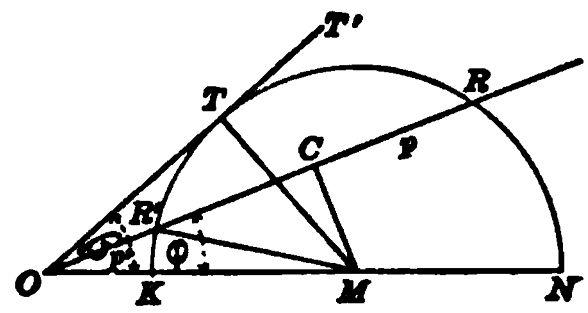


Fig. 220

Let $OR=p$, $OR'=p'$, ϕ their common obliquity, and $\beta=$ the *maximum obliquity* the material can be relied upon to stand. Required, the value of the ratio $\frac{p'}{p}$. See Fig. 220.

The only known quantities in the above problem are ϕ and β.

Assume a normal ON', draw OR, making the angle at O equal to ϕ; and draw OT', making the angle $T'ON'=\beta$. With *any point M* in ON' as a center, draw a circle tangent to OT'. The ratio of p' to p is found from the figure.

$$OR=OC+CR \text{ is a value of } p;$$
$$OR'=OC-CR \text{ is a value of } p'.$$
$$OC=OM\cos\phi, \qquad CR=\sqrt{(MR)^2-(CM)^2}$$

but $MR=MT=OM\sin\beta$, and $CM=OM\sin\phi$

$$\text{hence } \frac{p'}{p}=\frac{OM\cos\phi-OM\sqrt{(\sin^2\beta-\sin^2\phi)}}{OM\cos\phi+OM\sqrt{(\sin^2\beta-\sin^2\phi)}}$$

$$\frac{p'}{p}=\frac{\cos\phi-\sqrt{\sin^2\beta-\sin^2\phi}}{\cos\phi+\sqrt{\sin^2\beta-\sin^2\phi}} \qquad (1)$$

If $\phi=0$, as in important cases it is, the conjugate stresses are the principal stresses p_y and p_x, and we have

$$\frac{p_y}{p_x}=\frac{1-\sin\beta}{1+\sin\beta} \qquad (2)$$

Also, if $\phi=\beta$, it is seen from the figure as well as from the formula that

$$p'=p$$

These formulas are used for finding the value of p' or p_y, when p, β and ϕ are known, in studying the stability of slopes, retaining walls, and foundations.

PART II.

CHAPTER XIII.

KINETICS. TRANSLATION OF SOLID BODIES UNDER THE ACTION OF CONSTANT UNBALANCED FORCES.

206. When the forces acting upon a body do not balance, *i. e.*, when they have an unbalanced resultant, the body is not at rest nor is it in a state of uniform motion. It is easy to think of a body moving thru empty space acted upon by no force whatever. There could be no force if there were no other body to act. There would, therefore, be nothing to increase, check or modify its motion; it would then, of necessity, continue indefinitely its motion, whether of translation or rotation, or both. However, such a thing is purely imaginary; there are no such bodies in the physical universe. All bodies are more or less acted upon either by attraction, repulsion, or surface contact, by every other body, near or far, large or small. The *joint action* of sun, moon and stars is by no means small, but as human constructions (with which alone applied mechanics is concerned) share that action and its effect in common with all that is on or within the earth, we are unconscious of it, and in our present theories of rest and motion we ignore it.

207. Uniform motion. Thus far in this book, it has been assumed that bodies acted upon by balanced forces were at rest. We now must see that they might as well have been moving uniformly; the conditions of stability would have been the same. In a measure, personal experience confirms this. In a railway car moving steadily on a smooth, straight track, or on a large ship moving steadily along in still water, we stand or move about as tho the car or vessel were at rest. The forces acting on us are so perfectly balanced that we suffer no change in our motion. But the instant the forces do not balance, the motion is modified and we feel the effect, by being instantly "thrown" forward, backward, or to one side. The motion is no longer "uniform," which means preserving the same speed and the same direction.

It is not always easy to see just how the forces balance, but the fact that they do balance is shown by the uniformity of their motion. A train of cars is pulled across the plain at constant speed. We can measure the tension in the draw-bar of the engine, but we cannot as easily see just where, and in what proportion, the resisting forces

act: the backward action of the rails under every wheel; the unequal pressure of the air due to winds; the friction of the air; the possible horizontal component of the earth's attraction. If all these combine to exactly balance the action of the locomotive, the motion is uniform; otherwise it is not uniform.

In this part of the book, we are concerned with *unbalanced* forces, and the effect they have upon the motions of the bodies upon which they act.

208. Translation and rotation. There are two kinds of motion. When a body so moves (without regard to what makes it move) that every point on or in it moves in the same direction at the same rate; that is, when every straight line on or in the body points continually in the same direction—the motion is called *translation.*

When a body so moves as a whole, that a single point of it retains its position, the motion at every instant is called *rotation.*

These definitions are clear enough, but the student must beware lest he make them mean too much. Thus, suppose a connecting rod (Fig. 221) with two equal cranks turning upon fixed centers, distant from each other by the exact length of the rod. When in motion, not only the crank pins, but *every point in or rigidly connected with the rod is moving in a circle*; but while the cranks have motions of rotation, the connecting rod has a motion of translation. The circles described by points in the rod are equal, the radius in every case being equal to that of the circle described by the centers of the crank pins. Points in the cranks describe circles of varying size, all having their centers in the axis of the shaft or bearing. Were the cranks and rod those on two drivers of a moving locomotive engine, the rod would still have a motion of translation, tho no point would move in a circle. The "trochoid" described by every point will be discussed later on.

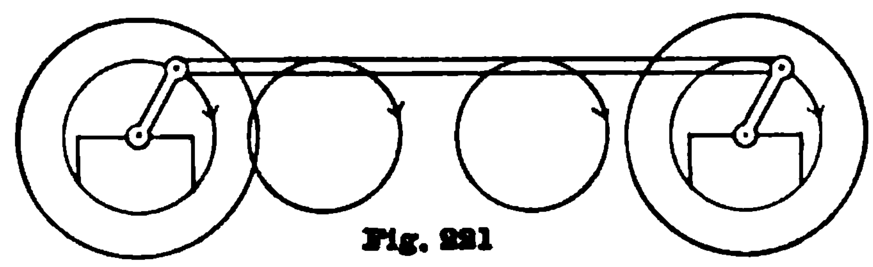


Fig. 221

209. Rotation. If a single point in a rigid body is fixed in position, while the body moves by any route, no matter how devious, from one general attitude to another, the change from first to last position may be produced by simple rotation about an axis passing thru O. The axis itself is found as follows (Fig. 222):

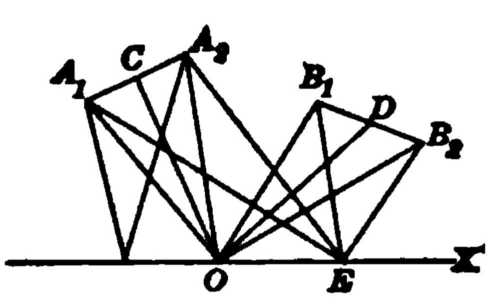


Fig. 222

Let O be a point in a rigid body which does not change its position while the body moves in accord with a certain set of conditions from No. 1 to No. 2. Suppose that point A_1, moves to A_2, while B moves from B_1 to B_2. As the body is

rigid, $OA_1 = OA_2$, $OB_1 = OB_2$ and $A_1B_1 = A_2B_2$, hence, if $A_1\ A_2$ and $B_1\ B_2$ be bisected by planes respectively perpendicular to those lines, they must both pass thru O and intersect in some line OX. Now it is clear that every point in OX might have retained its distance from A and B during that movement, for take any point in OX, as E. Since it is in the plane which bisected $A_1\ A_2$, the distances EA_1 and EA_2 are equal. In like manner we see that $EB_1 = EB_2$. Hence, on account of these equalities, E may have retained its position. Hence, OX suffices as the axis of rotation. If it be pointed out that A and B might have reached their second positions by zigzag courses, it can be added that every element in that varied motion had an axis of its own, and that the whole movement could have been effected by rotation about one axis.

Not all motions are simple, either pure translation or pure rotation. May bodies have motions which are combinations of the two. Note the motions of screws and nuts when turned, the motion of projectiles, and in general "secondary" members of machines (those which are guided in their motions by members which are themselves moving).

The discussion of the motion of bodies, without regard to the forces which cause them, comes under the head of *kinematics*, which will be touched upon later in this book.

210. Time. Motion takes time. In Statics we had no thought of the lapse of time. Henceforth every problem will involve the thought of a never-ceasing, constant flow of the stream of time. All other quantities vary according to different laws, time alone has a constant increment. Hence, in dynamic problems, t, the time elapsed, or "elapsed time," is the independent variable, whose differential, dt, is constant.

The unit of elapsed time is a second, or its multiple, the civilized world over. Men of science differ in their units of length, their units of force, their units of mass, as they do in their units of commercial value; but in their unit of time they agree. In engineering, the most common unit is the second, the next in order is the minute, and still less frequently the hour.

In speaking of time, we shall always mean "elapsed time," between two events, from a start to a finish, or the interval between two dates. The quantity t never means a date.

211. Distance; the foot. Motion requires room, or space, or, more accurately, one of the dimensions of space, length or distance. The unit of length, by means of which all linear motion will be measured in this book, is the foot, which is defined by the International

Bureau of Heights and Measures, as 0.3048 of a French meter. A meter is 39.37 American and British inches. Both the meter and foot are legal in this country.

212. Velocity. Velocity measures the rate of a body's motion. If the motion is uniform, it measures the distance moved during a unit of time, which is found by dividing the number of units of length in a given distance moved by the number of units in the time occupied in the moving; in algebraic language

$$v = \frac{s}{t} \textit{ (when } v \textit{ is constant).}$$

In former equations every letter meant a *number* of units of some particular thing; the statement will hold for future equations, and they will signify nothing beyond abstract arithmetic unless the reader has a clear idea of every unit involved. No one should be misled by the elliptical statement that we "divide distance by time." When the terms of a fraction are associated with different units, the denominator (divisor) is an abstract number; if they are associated with the same unit, they form simply an abstract ratio. For example, $s_1 = 13$ feet, divided by $s_2 = 14$ feet, is merely the abstract ratio $\frac{13}{14}$; but when we have: $\dfrac{s = 13 \text{ feet}}{t = 2 \text{ sec}}$, this means: $\frac{1}{2}$ of 13 feet *in one second.*

When v is not constant, its value at any instant is expressed by the differential co-efficient $v = \dfrac{ds}{dt}$ (1)

This is the first equation in dynamics. It is always true whether the motion is uniform or not. Its derivation is simple. Suppose a body has already moved a distance s in the time t, and that it moves an additional distance Δs in the additional time Δt. The fraction $\dfrac{\Delta s}{\Delta t}$ gives the *average* velocity during that additional time, but it may not be the actual velocity at either the beginning, end or middle of Δs; but it is the velocity at *some* intermediate point. If now both Δt and Δs are diminished until they become dt and ds, it is evident that the value of the quotient $\dfrac{ds}{dt}$ gives the value of v at the point when the distance was s. Hence follows formula (1).

213. Acceleration. 1. When the velocity changes, the change is always gradual. The change may be rapid or slow, but it proceeds

gradually at some rate or with a varying rate. The *rate of change* in a body's motion, which is best defined by its mathematical equation:

$$a = \frac{dv}{dt}$$

is called *acceleration*, and it depends partly upon the mass of the body moved, and partly upon the magnitude of the unbalanced force which causes it to move. We know by experience that with the same mass the greater the unbalanced force the more rapidly does the velocity change. Suppose a light truck is being drawn along a platform at a uniform velocity, the forces acting on it being balanced. Now, suppose a child steps up behind it and gently pushes, the pull in front continuing unchanged; the velocity at once slowly increases, showing that the acceleration is small. If now the child is replaced by a man who pushes much harder, the change is more rapid, and a is greater. In short, we recognize Newton's second law of dynamics, that **the acceleration of the velocity of a moving body, produced by the action of an unbalanced force, is proportional to that force.** That is to say, if the force F_1 (always unbalanced) acting on a certain body causes an acceleration a_1, and another force F_2 (under similar conditions) causes an acceleration a_2, we must have the proportion:

$$\frac{F_1}{F_2} = \frac{a_1}{a_2}$$

2. If now we have, by careful experiments, found the acceleration which a known unbalanced force acting upon a body produces, this equation enables us to find the acceleration another known force would produce in the motion of the same body; for $a_2 = \frac{F_2}{F_1} a_1$. That is, knowing F_1 and a_1, we can find the acceleration a_2 for an assigned value of F_2; or we can find the value of F_2 for an assigned value of a_2;

$$F_2 = \frac{a_2}{a_1} F_1.$$

Now, we do know with great accuracy the acceleration which the *unbalanced pull* (attraction) *of the earth* is *able* to give to a solid body upon the earth's surface. Take note that the body must be in a vacuum, free from all magnetic influences, and at a certain level and latitude (at the sea-level, and 45° north). Under those conditions the pull of the earth is exactly W standard pounds of force, and the acceleration it can cause, when wholly unbalanced, is 32.1740 feet per sec. in a velocity of feet per sec. This number is generally

called g, and like other accelerations, is often read *g ft. per sec. per sec.*, or $g \frac{\text{ft.}^2}{\text{sec.}^2}$. However, when the units of length and time are well understood, it will be sufficient to say that the acceleration is so many (naming the number) *feet per second.**

3. The reason why the sea level and the latitude (not far from Paris) are specified is that when one goes high in the air, or into a deep mine; far north, or nearer the equator; the earth's pull on a body is sensibly different, and consequently g will be different. To be above or below the sea level, and nearer the equator, makes the actual pull of the earth less; to go north at the sea level makes it greater. As the ratio of the number W, to the number g, is always the same, for the same body, and as these numbers are rarely separated in real problems of applied mechanics, variations in W due to local conditions can generally be neglected. If, however, g is used by itself its local value may well be used. The value of g for the City of St. Louis is very nearly 32.1.†

214. The measure of mass. N. B. Thruout this Chapter the letter F will always represent an *unbalanced* force, or the *unbalanced resultant* of two or more forces. Returning now to our equation which declares the equality of two ratios, and substituting W for F_1, and g for a_1, we have, omitting subscripts,

$$\frac{F}{W} = \frac{a}{g} \tag{1}$$

This is a most important fundamental equation, and one to which, or to its immediate derivatives, the student will at once return when he has a real problem in dynamics to solve.

The first derived form is

$$F = \frac{W}{g} a \tag{2}$$

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* The student who consults other writers must not be confused by such expressions as: an acceleration of "13 feet per second per second"; or "13 ft. per second square"; or "13 ft.-pr-sec-pr-sec." These expressions are exceedingly elliptical, tho they are based on a desire to define velocity and acceleration at the same time. The above quoted expressions all mean: *an increase of 13 ft. per second in a velocity of feet per second.* When it is perfectly understood that velocity means so many feet per second, that understanding need not be constantly put into words.

† Some eminent writers make the very injudicious statement that "g = the force of gravity,"—to the great confusion of the minds of students. The number g is no more the force of gravity than is any other number, 10, 13, 62.2, or 1000. Every number measures the force of gravity acting on a certain mass: g measures the force acting on a unit of mass.

in which the ratio $\frac{W}{g}$ explicitly appears. It is quite universally represented by one letter, m, which stands numerically for the *mass* of the body which weighs W. Hence the simple formulas $m = \frac{W}{g}$ and

$$F = ma. \tag{3}$$

a formula used in all systems of units, and should be memorized as: The (unbalanced) force (F) is (numerically) equal to the number of units of mass in the body acted upon, multiplied by the number of units in the acceleration produced, or elliptically: **The unbalanced force equals mass times acceleration.**

215. Impulse and momentum.

Since
$$a = \frac{dv}{dt}$$

the last equation in **214** gives
$$Fdt = mdv$$

which some writers regard as the fundamental form. The first member being (numerically) the product of an *unbalanced* force by the time during which it acts upon a free body, and is called an "*Impulse.*" The second member, being a differential of mv (which is called the *momentum* of a moving body), represents the *change* in the momentum produced by the Impulse in the time dt; so that equation states the general law, that **A change of momentum is equal (numerically) to the Impulse producing it;** or **An Impulse is equal (numerically) to the momentum it produces.**

If F is constant, the equation can be integrated, the integration being from t_1 and v_1, to t_2 and v_2;

$$F(t_2 - t_1) = m(v_2 - v_1)$$

The time of action is, of course, the interval between the interval t_1 (from an arbitrary starting instant), and the finishing interval t_2. The momentum at the start was mv_1, and the momentum at the finish was mv_2, so that the *gain* in momentum is the difference shown.

It should be noted that the word "impulse" does not of necessity mean that the time of an action is short, but it may be so.

216. The equations of dynamics. Before solving the problems of dynamics, it will be well to bring together the equations embodying definitions and the fundamental principles as applied to the translation of solid bodies.

$$\left.\begin{array}{lr}\text{By definition} & v=\dfrac{ds}{dt} \\ \text{By definition} & a=\dfrac{dv}{dt}=\dfrac{d^2s}{dt^2} \\ \text{Eliminating } dt, & vdv=ads.\end{array}\right\} \qquad (1)$$

These are for pure motion without reference to force.

For motion caused and modified by unbalanced forces, *acting in the direction of the motion*:

$$\left.\begin{array}{l}\dfrac{F}{W}=\dfrac{a}{g} \\ F=\dfrac{W}{g}a=ma=m\dfrac{dv}{dt} \\ Fdt=mdv.\end{array}\right\} \qquad (2)$$

These equations are always true whether F be constant or not.

It is self-evident that when a body is compelled by an unbalanced force to move, it moves in the direction in which the force acts. In what immediately follows, it will be assumed that the motion is translation in right lines.

Motion Under Constant Forces.

When the resultant unbalanced force acting upon a body is *constant*, the acceleration is constant, and the equations of motion are readily integrated. Multiplying Eq. (1) by dt and integrating

$$\left.\begin{array}{l}\displaystyle\int_{v_o}^{v}dv=a\int_{o}^{t}dt \\ v=v_o+at=\dfrac{ds}{dt}\end{array}\right\} \qquad (3)$$

$$\left.\begin{array}{l}\displaystyle\int_{o}^{s}ds=v_o\int_{o}^{t}dt+a\int_{o}^{t}tdt \\ s=+v_ot+a\dfrac{t^2}{2}\end{array}\right\} \qquad (4)$$

$$\left.\begin{array}{l}\displaystyle\int_{v_o}^{v}vdv=a\int_{o}^{s}ds \\ v^2-v_o^2=2as\end{array}\right\} \qquad (5)$$

If when $t=0$, s was s_0, the *lower* limit for $\int ds$ would be $\int_{s_0} ds$, and we should have had

$$s=s_0+v_0t+a\frac{t^2}{2} \tag{6}$$

If $v_0=0$, and $s_0=0$ (which means that the body moves from rest), the above equations become

$$\left.\begin{aligned} v&=at \\ s&=\tfrac{1}{2}\,at^2 \\ v^2&=2as \end{aligned}\right\} \tag{7}$$

These integral equations are not to be used unless the force is *constant.*

Eqs. (3) to (6) are the general equations for the rectilinear motion of bodies under the resultant action of constant unbalanced forces. The motion may be in *any direction,* up or down, inclined or horizontal. The student must have no bias in favor of vertical motions. Great care must be taken to discover in every problem the correct value of the resultant unbalanced force. Forces acting *with the motion* are positive; those acting *against the motion* are negative. The student must always keep in mind, while considering the motion of a particular body, *not* how *it acts* upon other bodies, but how *other bodies act upon it.*

Problems.

217. Problem to illustrate the use of formulas. 1. A body weighing 400 pounds starts from rest and moves upon a rough horizontal plane, being drawn by a motor which maintains a *constant horizontal pull* upon it of 180 pounds, until the velocity attained is expressed by: "A mile a minute." As soon as that velocity is reached, the draw-bar is released and the body moves on till it comes to rest. Find distances, velocities and times. The co-efficient of friction is $\frac{1}{10}$. The only unbalanced force to be ignored is that of the resistance of the air. The direct action of the earth's pull is *balanced* by the vertical upward action of the plane. The moving body we will call B.

There are three forces acting on B, viz.: the motor, the earth (by attraction of gravity), and the rough plane. The motor acts *positively* 180 lbs. The earth's action of 400 lbs. is exactly balanced by the *normal* action of the plane, and hence does not affect F. The plane, by friction, acts tangentially *against* the motion by an amount $\frac{1}{10}$ of 400 lbs., which is negative. Hence

$$F=+180-40=140 \text{ lbs.};\ v_0=0;\ t_0=0.$$

By formulas of **216**, $a = \frac{Fg}{W} = \frac{140}{400}g = \frac{7}{20}g$

$$v = at = \frac{7}{20}gt$$

"A mile per minute" means 88 ft. per sec. $= v_1$; hence the time during which F acts, t_1, is

$$t_1 = \frac{20 \times 88}{7g} \text{ seconds,}$$

and the space passed over during t_1 is by $v_1^2 = 2as_1$

$$s_1 = \frac{(88)^2 \times 10}{7g} \text{ feet.}$$

When the draw bar is released, the conditions change and our problem enters upon a *second epoch,* and we must begin again by finding the new unbalanced force, F_2. The motor has ceased to act. The pull of the earth, W, is balanced as before; all that remains to make F_2 is the friction which is (−40 lbs.): hence $F_2 = -40$.

Proceeding as before, we find a_2.

$$a_2 = \frac{(-40)g}{400} = -\frac{g}{10}.$$

We have now $v_0 = 88$, since the new epoch begins with the initial velocity 88, and as $v = v_0 + a_2t$, we shall have, when the body stops, and $v = 0$:

$$o = 88 - \frac{g}{10}t_2$$

$$t_2 = \frac{880}{g} \text{ seconds,}$$

which gives the time occupied by the second epoch.

For s_2 we have from (5) **216**

$$v^2 - v_0^2 = 2as$$

$$o - (88)^2 = -\frac{g}{5}s_2$$

$$s_2 = \frac{5(88)^2}{g} \text{ feet,}$$

which is the distance moved during the second epoch.

The whole time $t = t_1 + t_2 = \frac{88}{g}\left(\frac{20}{7} + 10\right)$ sec.

The whole distance $s = s_1 + s_2 = \frac{(88)^2}{g}\left(\frac{10}{7} + 5\right)$ feet.

(The relation between s and t is interesting.)

Had we substituted m for $\frac{400}{g}$ all of our results would have been functions of m instead of functions of g. Let the student derive them all, thus

$$a_1 = \frac{140}{m} \qquad a_2 = -\frac{40}{m} \text{ etc.}$$

The inaccuracy of weights and measures and the uncertain value of the co-efficient of friction in engineering, hardly justify the decimal in the value of g, and the value 32 will generally suffice. Results purely numerical can be obtained by substituting 32 for g above.

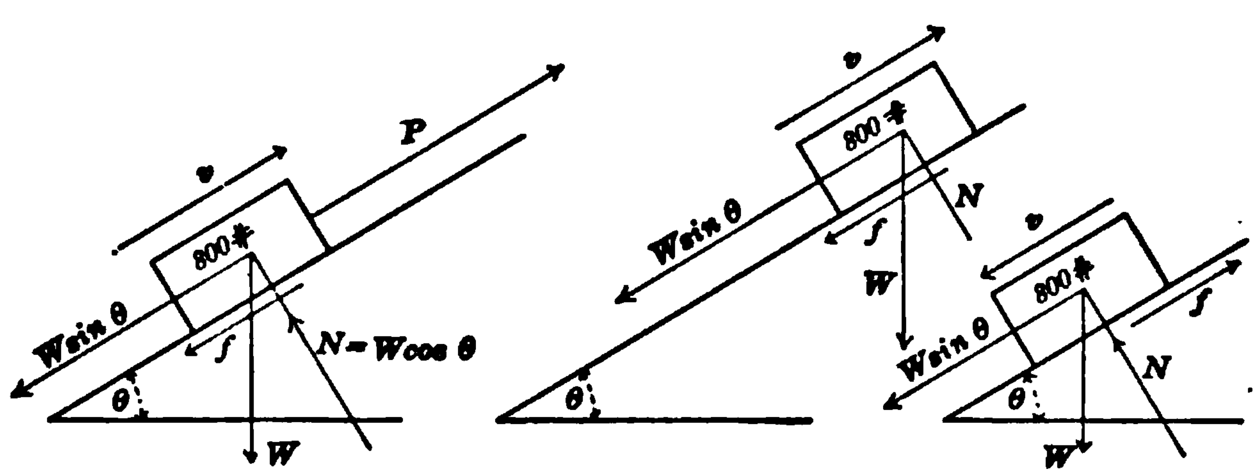
Many problems consist of *epochs* which the student should carefully distinguish.

Problem 2. Suppose a body is being hauled from rest up a rough inclined plane by a *constant* tension in a rope, which, at a certain point on the plane, breaks; the body soon stops and may slide back. While in action, the rope is parallel to the plane. Discuss the entire movement.

Here there are three epochs, and we shall have F_1, F_2, F_3; a_1, a_2, a_3; t_1, t_2, t_3; and s_1, which is given, s_2, s_3. The student should picture in his mind distinctly, the beginning, the current (or general) instant, and the end of each epoch before he puts pencil to paper, or looks up a formula.

Epoch I. At the beginning $s = 0$, $v = 0$, $t = 0$.

At the current instant, there are general increasing values of s, v and t. Three forces are acting, as before, but one component of the weight acts *against* the motion, viz.: $W \sin\theta$; the other component $W\cos\theta = N$ is balanced by the normal reaction of the plane; and the resistance due to friction is $fN = fW\cos\theta$, in which f stands for the co-efficient of friction.



Epoch 1. *Epoch 2.* *Epoch 3.*

Fig. 228

We therefore have $F_1 = +P - W\sin\theta - fW\cos\theta$

and a_1 follows at once, from the formula, $a_1 = \dfrac{F_1 g}{W}$.

As s_1 is given, the student finds v_1 and t_1.

At the end of the epoch, when the rope breaks, $t = t_1$, $s = s_1$, and $v = v_1$, which becomes v_o for the second epoch.

Epoch II. It is best to let time and space begin over again from zero, but $v_o = v_1$.

$F_2 = -W\sin\theta - fW\cos\theta$, since P is no more; a_2 follows at once, also negative.

At the end of Epoch II the body stops and v is zero. Now find t_2 and s_2, as in Problem I.

Epoch III begins with $v=0$, $t=0$, $s=0$, in general $F_3 = -W\sin\theta + fW\cos\theta$, the positive direction for forces and motion being unchanged. The friction is always *against* the motion. If $W\sin\theta$ is greater than $fW\cos\theta$, the body is now sliding down the plane, and *the friction acts up the plane.*

$$a_3 = \frac{F_3 g}{W} = (-\sin\theta + f\cos\theta)g.$$

Now, if $s_3 = -(s_1 + s_2)$, the body returns to its starting point; t_3 is found from the Eq. $s_3 = \frac{1}{2}a_3 t_3^2$

and v_3 (final) from
$$\left.\begin{aligned} v_3 &= a_3 t_3 \\ \text{or}\quad v_3 &= \sqrt{2a_3 s_3} \end{aligned}\right\} \text{both negative.}$$

Problem 3. In this problem let the student put $W = 800$ lbs., $\theta = 30°$, $P = 600$ lbs., $f = \frac{1}{8}$ and $s_1 = 120$ ft., $g = 32$, and then find numerical values for all the F's, a's, t's and s's.

In the first epoch: $F = P - W\sin\theta - fW\cos\theta$.
In the second epoch: $F_2 = -W\sin\theta - fW\cos\theta$.
In the third epoch: $F_3 = fW\cos\theta - W\sin\theta$.
In No. 1, a_1 is positive, $v > o$, $s > o$.
In No. 2, a_2 is negative, $v > o$, $s > o$.
In No. 3, a_3 is negative, $v < o$, $s < o$.

Final values of v for one epoch, are initial values for the next epoch.

The t is *always* positive.

If t_3 be 10 seconds, where will W be?

Problem 4. If, in this last example, $\theta = \dfrac{\pi}{2}$, $W = 800$ lbs., and $P = 1{,}000$ lbs., the plane's action will vanish, and the motion will be vertical. There will be three epochs, as before.

In Epoch I; $F_1 = P - W = 200$ lbs.

$$a_1 = \frac{200g}{800} = \frac{g}{4} = 8, \text{ etc.}$$

In Epoch II; $F_1 = -800$

$$a_2 = -g = -32, \text{ etc.}$$

In Epoch III; if we take $s_3 = s_1 + s_2$ positive downwards, we shall have

$$F_3 = 800$$

$$a_3 = g, \text{ etc.}$$

under what is known as the case of "falling bodies" where $a = g$ and $F = W$; t_3 can easily be found.

218. Falling bodies, and sliding on smooth planes. When a body falls freely, as in a vacuum, and s is positive downward, g is positive. If a body is rising, and W is positive, g is negative. Engineers have very little occasion to solve problems of bodies falling *in vacuo.*

If in the problem of a body on an inclined plane, $f = 0$; that is, if the plane is "*smooth*" (an impossible case), we have in Epoch III, the ideal case of a body sliding down a smooth plane. If W is positive

$$F = W \sin\theta$$

$$a = g \sin\theta$$

$$v = v_0 + gt \sin\theta$$

$$s = v_0 t + \frac{t^2}{2} \cdot g \sin\theta$$

$$v^2 - v_0^2 = 2gs \sin\theta$$

219. The motion of trains. When the different members of a train are so connected that each is moving in a straight line with which its F is parallel, all having the same velocity and acceleration, the whole train may be treated as one body; and again, each member may be considered separately.

Problem 3. Suppose a train of three cars is drawn from a station on a steady up-grade by a motor which maintains a constant pull, P, for the distance S_1. Let the weights of the cars be W_1, W_2, W_3; the co-efficient of resistance (friction, air, etc.) be f_1, f_2, f_3, respectively, and let the up-grade be so gentle that we may take $\cos\theta = 1$, $\sin\theta = \tan\theta = 0.01$.

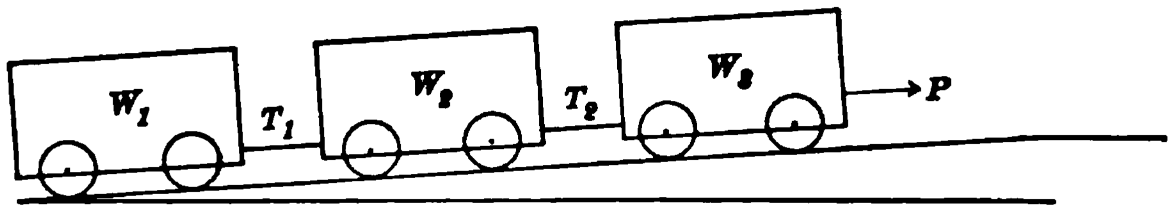


Fig. 224

We are to find how long it will take the train to reach the top of the slope, distance s_1 from the station; what velocity the train will have when it reaches a level plain at the summit; what the draw-bar tension must be on that level plain to maintain uniform velocity along the level plain; and what the tensions T_1 and T_2 are, on the way up, and later on the plateau.

This problem may sound complicated, but it consists of steps each of which is simple and easy. There are, of course, two epochs. Take first the whole train as a unit.

Epoch I. $$F_1 = P - (W_1 + W_2 + W_3) \times \tfrac{1}{100} - (f_1 W_1 + f_2 W_2 + f_3 W_3)$$

$$a_1 = \frac{F_1 g}{W_1 + W_2 + W_3} = \frac{P_1 - \frac{1}{100}\Sigma W - \Sigma(fW)}{\Sigma W} g$$

$$t_1 = \sqrt{\frac{2s_1}{a_1}}$$

$$v_1 = \sqrt{2a_1 s_1} = a_1 t_1$$

Now, consider the condition of the car W_1. We know its acceleration, a_1, as a member of the train. We can write the F of W_1 as follows:

$$F_1' = T_1 - W_1 \tfrac{1}{100} - f_1 W_1$$

$$\frac{a}{g} = \frac{T_1 - W_1 \frac{1}{100} - f_1 W_1}{W_1}$$

Now since $\frac{a}{g}$ is the same here as above, we have

$$\frac{T_1 - W_1(\frac{1}{100} + f_1)}{W_1} = \frac{P_1 - (\Sigma W)\frac{1}{100} - \Sigma(fW)}{\Sigma W}$$

and T_1 is readily found.

We can find T_2 by considering the condition of the train of two cars, W_1 and W_2, moving with an acceleration already known; or we may consider the condition of W_3 alone, as follows:

$$F_1'' = P - T_2 - W_3(\tfrac{1}{100} + f_3)$$

and solve for T_2, after equating the acceleration of W_3 to that of the train.

The draw-bar tension while the train is moving on the level plateau, is found from the equation for F_2 (which must be zero).

$$F_2 = 0 = P_2 - \Sigma(fW)$$

$$P_2 = f_1 W_1 + f_2 W_2 + f_3 W_3$$

As each car has uniform motion, it is moving under balanced forces: Accordingly

$$T_1 = f_1 W_1$$
$$T_2 = f_1 W_1 + f_2 W_2$$

220. The following ideal problem illustrates some very important points:

Problem 4. A body weighing W_1 pounds, rests on a smooth horizontal plane. A cord, strong, flexible and weightless, connects W_1 with W_2, as shown in the figure (Fig. 225), in its course passing over a *smooth* guide at G. Another cord, T_0, connecting W_1 with a fixed support at A, prevents motion. While thus arranged, the tension in the two cords is the same, namely: $T_0 = T = W_2$, since there is no friction, and the weight of W_1 is completely balanced by the *lift* of the plane.

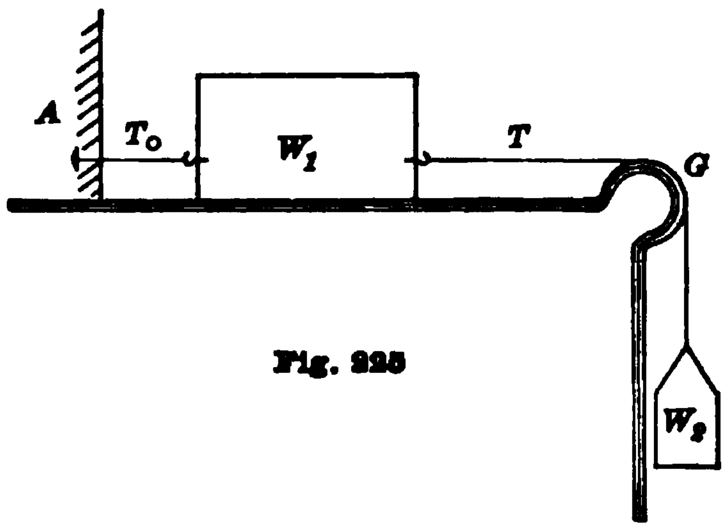


Fig. 225

Now, suppose the anchor cord at A is burned off by an electric spark. What is the *immediate* value of the acceleration, and the tension T?

Solution. The two bodies thus connected form a train, and the motor is the earth's pull upon W_2. As there is no friction, and no force acting against the motion, and unbalanced force F is

$$F = W_2$$

and hence
$$\frac{W_2}{W_1+W_2} = \frac{a}{g},$$

and a is found. We can find T by considering the motion of either W_1 or W_2. Taking W_2, we have the unbalanced force acting

$$F_2 = W_2 - T$$

Hence
$$\frac{W_2 - T}{W_2} = \frac{a}{g}$$

equating the two expressions for $\frac{a}{g}$, we have

$$\frac{W_2}{W_1+W_2} = \frac{W_2 - T}{W_2}$$

hence
$$T = \frac{W_1 W_2}{W_1 + W_2}.$$

Or, taking the condition of W_1, we have

$$F=T$$

$$\frac{a}{g}=\frac{T}{W_1}=\frac{W_2}{W_1+W_2}$$

$$T=\frac{W_1W_2}{W_1+W_2} \text{ as before.}$$

If numerical values are given to W_1 and W_2, the student will more fully realize that T must be less than W_2 if the train starts and maintains an accelerated motion. Thus if

$$W_1=W_2=100 \text{ lbs.}$$
$$T=50 \text{ lbs.}$$

The *instant* the spark burns the anchor cord, the tension in the other cord drops from 100 lbs. to 50 lbs., and both bodies start with an acceleration which is $\frac{g}{2}$, and remains so.

221. No body can be started in any direction without an unbalanced force. No body can be lifted from the floor to a table without a lift *in excess of its weight*; the excess depends on the magnitude of the acceleration given. If P is the lift, the unbalanced force which starts a body from the floor to the table is

$$P-W$$

and the initial acceleration is

$$a=\frac{P-W}{W}g \text{ and } P=W+\frac{W}{g}a=W+ma.$$

The first term in the last member is just to balance the pull of the earth; the second term is *mass×acceleration*; this second term varies with the magnitude of the action, and the quickness of the "start." If the start is what we call "quick," as by a "yank" or "jerk," the values of a and ma are large; in fact, ma may be much larger than W. When an elevator starts up with a quick jerk, the tension in the wire cable may be many times the weight of the car and its contents.

222. Problem 5. A train may consist of two bodies, A and B, connected by an ideal cord over a *smooth* plug or guide, as shown. (A smooth plug is preferred to a pully with frictionless bearings, because it takes *an unbalanced couple* to turn the pulley, and that brings in a principle not yet considered.) See Fig. 226.

Initially, the heavier body rests on a shelf, and the cord thruout its length sustains a tension, W_2. The two bodies (the cord and the earth) act on B, and balance; three bodies (cord, shelf and the earth) act on A, and they balance; (How much does the shelf act?). All these forces are acting but there is no motion; it is a case of static equilibrium.

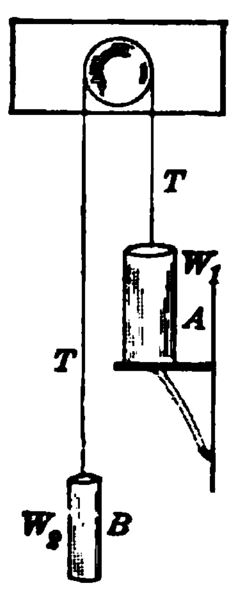


Fig. 226

Now, knock out the shelf. Instantly motion begins, the train starts. A moves down and B moves up. These motions indicate the directions in which the two unbalanced forces act.

The F_1 for A is $W_1 - T$;

The F_2 for B is $T - W_2$;

the accelerations caused are of equal magnitude; hence

$$\frac{a}{g} = \frac{W_1 - T}{W_1} = \frac{T - W_2}{W_2} = \frac{W_1 - W_2}{W_1 + W_2}$$

Hence $$T = \frac{2W_1 W_2}{W_1 + W_2}$$

Having found a, all questions as to v, s and t are readily answered.

The third expression for $\frac{a}{g}$, above, can be found by considering the train *as a whole*. Let $W_1 = 6$ lbs. and $W_2 = 3$ lbs., and note that one F_1 is twice as large as F_2, as it clearly *ought* to be.

Problem 6. Fig. 227.

Find expressions for a and T.

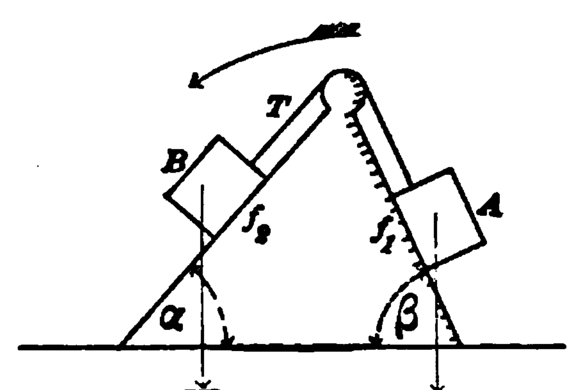


Fig. 227

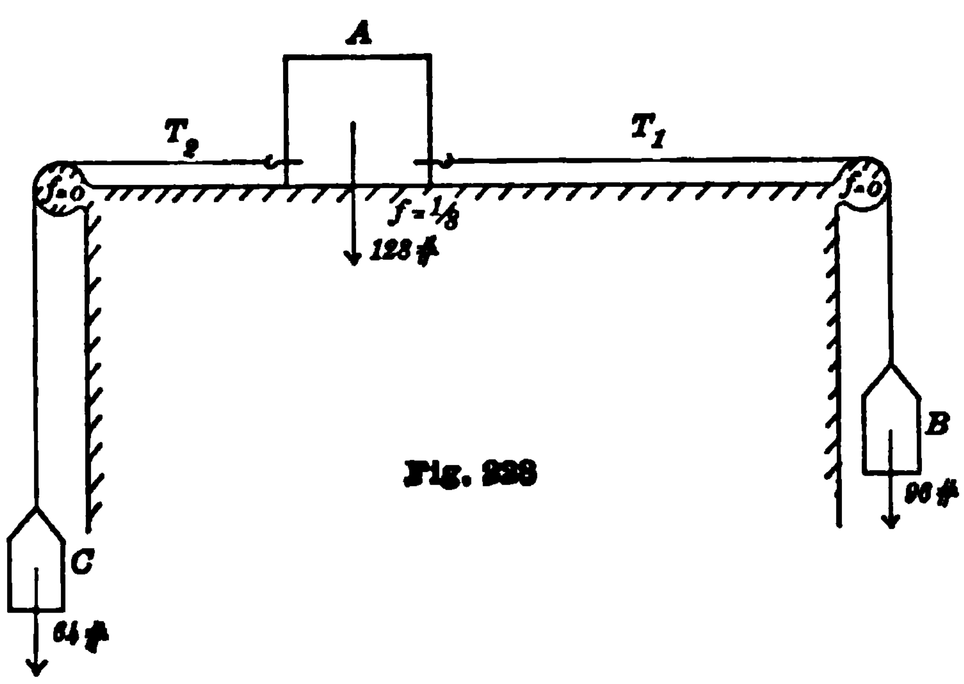


Fig. 228

Problem 7. A body, A, is clamped upon a rough horizontal plane; two other bodies, B and C, are suspended by ideal cords connected to A as shown. Fig. 228.

The cords are ideally strong and imponderable; $g = 32$; the co-efficient of friction for A and the plane if $\frac{1}{8}$.

Epoch I. A is unclamped and the train moves 10 sec.; then the cord to B is cut.

Epoch II. A and C continue motion till they stop.

Epoch III. A and C move in the opposite direction for 8 sec.; then the cord to C is cut.

Epoch IV. A moves till stopped by friction.

How far is A from its original position? and in which direction? Find a value for T_2 for each epoch, I, II, and III.

225. The angle of minimum draft. Closely related to problems of motion are those involving balanced forces acting upon bodies which are *upon the point of starting*.

When a body, weighing (with the load it carries) W, rests on a rough horizontal plane, what is the least lateral force that can move it?

THERE ARE THREE CASES. I. If the lateral force is horizontal, $P = f_1 W$ where f_1 is the co-efficient of *starting* friction; after the start, P_1 will continue to move the body if it equals fW, where f is the co-efficient of *sliding friction.*

CASE II. If P acts upward with an inclination θ, one component, $P \sin\theta$, tends to lift the body (thereby diminishing N, the normal action between body and plane), and the other component, $P\cos\theta$, must be on the point of overcoming the starting friction, hence

$$P\cos\theta = (W - P\sin\theta)f_1$$

$$P = \frac{f_1 W}{\cos\theta + f_1 \sin\theta}$$

It thus appears that P depends not only on f_1 but on θ, and that it varies with θ.

Let us find the value of θ, for which P is a minimum. Differentiating with respect to θ, and remembering that $dP = 0$ when P is a minimum, we have

$$P(f_1 \cos\theta - \sin\theta) = 0$$

or

$$\tan\theta = f_1$$

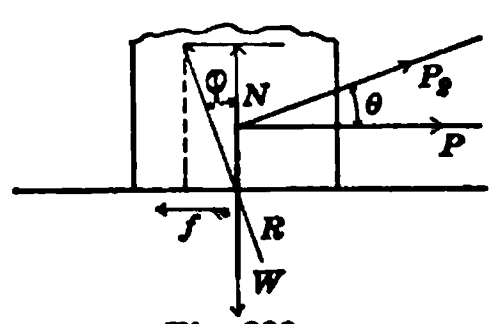


Fig. 229

That is to say, $\theta = \text{arc}\ \tan f_1 = \phi$, the "angle of repose," or the "angle of friction." This result is shown graphically by drawing the diagonal, R, of W, and $f_1 W$, and noting that the line of action of P when it is a minimum is perpendicular to R.

If we substitute for ϕ for θ, and $\tan\phi$ for f_1 in the denominator, we get the minimum force

$$P_2 = \frac{\tan\phi\, W}{\cos\phi + \dfrac{\sin^2\phi}{\cos\phi}} = W\sin\phi \qquad (3)$$

so both the magnitude and the direction of the *least* force are found.

The practical importance of this result can hardly be over-estimated. Every experienced teamster knows, in a general way, that on rough ground it helps to pull *up* as well as *forward*, and the *rougher the greater the proper inclination.* Some times when the traces are inclined, a driver, on striking a stretch of "heavy road," will shorten the traces so as to increase the inclination, strictly in accord with theory.

CASE III. When θ is negative or measured below the horizontal, the normal action is increased, and $P\cos\theta=(W+P\sin\theta)f_1$

or
$$P=\frac{f_1 W}{\cos\theta-f_1\sin\theta}$$

from which we see, not only that P, to start the body, is greater than f_1W, but that if $\theta=\frac{\pi}{2}-\phi$, $P=\infty$; which means: if P acts in a line parallel to the resultant reaction of the plane, it will not be able to move the body along the plane, however large it (P) may be.*

226. Problems of quickest descent. Let OAB, Fig. 230, be a circle in a vertical plane, with O the topmost point. Prove that the time of descent of all bodies sliding from O down *smooth chords* to the circumference is the same.

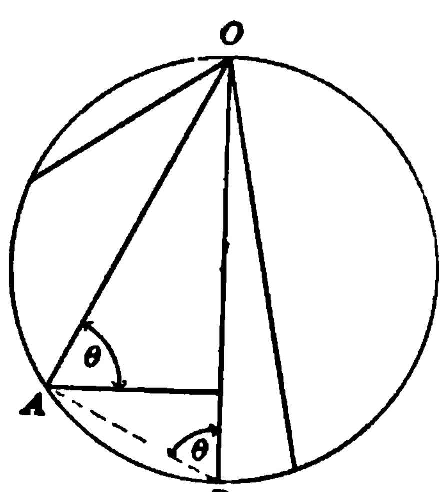


Fig. 230

Take the chord, OA, whose inclination is θ. Here $s=2r\sin\theta$, and $a=g\sin\theta$, hence $t=2\sqrt{\frac{r}{g}}$ which is independent of θ; and hence the time is the same for all chords, *i. e.*, that of a free fall from O to B.

APPLICATIONS. 1. **Without friction.** If water flows, or bodies slide (not roll) in a "*smooth*" and straight tube, trough or spout, to meet a given inclined plane (as from O to the given plane AB), Fig. 231, in the *shortest time*, how is the tube or spout located?

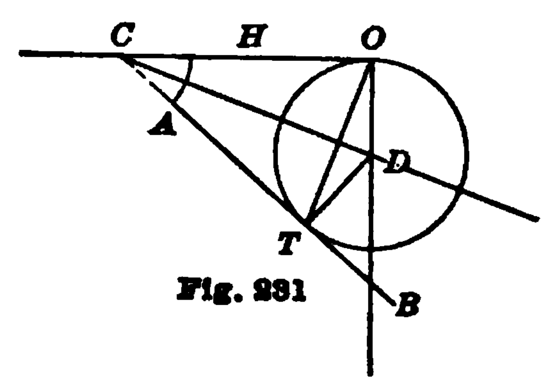


Fig. 231

Draw OH horizontal, OD vertical; produce BA to C; bisect the angle at C to D; draw DT perpendicular to AB, and OT is the spout. Prove it.

The simpler construction is to draw from O a line perpendicular to the bisector of the angle C; it will be the quickest straight line descent, if there be no friction.

* The "jamming" of gears with rough ungreased teeth, too few in number, is quite analogous to this.

2. If a body is to slide or flow along a smooth straight guide from a given point to the circumference of a given circle, determine the position of the guide.

If O is without (or within) the circle: Lay off from O, vertically, up (or down) the length of the radius r. Draw OT parallel to RC. OT is the required guide.

The proof is left to the student, as is the indicated construction when O is *inside* the circle C. See Fig. 232.

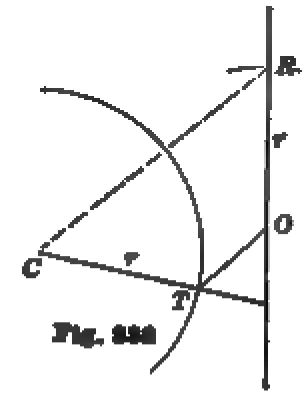


Fig. 233

3. With friction. If, in Problem 1, there be friction, and ϕ be the Angle of Repose, draw OC with an inclination ϕ, and draw OT perpendicular to the bisector of C as before. Prove this.*

4. If, in Problem 2, there be friction, with ϕ as the angle of repose, the line, OR, should be laid off up (or down) according as it is without (or within) the circle, making an angle ϕ with a vertical on the right (or the left) of the vertical, as the proposed position of T is to be on the left (or the right). The line, OT, is always parallel to CR. This construction should be proved. Fig. 234.

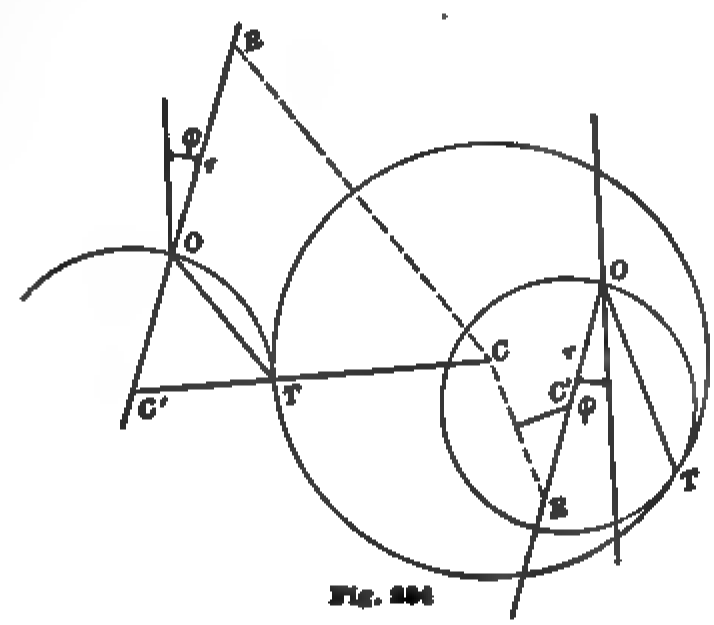


Fig. 234

* While the proof of this construction is simple, it may be best to give it as a last resort of the student who fails to hit upon it by himself. Fig. 233.

Taking the construction as given in the text draw from O a perpendicular to OC till it cuts the bisector in D. With D as a center draw the circle OT. Take now any chord OP making with the line OH an angle θ less than 90°. Now we will find the time required for sliding down this chord to the circle at P. The length of the chord is

$$s = 2r\sin(\theta-\varphi) = 2r\,(\sin\theta\cos\varphi - \cos\theta\sin\varphi)$$

The acceleration of the sliding body is

$$a = (\sin\theta - \cos\theta\tan\varphi)\,g$$

Hence we have for the square of the time

$$t^2 = \frac{2s}{a} = \frac{4r}{g}\cdot\frac{\sin\theta\cos\varphi - \cos\theta\sin\varphi}{\sin\theta\cdot - \cos\theta\tan\varphi}.$$

$$t^2 = \frac{4r\cos\varphi}{g} = \frac{2\cdot OE}{g} = \frac{2h}{g}$$

It thus appears that the time is independent of θ and just equal to the time of for a free fall from O to E. As the time for all chords is the same, it is evident that the chord OT is the path for the shortest time to AB.

227. Cautions. When the student takes up *Mechanism*, he will find many instances of motion and unbalanced forces, which will require the use of the principles of this chapter. Motion, both translation and rotation, of machinery must be discussed, but he is not yet fully prepared for such discussions.

The most difficult step, in the solution of problems like those of this chapter, is the finding of correct expression for the unbalanced force F. In dealing with the weight of a body, the student must not forget that *weight is force*, and that when W, or any part of it, is balanced by another force, the *mass* of the body whose motion is under discussion, is not thereby affected. Again, if a body is in any way supported, the *support acts*, and its action (or force) is not to be neglected in making up F. Of course, when forces balance, they have no effect upon the motion of a solid body, tho they may effect internal stresses.

Examples. 1. Three weights, 7 lbs., 5 lbs. and 4 lbs., are connected by a strong thread which passes over a smooth peg as shown in Fig. 235. Starting from rest, the train moves 16 feet, when the thread between 5 lbs. and 4 lbs. is burned off. After a little, the motion of 5 lbs. and 7 lbs. is reversed. In how many seconds, counting from the start up, will the 7 lbs. return to the starting position?

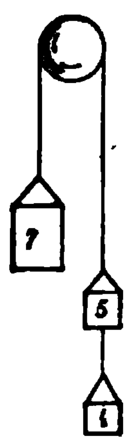


Fig. 235

2. A bucket of water is descending with an acceleration of $\frac{1}{2}g$. A block of marble, weighing in the air 4 lbs., is immersed in the water. Assuming that the specific weight of the marble is 2.4, what is the mutual action between the block and the bottom of the bucket during the descent?

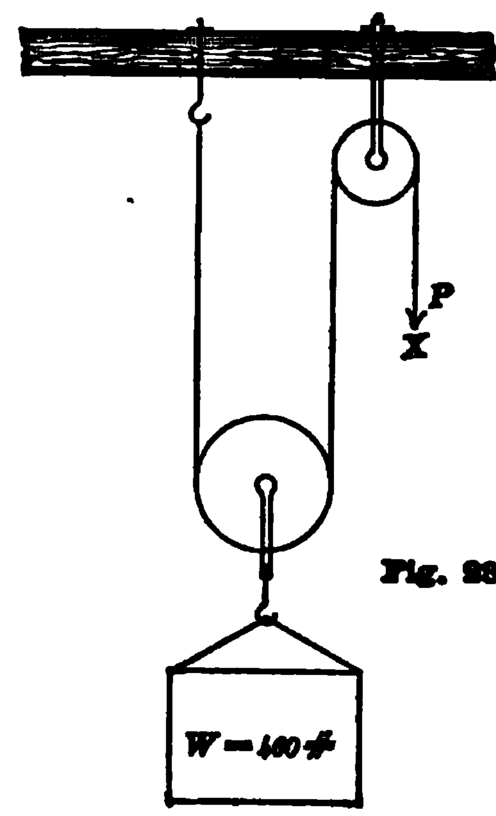


Fig. 235 (a)

3. A pull x at P must exceed what number of pounds to start $W = 460$ lbs. upwards, if friction at starting is one fourth of x? See Fig. 235a. Ans. $306\frac{2}{3}$ lbs.

CHAPTER XIV.

THE TRANSLATION OF BODIES UNDER THE ACTION OF VARYING UNBALANCED FORCES.

228. General formulas. The general definitions for velocity and acceleration, and two very important formulas of the last chapter, are the basis of all the work of this chapter. The equations are *necessarily*

differential, and the reader must never use the integrations of the last chapter in this chapter: he must always begin with the following:—

$$v=\frac{ds}{dt};\quad a=\frac{dv}{dt}=\frac{d^2s}{dt^2};\quad vdv=ads;$$

$$\frac{F}{W}=\frac{a}{g}\text{ or }F=ma.$$

in which F is always the *unbalanced force*, and $W=mg$ is the weight of the moving body. Since in every case in this chapter dF/dt is not zero, a is no longer constant, and *the integrations* of (**216**) *do not hold.*

The new methods to be employed will be best shown by the solution of some ideal problems.

229. An ideal chain (slender, smooth, flexible, but heavy) is stretched along a level plane with an over-hang, s_1; the entire length is l, and the weight per foot is w. A clamp C holds it in position until the initial conditions are examined.

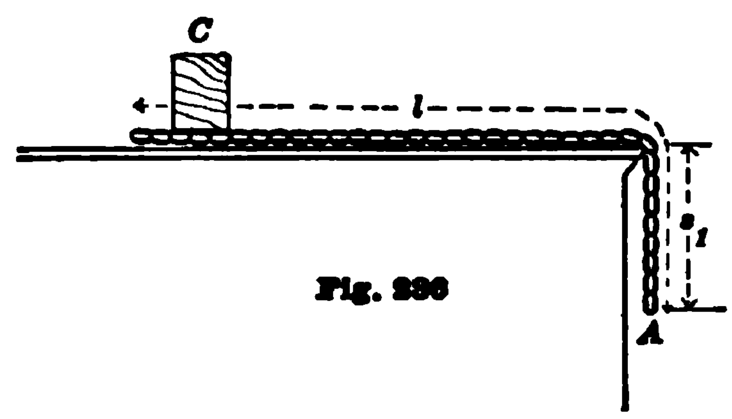


Fig. 236

The total weight of the chain is lw and its mass is $\frac{lw}{g}$.

When the clamp C is lifted, the chain will move, and the end over-hang will increase. The unbalanced force will always be the over-hanging portion, until the chain is wholly off. Don't write an equation for the condition until the body (the chain) is well in motion with a value of v, an elapsed time t, and a new overhang s.

The unbalanced force F under *current* conditions is

$$F=sw.$$

Hence
$$\frac{F}{W}=\frac{sw}{lw}=\frac{a}{g}\quad\therefore\ a=\frac{g}{l}s$$

The fundamental equation now to be used is $vdv=ads$, hence

$$\int_0^v vdv=\frac{g}{l}\int_{s_1}^{s} sds.$$

The limits must correspond, and the student should (mentally) say: "When v was zero, s was s_1, etc."

Integrating and simplifying,

$$v^2=\frac{g}{l}(s^2-s_1^2)$$

$$v=\sqrt{\frac{g}{l}}\,(s^2-s_1^2)^{\frac{1}{2}}$$

When $s=l$, the chain is off the platform, and enters a second epoch (with which we are not now interested) with a velocity

$$v_1=\sqrt{\frac{g}{l}}\,(l^2-s_1^2)^{\frac{1}{2}}$$

To find the time occupied by the motion, we must use the *current value* of v in the fundamental equation $v=\dfrac{ds}{dt}$.

$$\int_0^{t_1} dt=\int_0^{v_1}\frac{ds}{v}=\sqrt{\frac{l}{g}}\int_{s_1}^{l}\frac{ds}{(s^2-s_1^2)^{\frac{1}{2}}}$$

$$t_1=\sqrt{\frac{l}{g}}\log_e\left(s+(s^2-s_1^2)^{\frac{1}{2}}\right)\Bigg]_{s_1}^{l}$$

$$t_1=\sqrt{\frac{l}{g}}\log_e\frac{l+(l^2-s_1^2)^{\frac{1}{2}}}{s_1}$$

It is interesting to note that if $s_1=0$, we have $v_1=\sqrt{gl}$ and yet $t_1=\infty$.

Had the chain been in a loose coil near the edge of the platform, the motion, if once started, would have been more rapid. A heavy rope or chain thus let loose, even when there is friction, is very dangerous.

230. Harmonic motion. Suppose a heavy body resting on a smooth level surface between smooth guides, is connected with an (ideal) imponderable coiled spring which is capable of indefinite extension and compression, with a constant "force" p. The "force" or "stiffness" of a spring is that external force which will stretch (or compress) the spring one foot (or one inch); and if the spring is already stretched (or compressed), the force p will, if added to the external force, add one foot (or one inch) to the extension (or compression). The spring is shown in the figure in a stretched condition, the elongation NA being $c-s$; the position at N is called the "neutral position."

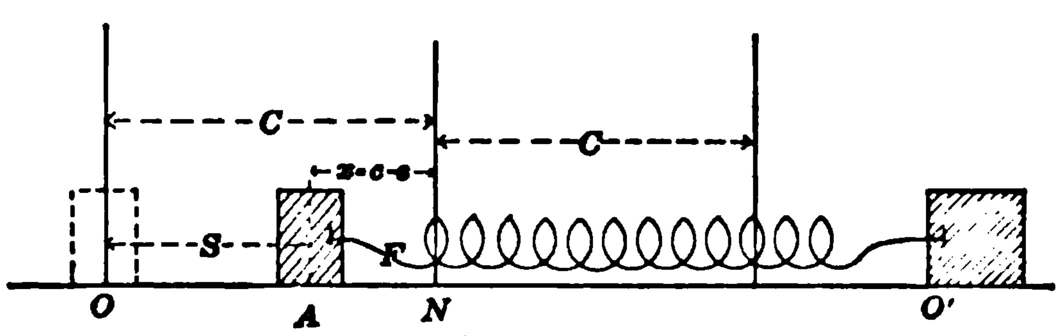


Fig. 237

The body was, by some external agency, drawn back to O, and then released. When it was at O the tension of the spring was cp. When released, the body started

towards N, and it is *now* in its *current position* having moved a distance s. The present value of F is

$$F=(c-s)p.$$

Hence
$$\frac{F}{W}=\frac{a}{g}\ \therefore\ a=\frac{(c-s)pg}{W}=\frac{(c-s)p}{m} \tag{1}$$

and
$$vdv=\frac{p}{m}(c-s)ds.$$

Integrating from $v=0$ and $s=0$ to current limits,

$$\frac{v^2}{2}=\frac{p}{m}\left(cs-\frac{s^2}{2}\right)$$

$$v=\sqrt{\frac{p}{m}}\,(2cs-s^2)^{\frac{1}{2}} \tag{2}$$

An examination of the above value of F shows that F is positive so long as $s<c$, and when $s=c$, $F=0$. When $s>c$, F and a are negative. The velocity v is therefore a maximum when $s=c$, so that the maximum value of v is

$$(Max)\ v=c\sqrt{\frac{p}{m}}=c\sqrt{\frac{pg}{W}}$$

We further find that $v=0$, when $s=2c$, which shows that the body will stop when the compression of the spring is equal to the original elongation. The instant it stops, it starts back, all conditions being reversed.

Under the assumed (imaginary or ideal) conditions, the body will continue to oscillate forming a "*straight pendulum.*"

The *time* of an oscillation will be found by integrating the general equation

$$\int_0^{t_1} dt=\frac{ds}{v}=\sqrt{\frac{m}{p}}\int_0^{2c}\frac{ds}{(2cs-s^2)^{\frac{1}{2}}}$$

$$t_1=\sqrt{\frac{m}{p}}\ \text{arc ver sin}\frac{s}{c}\Bigg]_0^{2c}=\pi\sqrt{\frac{m}{p}}=\pi\sqrt{\frac{W}{gp}} \tag{3}$$

It is seen that the time is proportional to the square root of the mass, inversely proportional to the square root of the stiffness of the spring, and *independent of c*, the "*amplitude*" of the oscillation. This property

---

* The formula for integration is $\int\frac{dz}{\sqrt{2kz-z^2}}=\text{arc ver sin}\frac{z}{k}$. If θ be the arc, then ver sin $\theta=\frac{z}{k}=1-\cos\theta$.

of *equal times* for all values of c gives the adjective "isochronous" to this pendulum and "harmonic" to the motion.

If the arc in the general equation for t_1 be called θ, we have as the general value of t:

$$t = \theta\sqrt{\frac{m}{p}} \tag{4}$$

θ being defined by the equation

$$\operatorname{ver}\sin\theta = \frac{s}{c} = 1 - \cos\theta$$

or

$$\cos\theta = \frac{c-s}{c}.$$

If a circle with radius c and center N, be drawn on the level plane, and an ordinate be drawn from any position of the oscillating body A, the intercepted arc determines θ, as shown in Fig. 238.

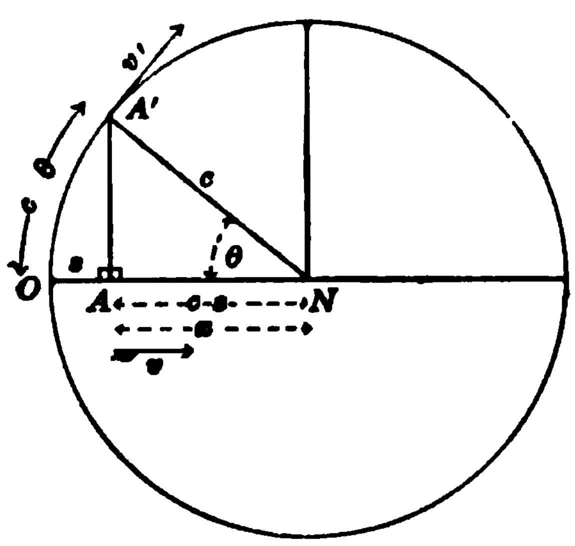


Fig. 238

When $s = 2c$, it is seen that $\theta = \pi$. Substituting for s the value $c(1-\cos\theta)$ in the values for a and v, and using the equations above, we have the following relations between the co-ordinates of the moving point A', and the position, acceleration, velocity and time of the body A:

$$\left.\begin{aligned}
s &= c\operatorname{ver}\sin\theta = c\,(1-\cos\theta)\\
ds &= c\sin\theta\, d\theta\\
t &= \sqrt{\frac{m}{p}}\,\theta\\
v &= \sqrt{\frac{p}{m}}\,c\sin\theta\\
a &= \frac{p}{m}\,c\cos\theta
\end{aligned}\right\} \tag{5}$$

If we differentiate equation (4), we have

$$d\theta = \sqrt{\frac{p}{m}}\,dt \text{ or } \frac{d\theta}{dt} = \sqrt{\frac{p}{m}} \tag{6}$$

which shows that θ *increases uniformly*; in other words, the point A' and the radius NA' go around the center N *at a uniform rate*. The "time-rate" of this angular motion is called its "*angular velocity*," and is quite generally represented by the Greek letter ω (omega),

and $$\omega = \frac{d\theta}{dt} = \sqrt{\frac{p}{m}}$$

If we let $x = c - s$, the distance from N, we have

$$x = c - s = c\cos\theta$$
$$dx = -c\sin\theta\, d\theta$$
$$d^2x = -c\cos\theta\,(d\theta)^2 = -x(d\theta)^2$$

but $d\theta = \omega dt$, hence we get at once

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

which is the differential equation of a Simple Harmonic Motion.

231. Vibration against constant friction. 1. Referring again to Fig. 237. Let the friction be fW. If s is the distance of the body A from O, and A is moving towards N so that ds, v and a are positive, we have

$$F = p\,(c - s) - fW$$
$$a = \frac{(-fW + p\,(c - s))}{m}$$
$$vdv = \frac{p}{m}\left(c - s - \frac{fW}{p}\right) ds$$
$$v^2 = \frac{p}{m}\left[\left(2c - \frac{2fW}{p}\right) s - s^2\right] \tag{1}$$

The velocity v is zero, not only when $s = 0$, but when

$$s = 2c - \frac{2fW}{p} \tag{2}$$

which shows that compared with the former case when $f = 0$, the double amplitude has been diminished by the quantity $\frac{2fW}{p}$, *which is independent of c*; this shows that the return oscillation starts at a less distance from the neutral point, and as the shortening of the amplitude is independent of c, the second oscillation is *shortened by the same amount*; hence, it will begin its third oscillation at the distance $c - \frac{4fW}{p}$ from N; and at the end of the nth oscillation it will stop with a central distance $c - \frac{2nfW}{p}$. When $c - \frac{2nfW}{p} < \frac{2fW}{p}$ the body will not start again.

2. To find t_1 we integrate $\int_0^{t_1} dt = \int_0^{2c-\frac{2fW}{p}} \frac{ds}{v}$ and have

$$t_1=\sqrt{\frac{m}{p}}\int_0^{2c-\frac{2fW}{p}}\frac{ds}{\left[\left(2c-\frac{2fW}{p}\right)s-s^2\right]^{\frac{1}{2}}}$$

$$t_1=\sqrt{\frac{m}{p}}\,\text{arc ver sin}\left.\frac{s}{c-\frac{fW}{p}}\right]_0^{2\left(c-\frac{fW}{p}\right)}=\pi\sqrt{\frac{m}{p}} \qquad (3)$$

which is identical with the time when $f=0$. Which shows that the *time* of oscillation is the same with friction as without it.

232. Ideal problem. A body weighing W lbs is attached to a spring balance, whose force (or stiffness) is p, Fig. 239. The body is partly supported by the spring which lifts ps_1, and partly by the shelf, $W-ps_1$.

The shelf is *suddenly* knocked away and the body descends. The space passed over is best measured from the shelf line. Consider the condition after a time t has elapsed. The distance moved is now s. The *unbalanced* force which causes the motion *now* is

$$F=W-p(s_1+s)$$

hence
$$a=\frac{W-p(s_1+s)}{m}$$

Fig. 239

$$vdv=a\,ds$$

$$\int_0^v vdv=\int_0^s\frac{1}{m}\left[W-ps_1-ps\right]ds$$

$$\frac{v^2}{2}=\frac{1}{m}\left[(W-ps_1)\,s-\frac{ps^2}{2}\right]$$

$$v=\sqrt{\frac{p}{m}}\left[2\left(\frac{W}{p}-s_1\right)s-s^2\right]^{\frac{1}{2}}$$

It is now seen that $v=0$, when $s=2\left(\frac{W}{p}-s_1\right)=c$. This measures the distance c, which the *weight will descend*, and then *stop* and *return*.

The time of fall is found by integrating the equation $dt = \dfrac{ds}{v}$

$$\int_0^{t_1} dt = \int_0^{c} \frac{ds}{v} = \sqrt{\frac{m}{p}} \int_0^{c} \frac{ds}{\left(2\left(\frac{W}{p} - s_1\right)s - s^2\right)^{\frac{1}{2}}}$$

$$= \sqrt{\frac{m}{p}} \text{ arc ver sin } \left.\frac{s}{\frac{W}{p} - s_1}\right]_0^c$$

$$= \sqrt{\frac{m}{p}} \cdot \pi$$

Hence
$$t_1 = \pi\sqrt{\frac{m}{p}} = \pi\sqrt{\frac{W}{pg}}$$

which is the same as for the horizontal spring, and is independent of s_1. All this means that, under ideal conditions, we have here a vertical, isochronous pendulum.

The interesting point is the maximum tension in the spring. The tension equals p times the total stretch.

$$T = p(s + s_1)$$

When s is a maximum $= 2\left(\dfrac{W}{p} - s_1\right)$

$$T = 2W - ps_1$$

If now $s_1 = 0$, which means that there is no initial stretch in the spring (that is, the shelf is raised s_1 or the spring-balance is lowered so that the index stands at zero when the shelf is knocked away), the total descent of W is $2\dfrac{W}{p}$, and the tension in the spring at the lowest point is

$$T_1 = 2W.$$

This means that when a load (weight W) is suddenly placed upon, or hung upon (without a preliminary drop), an unloaded (and imponderable) spring, the tension or load pressure produced at the end of an oscillation is equal to $2W$.

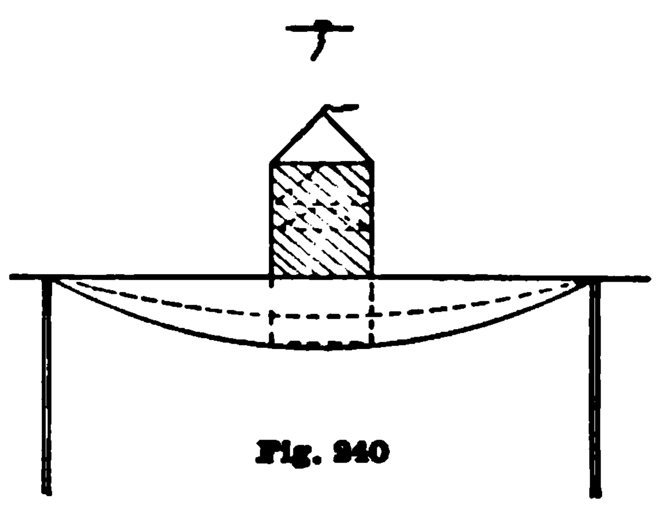


Fig. 240

233. A horizontal elastic bar supported at the ends is a light spring, tho not imponderable. Fig. 240. A heavy weight, suspended by a cord, and in con-

tact with, but not pressing upon, the bar at the center, is suddenly released and the bar is made to spring down and up. If the *first descent* is automatically recorded, it will be found to be *nearly* double the static deflection produced when the bar and weight are at rest. A heavy weight on a spring balance which has a long spiral would perhaps show a nearer approximation to $\frac{2W}{p}$.

Remarks. The problems involving a spiral spring were called ideal, because no account was taken of the mass of the spring itself. Still, it is evident that it is prudent in practice to place heavy bodies upon springs *gradually* and not suddenly. For example, if a man is to walk a plank which seems barely strong enough to support his weight, he should move *along slowly*. Engineers driving heavy locomotives over weak bridges are told to *slacken speed*. The greatest tax upon the bridge comes, not when the locomotive is in the center, but later when the heavy machine is being *lifted quickly* to the top of the far abatement.*

234. Damped vibration. Referring once more to the horizontal spring, Fig. 237, instead of a *constant* retarding force like friction, assume that the retarding force is proportional to the velocity, say, kv. If the vibrating body started from a point distant c from the neutral center, and has moved the distance $c-x=s$, in general, so that $ds=-dx$, we have

$$F=px-kv \tag{1}$$

in which v is multiplied by k to produce the retarding force.

$$\frac{F}{W}=\frac{a}{g} \therefore a=\frac{px-kv}{m}=\frac{d^2s}{dt^2}=-\frac{d^2x}{dt^2} \tag{2}$$

Substituting for v its equal, $\left(-\frac{dx}{dt}\right)$, we have the differential equation

$$\frac{d^2x}{dt^2}+\frac{k}{m}\cdot\frac{dx}{dt}+\frac{px}{m}=0 \tag{3}$$

* The oft-quoted argument:—that a rapid skater who goes over thin ice with safety, while the slow skater breaks thru, thus proves the contrary of what is said in the text,—is not relevent to the case of a plank or bridge supported at the ends. The ice has a continuous support and it cannot deflect without moving a large mass of water. It takes *time* to sensibly move large masses of water by a gentle pressure, and it requires a heavy pressure to move it quickly, as one can prove by striking the water with the flat surface of a paddle.

The reader will recall the shrewd reply of George Stephenson to the M. P. who questioned the strength of a rail under the wheel of a *six-ton locomotive* moving at the high rate of *12 miles per hour!* The reply was fallacious, but it *worked.* See Smiles's Life of Geo. Stephenson.

For the general integral of this equation, the student is referred to a later study (which he ought not to neglect), viz.: Differential Equations. He will find the integral in this form:

$$\left.\begin{aligned} x &= A\epsilon^{h_1t}+B\epsilon^{h_2t} \\ \text{in which}\quad h_1 &= \frac{-k+\sqrt{k^2-4pm}}{2m} = -\mu+\sqrt{\mu^2-b^2} \\ h_2 &= \frac{-k-\sqrt{k^2-4pm}}{2m} = -\mu-\sqrt{\mu^2-b^2} \end{aligned}\right\} \tag{4}$$

where for brevity $\mu=\frac{k}{2m}$ and $b^2=\frac{p}{m}$. The quantities A and B are constants of Integration to be determined by initial conditions. There are three cases.

Case I. When, as generally would be the fact, $k^2<4pm$, the radical involves $\sqrt{-1}$ as μ^2 is less than b^2.

In this case, let $\sqrt{b^2-\mu^2}=\beta$, then the general integral takes the form

$$x=\epsilon^{-\mu t}\Big(A\cos(\beta t)+B\sin(\beta t)\Big). \tag{5}$$

To find A, we know that when $t=0$, $x=c$, in (4) and (5)

hence $$A=c$$

We also know that when $t=0$, $v=0$.

Differentiating (5)

we get
$$\frac{dx}{dt}=-\mu\epsilon^{-\mu t}\Big[A\cos(\beta t)+B\sin(\beta t)\Big]+$$
$$\epsilon^{-\mu t}\Big(-A\,\beta\sin(\beta t)+B\,\beta\cos(\beta t)\Big) \tag{6}$$

and when $t=0$, this becomes.

$$o=-\mu A+B\beta$$

hence
$$\left.\begin{aligned} B &= \frac{c\mu}{\beta}=\frac{ck}{2m\sqrt{b^2-\mu^2}} \\ B &= \frac{ck}{\sqrt{4pm-k^2}} \end{aligned}\right\} \tag{7}$$

$$A=c.$$
$$\mu=\frac{k}{2m}$$
$$\beta=\frac{\sqrt{4pm-k^2}}{2m}$$

The oscillations are isochronous, but the amplitudes diminish at a diminishing rate and the body comes to rest at the center only after an infinite time.

The first oscillation is completed in the time

$$t=\frac{\pi}{\beta}=\frac{\pi\cdot 2m}{\sqrt{4pm-k^2}}, \qquad \text{when } x=-A\epsilon^{\frac{-\mu\pi}{\beta}}. \tag{8}$$

The first amplitude will equal

$$c+(-x)=c(1+\epsilon^{\frac{-\mu\pi}{\beta}}) \tag{9}$$

CASE II. If the *radical is real*; that is, if $\mu^2>b^2$, the integral as given is in proper form. Eq. (4).

Since $x=c$ when $t=0$, the equation gives

$$c=A+B.$$

When $v=0$, that is when $-\frac{dx}{dt}$ is zero, and $t=0$, we have

$$o=Ah_1\epsilon^{h_1t}+Bh_2\epsilon^{h_2t}$$

$$o=Ah_1+Bh_2$$

Hence

$$\left.\begin{aligned} A&=c\frac{h_2}{h_2-h_1} \\ B&=c\frac{h_1}{h_1-h_2} \end{aligned}\right\} \tag{10}$$

There is no oscillation, as v becomes zero only when $t=\infty$.

CASE III. When the radical is zero; that is, when $k^2=4pm$. This is what is known as a case of equal roots, and the general solution is

$$x=\epsilon^{\frac{-kt}{2m}}(At+B) \tag{11}$$

To find B, we let $t=0$, and $x=c$,

whence $B=c.$

To find A, we put $\frac{dx}{dt}=0$ and let $t=0$

$$o=-\frac{k}{2m}\epsilon^{\frac{-kt}{2m}}(At+c)+A\epsilon^{\frac{-kt}{2m}}$$

$$A=\frac{ck}{2m}.$$

Hence $$x = c\epsilon^{\frac{-kt}{2m}}\left(\frac{kt}{2m}+1\right) \tag{12}$$

There is here no oscillation as the body comes to rest only when $t=\infty$.

In every case, since all constants are now known, and since $v=-\frac{dx}{dt}$, the quantities v, s and x are found for any value of t.

235. Motion of translation in a circular path. It was shown **208** that a solid body could have a motion of translation tho every point in it described a circle. If the paths were circles, no two points had the same circle, and all circles were equal. In the present case, we will take the best representative point of a solid body, viz.: its center of gravity, as tho the whole mass of the body were concentrated there.

Now, no such body can be moving in a circle under balanced forces: there must always be a resultant unbalanced force.

We will first suppose that the C. G. of the body is forced to move in the arc of a vertical circle by the joint action of gravity, and a *smooth* circular guide. See Fig. 241. The body (that is the point C) was originally at C_1, and the ideal radius C_1O, made an angle α with the vertical OA. The body as represented is sliding down the guide with a *velocity in its path* of v. (The body must *not roll*, it must *slide*. The guide is *smooth*; friction would make it roll. Rolling bodies do not have motions of translations alone; they combine rotation with translation, a subject not yet discussed in this book. We have as yet no equations which apply to rolling or rotating bodies. In this case the point H on the top of the body is always on top.)

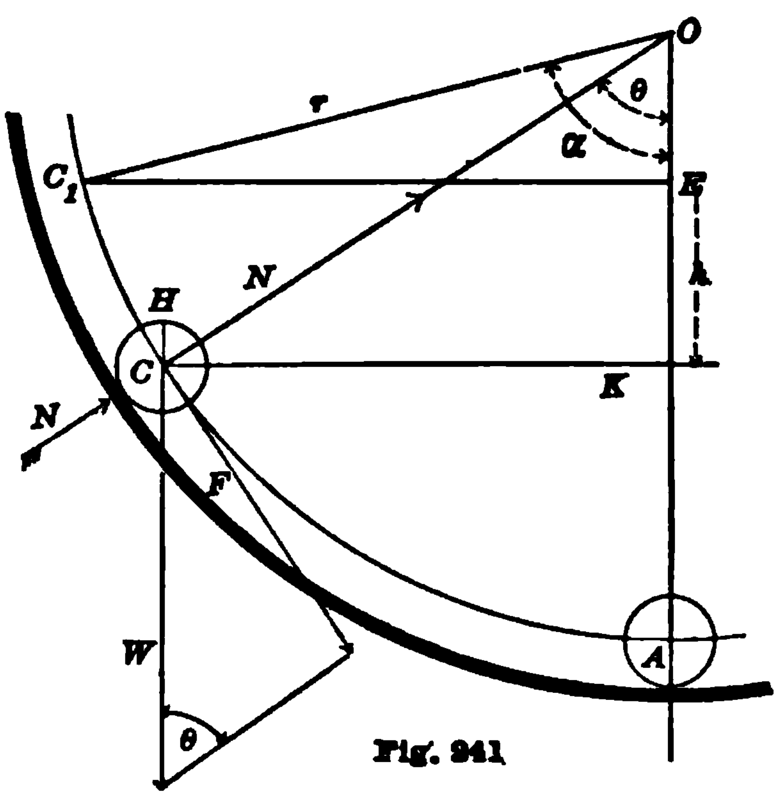


Fig. 241

Two forces act on the body: W down, and N (the action of the guide) towards O. The tangential component of W is $F=W\sin\theta$, the *unbalanced force* in the direction of its motion. The unbalanced portion of the guide's action does not affect the velocity.

Hence $$\frac{W\sin\theta}{W}=\frac{a}{g}$$

$$a=g\sin\theta$$

$$vdv=g\sin\theta\, ds$$

Now, ds is an element of the arc CA, and it subtends a differential of θ at a unit's distance from O. But $d\theta$ is negative; hence $ds = -rd\theta$, and

$$vdv = -gr \sin\theta \, d\theta$$

The limits of integration for v are: o and v; for θ they are a and θ; hence

$$\frac{v^2}{2} = gr\cos\theta\Big]_a^\theta = gr(\cos\theta - \cos a)$$

$$v = \sqrt{2g}\,(r\cos\theta - r\cos a)^{\frac{1}{2}} \qquad (1)$$

$$v = \sqrt{2g}\,(OK - OE)^{\frac{1}{2}}$$

$$v = \sqrt{2gh}, \text{ if } h = OK - OE. \qquad (2)$$

from which it appears that the velocity of the sliding body is (under the ideal conditions) due solely to the vertical descent of the body. When the body reaches A, its velocity will be

$$v = \sqrt{2g.EA} = \sqrt{2gr(1-\cos a)}$$

just as tho the body had had a free fall (in the vacuum) from C_1 to the level of A. The time, however, is very different.

We find the time of descent by integrating the eq.

$$\int dt = \int \frac{ds}{v} = \frac{1}{\sqrt{2gr}} \int \frac{ds}{(\cos\theta - \cos a)^{\frac{1}{2}}} = -\sqrt{\frac{r}{2g}} \int \frac{d\theta}{(\cos\theta - \cos a)^{\frac{1}{2}}} \qquad (3)$$

236. It is easy to get **an approximate value of** t as θ changes from a to zero, which is very accurate for small values of a; but it is a little difficult (by means of an integration into a series) for larger values of a up to π.

Develop $\cos\theta$ and $\cos a$ by McLauren's Theorem

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{\underline{|4}} - \text{etc.}$$

$$\cos a = 1 - \frac{a^2}{2} + \frac{a^4}{\underline{|4}} - \text{etc.}$$

$$(\cos\theta - \cos a) = \frac{1}{2}(a^2 - \theta^2) - \frac{1}{24}(a^4 - \theta^4) + \text{etc.}$$

When a is small, we may neglect, when we substitute, all after the first parenthesis. Substituting, we have

$$t = -\sqrt{\frac{r}{g}} \int_a^\theta \frac{d\theta}{(a^2 - \theta^2)^{\frac{1}{2}}} \qquad (4)$$

To get the time of descent to A, we integrate from $\theta=a$ to $\theta=0$.

$$t_1=-\sqrt{\frac{r}{g}}\,\text{arc}\sin\frac{\theta}{a}\Big]_a^0=\frac{\pi}{2}\sqrt{\frac{r}{g}}.$$

If the guide is continuous beyond A, the body will be stopped by the negative value of the unbalanced force, $W\sin\theta$.

$$v=0 \text{ when } \cos\theta=\cos a$$

which gives two values of θ, viz.:

$$\theta=a \text{ and } \theta=-a$$

The value was $+a$ at the start, and will be $(-a)$ at the stop. Hence to find the time from start to stop, we give the limits $+a$ and $-a$.

$$t_1=-\sqrt{\frac{r}{g}}\,\text{arc}\sin\frac{\theta}{a}\Big]_{+a}^{-a}=\sqrt{\frac{r}{g}}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]=\pi\sqrt{\frac{r}{g}} \qquad (5)$$

The guide in Fig. 241 may be replaced by an imponderable inelastic cord CO, thereby forming a simple pendulum, provided we neglect the slight rotation of the moving body; and (5) is the approximate formula for a "*simple pendulum*" which is an ideal heavy ball *without size*, and an imponderable radius rod r swinging in a vacuum. The tension in the rod replaces the normal action of the smooth guide.* As the size of the initial angle does not appear in the value of t_1, it can be *any very small angle*, without sensibly affecting t_1.

237. To derive a more accurate value of t_1 for any value (less than π) of a, we proceed as follows:

Taking again the equation

$$t=\frac{1}{\sqrt{2g}}\int\frac{-r\,d\theta}{(r\cos\theta-r\cos a)^{\frac{1}{2}}}$$

we put

$$\theta=2\phi$$
$$d\theta=2d\phi$$
$$\cos\theta=1-2\sin^2\phi$$
$$r-r\cos a=h=EA. \quad \text{Fig. 241.}$$

* While the motion of a simple pendulum is not strictly translation, yet as the heavy ball was assumed to be "without size," its *angular motion* while oscillating may be neglected. The simple pendulum is a purely ideal thing; there are no real simple pendulums. All pendulums are *compound* tho they may approximate the "simple". See Chapter XV for the Compound Pendulum.

Then
$$t = -\frac{1}{\sqrt{2g}}\int\frac{2r\,d\phi}{(h-2r\sin^2\phi)^{\frac{1}{2}}}$$

$$= -\sqrt{\frac{r}{g}}\int\frac{d\phi}{\left(\frac{h}{2r}-\sin^2\phi\right)^{\frac{1}{2}}} \tag{6}$$

If we write $k^2=\frac{h}{2r}$, so that $k=\sin\frac{a}{2}$; and make $\sin\phi = k\sin\beta$ we have
$$d\phi = \frac{k\cos\beta\,d\beta}{\cos\phi}$$

$$t = -\sqrt{\frac{r}{g}}\int\frac{k\cos\beta\,d\beta}{\cos\phi\;k\cos\beta}$$

$$t = -\sqrt{\frac{r}{g}}\cdot\int\frac{d\beta}{(1-k^2\sin^2\beta)^{\frac{1}{2}}} \tag{7}$$

When $\theta=a,\ \beta=\frac{\pi}{2}$

and when $\theta=0,\ \beta=0$

hence $\int_0^{\frac{t_1}{2}} dt$ corresponds to $-\int_a^{o} d\theta$ and $+\int_0^{\frac{\pi}{2}} d\beta$

hence
$$\frac{t_1}{2} = +\sqrt{\frac{r}{g}}\int_0^{\frac{\pi}{2}}(1-k^2\sin^2\beta)^{-\frac{1}{2}}\,d\beta$$

Developing by the binomial theorem and integrating term by term, we obtain, after substituting for k^2, the following formula for any value of h up to $2r$.

$$t_1=\pi\sqrt{\frac{r}{g}}\left[1+\left(\frac{1}{2}\right)^2\frac{h}{2r}+\left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\left(\frac{h}{2r}\right)^2+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)^2\left(\frac{h}{2r}\right)^3+\text{etc.}\right] \tag{8}$$

in which $h=EA$, the vertical descent.

If h is greater than r, the guide should be inside, or the rod of the simple pendulum should be capable of resisting thrust.

It is seen that if h is so very small that when compared with unity the various multiples of h may be neglected, Eq. 8 reduces to

$$t_1 = \pi\sqrt{\frac{r}{g}}.$$

288. Sliding down a cycloid. Let the body slide down a smooth guide so that the C. G. describes the arc of a cycloid in a vertical plane, the base of the cycloid being horizontal. Fig. 242. Instead of a smooth cycloidal guide, we may use the normal tension along the *radius of curvature* ρ, thereby making the device a *simple cycloidal pendulum.*

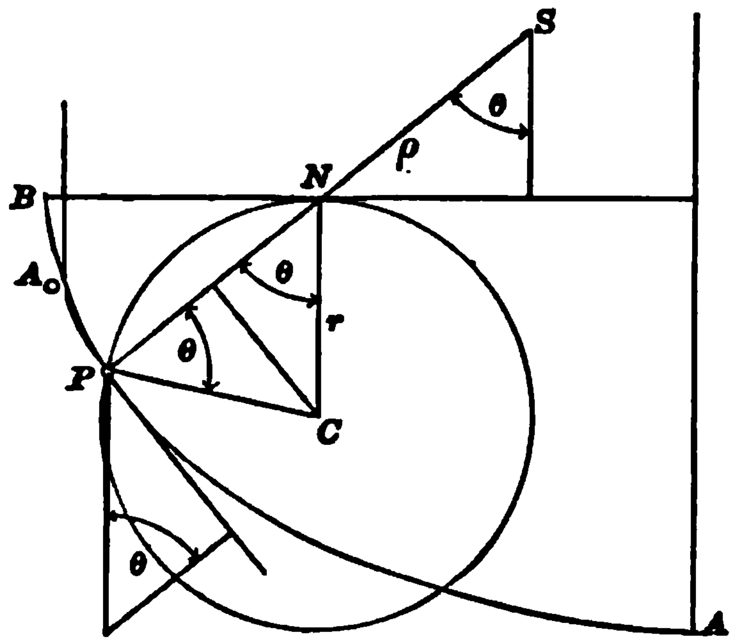


Fig. 242

As in the case of the circle,

$$F = W \sin\theta$$

$$a = g \sin\theta$$

$$v dv = g \sin\theta\, ds = -g\rho \sin\theta\, d\theta,$$

and since by the law of the cycloid the radius of curvature $\rho = PS = 2PN = 4r\cos\theta$

$$v dv = -4rg \sin\theta \cos\theta\, d\theta$$

Hence
$$\frac{v^2}{2} = -4rg \left.\frac{\sin^2\theta}{2}\right]_a^\theta = +2rg\,(\sin^2 a - \sin^2\theta)$$

$$v = 2\sqrt{rg}(\sin^2 a - \sin^2\theta)^{\frac{1}{2}}$$

$$dt = \frac{1}{2\sqrt{rg}} \cdot \frac{-4r\cos\theta\, d\theta}{(\sin^2 a - \sin^2\theta)^{\frac{1}{2}}}$$

$$\frac{t_1}{2} = -2\sqrt{\frac{r}{g}} \arcsin \left.\frac{\sin\theta}{\sin a}\right]_a^0 = \pi\sqrt{\frac{r}{g}}$$

which is independent of the initial angle a. Hence t_1, the time of a full oscillation, is

$$t_1 = 2\pi\sqrt{\frac{r}{g}}$$

The radius of curvature of the cycloid at the point A is $\rho = 4r$. If we write $l = \rho = 4r$, we have

$$t_1 = 2\pi\sqrt{\frac{4r}{4g}} = \pi\sqrt{\frac{l}{g}}.$$

which is the time for a simple pendulum whose length is l.

The result shows that the time is independent of the position of the point of starting, and that the cycloidal pendulum is isochronous.

It will be shown under the head of *rolling bodies* that a sphere *rolling* down a guiding surface parallel to the above cycloid, has an isochronous descent, *provided* the path of the *center* is the cycloid itself.

239. A sliding contest. It has long been known that the time occupied by a body sliding from A_0 to A along a smooth guide will be less for a cycloid than for any other guide. It may be of value to compare the times for a slide from A_0 to A along three different ideal paths: a smooth inclined plane; a smooth circular arc; and a smooth cycloid; the curves being tangent to a horizontal line at A. Fig. 243. Let r be the radius of the rolling circle which generates the cycloid, then

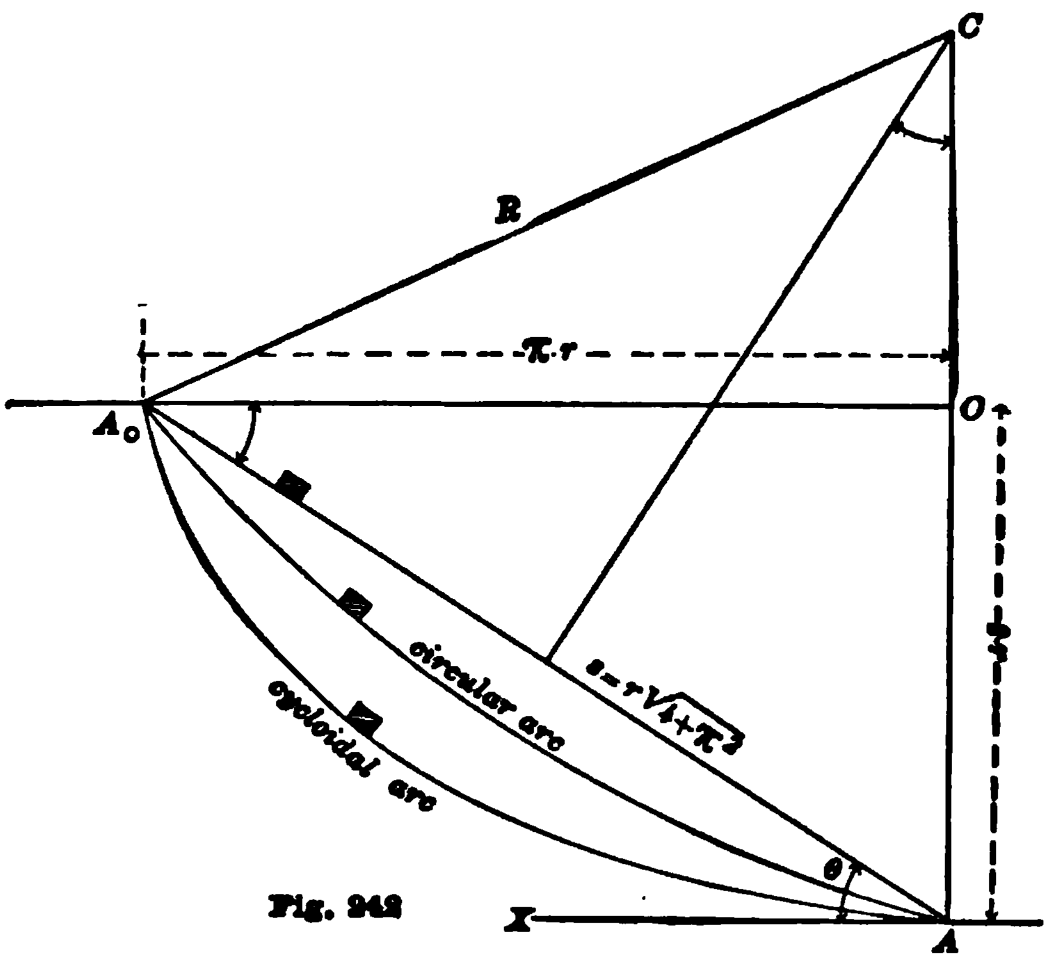


Fig. 242

$$AO = 2r$$

$$A_0O = \pi r.$$

The radius of the circle tangent to AX at A, and passing thru A_0 is

$$R = \frac{s^2}{4r} = \left(1+\frac{\pi^2}{4}\right)r =$$

For the plane, $\theta = \text{arc}\sin\left(\frac{2r}{s} = \frac{2}{\sqrt{4+\pi^2}}\right)$, and $\sin\theta = \frac{2r}{s}$.

For the inclined plane. $s = \frac{1}{2} g \sin\theta t^2$.

Hence
$$t^2 = \frac{2s^2}{g2r} = \frac{r}{g}(\pi^2+4)$$

$$t \text{ (for the plane)} = \pi\sqrt{\frac{r}{g}}\left(1+\frac{4}{\pi^2}\right)^{\frac{1}{2}}$$

$$= \pi\sqrt{\frac{r}{g}} \times 1.18 \tag{1}$$

The time for the cycloid is

$$t = \pi\sqrt{\frac{r}{g}}. \tag{2}$$

The time for the circular arc is found by taking the time of oscillation of a pendulum whose length is $R=\left(1+\frac{\pi^2}{4}\right)r$. From (Eq. 8), section 237, we have, since $h=2r$;

$$t=\frac{\pi}{2}\sqrt{\frac{r}{g}}\left(\sqrt{1+\frac{\pi^2}{4}}\right)\left(1+\frac{1}{4+\pi^2}+\frac{9}{64}\cdot\frac{16}{(4+\pi)^2}+\text{etc.}\right. \quad (3)$$

which is greater than the time for the cycloid by an exceedingly small quantity.

The valuable suggestion lies in the fact that the time for the smooth plane is *eighteen per cent greater* than for the curves.

The velocity with which the body reaches A is the same for all.

240. Miscellaneous problems. Thus far in our problems, the unbalanced force and the mass of the moving body have been given, and we have found two of the variables, v, s and t. Let us now suppose that we have given m, v_0 and s to find an expression for F.

1. Suppose a car whose mass is m, moving on a horizontal track, strikes a spring with a velocity $v_0 \frac{\text{ft}}{\text{sec}}$; what must be the "force" of the spring per foot which will stop the car if compressed 6 inches? Omit for the present the energy due to rotation in the wheels, and assume that the spring is adequately supported.

Solution. Let p be the "force" required. Let us suppose the car has already compressed the spring a distance (in feet) s. Then at *that instant*

$$F=-ps$$

$$\frac{-ps}{W}=\frac{a}{g}\;\therefore\; a=\frac{-p}{m}\cdot s$$

$$vdv=ads=-\frac{p}{m}sds$$

$$\left.\frac{v^2}{2}\right]_{v_0}^{o}=-\left.\frac{p}{m}\cdot\frac{s^2}{2}\right]_{o}^{\frac{1}{2}}$$

$$o-v_o^2=-\frac{p}{4m}$$

$$\therefore\; p=4mv_0^2$$

$$\text{or } p\,\frac{\text{lb}}{\text{inch}}=\frac{mv_o^2}{3}$$

2. *Numerical example.* Find p in lbs. per inch, if the car weighs 16 tons and is moving at the rate of 4 miles per hour.

3. Suppose a projectile weighing 500 pounds is discharged from a "smooth-bore" gun with a velocity of 2,000 feet per sec. Suppose further that the distance moved in the chamber of the gun is 20 feet and that the powder burns at such a rate that the pressure of the gas behind the shot is maintained constant. Suppose further, that the atmospheric pressure and the friction as the shot moves along, are constant (neither of which assumptions is very near the truth). What must be the *excess* of the gas pressure over the friction and atmosphere? That is, what must be the *unbalanced force?* Assume that the gun *does not recoil.* Ans. 1,562,500.

4. Suppose the total unbalanced pressure against the projectile in the above gun decreases uniformly to one-half of its initial value, at the muzzle. What was F at the start?

Solution. Consider the conditions after the shot has moved a distance s. The pressure has fallen from F_1 to

$$F = F_1\left(1 - \frac{s}{2l}\right)$$

Hence to get the acceleration *at that time,* we have

$$\frac{\frac{F_1}{2l}(2l-s)}{W} = \frac{a}{g} \qquad \therefore \qquad a = \frac{F_1(2l-s)}{2lm}$$

$$\int_0^{2000} v\,dv = \frac{F_1}{2lm}\int_0^l (2l-s)\,ds$$

$$\left.\frac{v^2}{2}\right]_0^{2000} = \frac{F_1}{2\,lm}\left(2ls - \frac{s^2}{2}\right)\Bigg]_0^l$$

$$(2000)^2 = \frac{F_1}{lm}\left(\frac{3l^2}{2}\right)$$

$$F_1 = \frac{2m(2000)^2}{3l} = \frac{2\times500\times4{,}000{,}000}{32\times20\times3}$$

$$= \frac{4{,}000{,}000{,}000}{1920} = 2{,}083{,}333\tfrac{1}{3}\text{ lbs.}$$

5. Assume that F_1 in the gun diminishes at the uniform *time-rate* till it is $\frac{F_1}{2}$ at the muzzle. At the time t, if t_1 is the full time of the motion in the gun,

$$F = F_1 - \frac{t}{t_1}\frac{F_1}{2} = \frac{F_1}{2t_1}(2t_1 - t)$$

$$a = \frac{Fg}{W} = \frac{F_1 g}{2Wt_1}(2t_1 - t)$$

$$\frac{d^2s}{dt^2} = \frac{F_1 g}{1000t_1}(2t_1 - t)$$

Multiplying by dt and integrating:

$$v = \frac{ds}{dt} = \frac{F_1 g}{1000t_1}\left(2t_1 t - \frac{t^2}{2}\right) + (H = 0)$$

Multiplying by dt and integrating again from $s=0$ and $t=0$, to $s=20$ and $t=t_1$:

$$20 = s = \frac{F_1 g}{1000t_1}\left(t_1^3 - \frac{t_1^3}{6}\right) = \frac{5}{6}\cdot\frac{F_1 g}{1000}t_1^2$$

$$v_1 = 2000 = \frac{F_1 g}{1000}\left(\frac{3}{2}t_1\right)$$

$$F_1 g t_1 = \frac{4{,}000{,}000}{3}$$

Whence

$$20 = \frac{4{,}000{,}000}{3}\cdot\frac{5}{6000}\cdot t_1$$

$$t_1 = \frac{18}{1000}$$

$$F_1 = \frac{4{,}000{,}000 \times 1{,}000}{32 \times 3 \times 18}$$

$$= 2{,}314{,}815 \text{ lbs.}$$

241. Motion produced by the attraction of another body. It was proved by Newton that *the mutual action between two homogeneous spheres was proportional to the product of their masses divided by the square of the distance between their centers.*

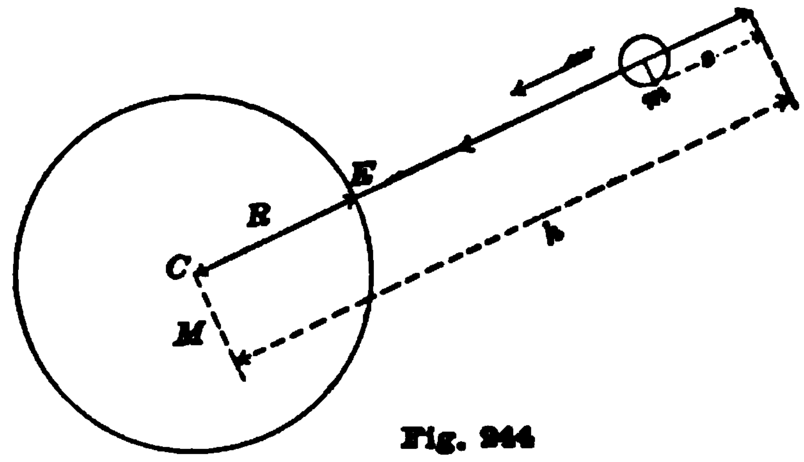


Fig. 244

Let M and m be the masses of two spheres, Fig. 244, whose centers were distant h, at a point of time when $t=0$; that is, when m started

to move towards M. At the time t, their distance apart is $h-s-s_1$, if we suppose both to move.

Let k be the mutual attraction between two units of mass, when a unit of distance apart. If the numbers of units be M and m, since each *unit* in one acts upon *each unit* in the other, the total number of actions between them will be Mm, and the total action when their centers are a unit of distance apart is Mmk; and if their distance apart be increased to $(h-s-s_1)$, the mutual action will be $\dfrac{Mmk}{(h-s-s_1)^2}$.

Now, let M be the mass of the earth, and m the mass of a body in space falling straight towards the earth undisturbed by all other bodies (an ideal condition). Under this action both M and m have an acceleration towards each other; but if m be small, the acceleration of M is so nearly infinitesimal that it may be neglected, and we may assume that the earth is absolutely at rest. Hence we make $s_1=0$, and have

$$F=\frac{Mmk}{(h-s)^2}. \tag{1}$$

Now we know what F is, when $h-s=R$; that is, at the surface of the earth; it is W, the *weight* of the mass m when measured in our standard pounds of force. Hence $Mmk=WR^2$, so that the general value of F at any distance is

$$F=\frac{R^2}{(h-s)^2}W$$

hence

$$\frac{F}{W}=\frac{a}{g}, \text{ and } a=\frac{R^2}{(h-s)^2}g.$$

$$\int_0^v vdv = \int_0^{h-R}\frac{R^2gds}{(h-s)^2}.$$

When the body reaches the earth's surface

$$v^2=2R^2g\frac{1}{h-s}\Bigg]_0^{h-R}=2R^2g\left(\frac{1}{R}-\frac{1}{h}\right) \tag{2}$$

If $h=\infty$, $v=\sqrt{2Rg}$ which represents the superior limit of velocity which a body could have falling from space to the earth's surface *under the conditions assumed.*

242. Problems in attraction. The law of mutual attraction of material bodies has just been stated, and the proposition usually proved in books on the integral calculus is that the resultant attraction of a

homogeneous sphere (or a homogeneous spherical shell) on an external mass, is the same as the attraction of an equal mass condensed to a material point at its center of gravity. The law holds for spheres alone. A few problems where the attracting bodies are homogeneous but not spherical will be in order at this point.

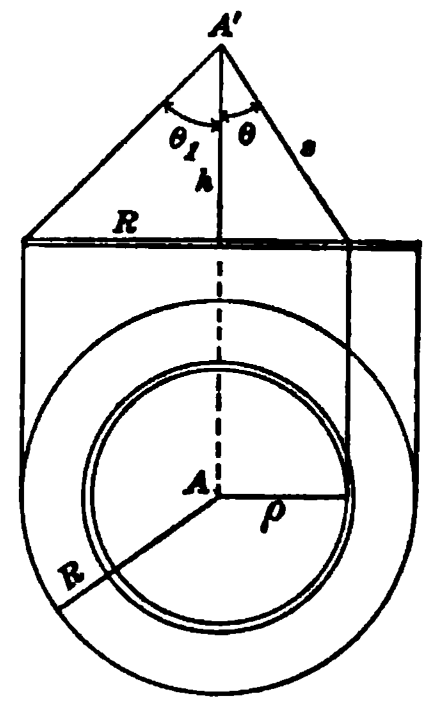


Fig. 245

1. To find the resultant attraction of a thin circular plate, radius R, upon a unit of mass at a point in the axis of the plate at a distance h.

Let the thickness of the plate be t, which is very small. Fig. 245. A ring with volume $2\pi\rho t d\rho$ and mass $2\delta\pi t\rho d\rho$ has a resultant attraction upon the unit of mass at A, if k is the mutual attraction between two units of mass a unit of distance apart,

$$dG = \frac{2k\delta\pi t\rho d\rho}{s^2}\cos\theta = 2k\delta\pi th\frac{\rho d\rho}{(h^2+\rho^2)^{\frac{3}{2}}}$$

and $$G = -2k\delta\pi th\cdot\frac{1}{(h^2+\rho^2)^{\frac{1}{2}}}\Bigg]_0^R = 2\pi k\delta th\left(\frac{1}{h}-\frac{1}{\sqrt{h^2+R^2}}\right) \qquad (1)$$

If $R = h\sqrt{3}$ or $h = \dfrac{R}{\sqrt{3}}$ we have

$$G = k\delta\pi t$$

If $R = \infty$ $$G = 2k\delta\pi t$$

which is independent of h. Hence it follows, that **the attraction of an infinite plane lamina upon an external mass is constant for every finite position of the external mass.** The attraction varies directly as the density of material, and for plates, as the thickness.

2. Since the attraction of an infinitely thin plate is the same for all points, that attraction upon exterior masses must also be *constant for thick plates and be proportional to the thickness.*

3. A body falling from A to the plane would have a constant acceleration (like bodies near the earth's surface) which would be $a = 2k\delta\pi t$; just as g is numerically equal to the earth's attraction upon a unit of mass on its surface.

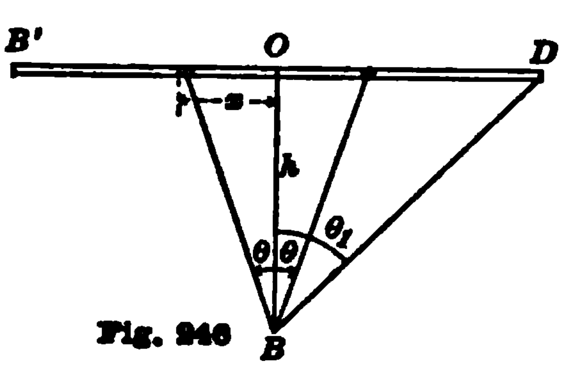


Fig. 246

243. The attraction of a small straight rod upon an external unit of mass in a perpendicular bisector. Fig. 246.

Let the cross-section of the rod be A, then $2\delta A dx$ is the mass of two symmetrical elements. Their resultant attraction upon a unit mass at B is

$$dG = \frac{2k\delta A dx}{h^2+x^2}\cos\theta = \frac{2hk\delta A dx}{(h^2+x^2)^{\frac{3}{2}}}$$

and

$$G = 2hk\delta A\int_0^{\frac{l}{2}}\frac{dx}{(h^2+x^2)^{\frac{3}{2}}}.$$

To integrate this let $x = h\tan\theta$

Then $dx = h\sec^2\theta\, d\theta$ and

$$(h^2+x^2)^{\frac{3}{2}} = h^3\sec^3\theta \text{ and}$$

$$\frac{dx}{(x^2+h^2)^{\frac{3}{2}}} = \frac{d\theta}{h^2\sec\theta} = \frac{\cos\theta d\theta}{h^2}$$

Hence

$$\int_0^{\frac{l}{2}}\frac{dx}{(x^2+h^2)^{\frac{3}{2}}} = \frac{1}{h^2}\int_0^{\tan^{-1}\frac{l}{2h}}\cos\theta d\theta$$

Since when $x=0$, $\theta=0$; and when $x=\dfrac{l}{2}$, $\tan\theta=\dfrac{l}{2h}$

$$\int_0^{\text{arc}\tan\frac{l}{2h}}\cos\theta d\theta = \sin\left(\text{arc}\tan\frac{l}{2h}\right) = \sin\theta_1 = \frac{l}{2\sqrt{h^2+\frac{l}{4}}} = \frac{l}{\sqrt{4h^2+l^2}}$$

Therefore,

$$G = \frac{2k\delta lA}{h(4h^2+l^2)^{\frac{1}{2}}} = \frac{2km}{h(4h^2+l^2)^{\frac{1}{2}}}$$

If $l=\infty$

$$G = \frac{2k\delta A}{h}$$

and the attraction is inversely proportional to h.

244. The forces acting between parts of a moving mechanism. Fig. 247 represents in skeleton a part of the mechanism of the steam

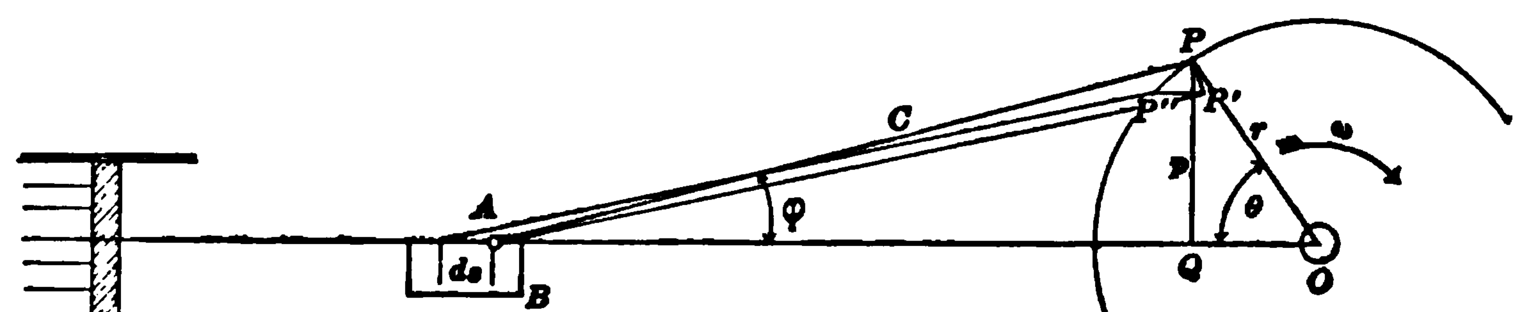


Fig. 247

engine. Assume that the speed of rotation (the angular velocity) ω of the crank is constant; and that the total steam pressure upon the piston, *less* the *friction of piston, piston rod,* and *cross-head* is S. Then the thrust up the connecting rod is *less than* $S \sec\phi$, since S is diminished by the force used in giving to the mass of piston, piston-rod, cross-head and one-half of the connecting rod, the acceleration due to their motion. The chief purpose of this section is to find this *accelerating force* which must be subtracted from S.

Suppose in the time dt, the cross-head moves ds, and the connecting rod moves to a new position, which it takes by two motions: a *translation equal to ds*, and a *rotation* $d\phi$. Meanwhile the crank-pin moves $rd\theta$. The three elementary movements of the crank pin form an infinitesimal triangle $PP'P''$ which may be enlarged for better illustration. See Fig. 248.

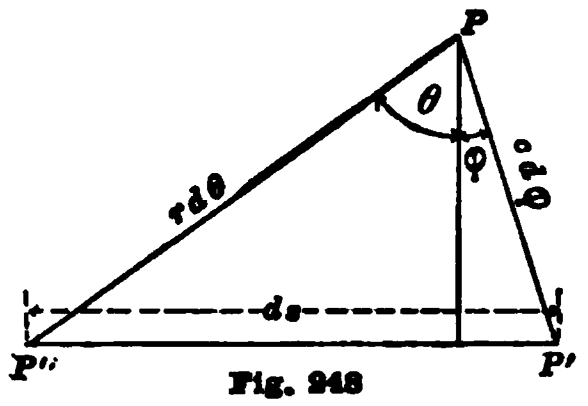


Fig. 248

$P''P' = ds$, $P'P = cd\phi$, and $P''P = rd\theta$. The figure shows that

$$ds = rd\theta \sin\theta + cd\phi \sin\phi$$

dividing by dt we have, since $\dfrac{d\theta}{dt} = \omega$,

$$v = \frac{ds}{dt} = r\omega \sin\theta + c \sin\phi . \frac{d\phi}{dt}.$$

But (Fig. 247) $c \sin\phi = r \sin\theta = PQ$, hence

$$v = r \sin\theta \left(\omega + \frac{d\phi}{dt} \right)$$

To find $\dfrac{d\phi}{dt}$, we differentiate the equation just used.

$$c \cos\phi d\phi = r \cos\theta d\theta$$

$$\frac{d\phi}{dt} = \frac{r \cos\theta}{c \cos\phi} . \omega = \frac{r\omega \cos\theta}{\sqrt{c^2 - r^2 \sin^2\theta}}.$$

Hence
$$v = r\omega \left(\sin\theta + \frac{\cos\theta \sin\theta}{(k^2 - \sin^2\theta)^{\frac{1}{2}}} \right) \tag{1}$$

in which $k = \dfrac{c}{r}$, a ratio which varies some in practice, but is rarely less than 4.*

* If k were very large, the crosshead A would have approximately a "harmonic motion," and the force we are seeking would be px, for a spring acting upon the mass of the reciprocating parts, with an amplitute of oscillation equal to r.

To find a the acceleration, we differentiate v.

$$\frac{dv}{dt}=a=r\omega\left(\cos\theta.\omega+\frac{\omega[(\cos^2\theta-\sin^2\theta)k^2+\sin^4\theta]}{(k^2-\sin^2\theta)^{\frac{3}{2}}}\right)$$

$$a=r\omega^2\left(\cos\theta+\frac{(2\cos^2\theta-1)k^2+\sin^4\theta}{(k^2-\sin^2\theta)^{\frac{3}{2}}}\right) \tag{2}$$

Hence S', which is to be subtracted from S in order to give the proper acceleration to the mass of the reciprocating parts, is $S'=ma$.

Hence $(S-ma)\sec\phi$ is the thrust up the connecting rod; and its component perpendicular to the crank-arm is

$$(S-ma)\sec\phi\sin(\phi+\theta)$$

and the turning moment (the "Torque") is

$$M=(S-ma)r\sec\phi\sin(\phi+\theta). \tag{3}$$

N.B. See Note I, Appendix.

Discussion of Results.

When $\theta=0$ or π (the common dead points),

$$a=r\omega^2\left(1+\frac{1}{k}\right) \text{or } a=-r\omega^2\left(1-\frac{1}{k}\right)$$

$$v=0$$

As θ increases, $a=$ decreases, but v increases.

When $\theta=\frac{\pi}{2}$, $\quad a=\frac{1-k^2}{(k^2-1)^{\frac{3}{2}}}r\omega^2=-\frac{r\omega^2}{\sqrt{k^2-1}}$; $\quad$ and $v=r\omega$

which is the constant velocity of the crank pin. This shows, since a is already negative, that the velocity has passed its maximum, and we properly conclude that at some point in the first quadrant (and in the fourth) of θ, the velocity of the reciprocating *parts* is greater than the velocity of the crank pin.

So long as a is positive, S' has a positive value, so that the torque is less than would be the case if the reciprocating parts were without mass.

When $a=0$, $S=0$.

When $a<0$, $S<0$ and S instead of being diminished is increased by the *positive quantity* $(-am)$, and the "torque" is increased during the stroke from the value of θ, which makes a zero, to the end when $\theta=\pi$.

Thus we see that the effect of heavy reciprocating parts is to somewhat equalize the torque, tho the supply of steam may be cut off and the pressure may become less, during the last part of a stroke.

CHAPTER XV.

Moments of Inertia of Solids. Rotation of Bodies. Translation and Rotation Combined.

245. Moments of inertia of solids. In Chapters IX and X we found and made use of the Moments of Inertia of *Surfaces* of action; we now are in need of the moments of inertia of *Masses.* Instead of multiplying an element of a *surface* by the square of its distance from an axis in its plane, we shall multiply an element of *mass* by the square of its distance from an axis in space.

The algebraic expression for an element of mass is

$$dm = \delta dV$$

Definitions. The **Density** of a mass is the number of units of mass in a unit of volume, and is represented by the letter δ (delta). Unless otherwise stated the density of the body under consideration will be constant, *i. e.*, the body will be *homogeneous.*

A clear concept of a **Unit of mass** is gained from the equation of definition already used

$$\frac{W}{g} = m$$

Since $g = 32+$, it is evident that the number m will be unity when W is numerically equal to g, *i. e.*, **A unit of mass weighs 32.16 lbs. under standard conditions.**

246. General formulas. The distance of an element of mass from the *axis* of moments will in general be called ρ, and the Moment of Inertia will still be called I, tho it now has a new meaning. Hence

$$dI = \rho^2 dm = \rho^2 \delta dV. = \delta \rho^2 dx dy dz.$$

$$I = \int \rho^2 dm = \delta \int \rho^2 dV = \delta \int\int\int \rho^2 dx dy dz.$$

The element of volume may be represented in various ways according to the kind of co-ordinates used, and according to the relation of the axis to the solid. The above formula requires three integrations; the form

$dV = y dx dz$ requires two integrations.
$dV = yl dx$ requires but one integration.

There are other forms using circular co-ordinates.

It is evident that when the element has a finite dimension, every portion of the element must have the same value of ρ. Care will be taken in what follows to select the simplest elemental form.

The Radius of Gyration is again introduced, and as before

$$I = mk^2.$$

The fitness of the name may now be shown. If we imagine the *entire mass* of a revolving body to be condensed into an infinitely thin cylinder whose radius is k, and whose axis is the axis of inertia of the body, we see that by definition mk^2 is its moment of inertia. Hence k is the radius of a thin cylinder *equivalent* to a given body in so far as concerns its *Moment* of *Inertia.*

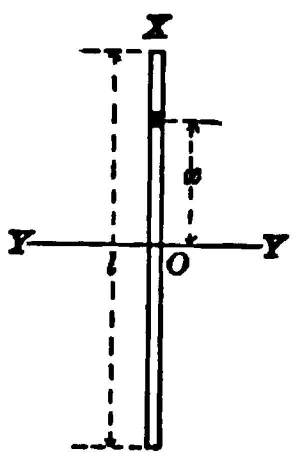


Fig. 249

The axis referred to will, in general, be an axis of symmetry, if there is one.

247. Problems.

Ex. 1. The I of a straight bar, or rod, whose cross-section is dA, and whose length is l. The axis is thru the center at right angles. The element lies between two consecutive cross-sections, and $\rho = x$; hence $dV = dAdx$; hence (Fig. 249):

$$I_o = \delta \int_{-\frac{l}{2}}^{+\frac{l}{2}} (x^2dx)dA = \delta dA \int_{-\frac{l}{2}}^{+\frac{l}{2}} x^2dx = \delta dA \left.\frac{x^3}{3}\right]_{-\frac{l}{2}}^{+\frac{l}{2}} = \delta dA \frac{l^3}{12} = m. \frac{l^2}{12} = mk_o{}^2.$$

The subscript ($_o$) to I and k signifies that the "*axis*" was thru the center of gravity.

Had the axis YY been at *end of the rod,* we should have had

$$I = \delta dA \int_o^l x^2dx = \delta dA \frac{l^3}{3} = m. \frac{l^2}{3} \cdot = mk^2.$$

Had the cross-section been finite, it is evident that all points in the element could *not* have been equally distant from OY. The problem for a bar or block with finite dimensions is therefore deferred.

In the above example, every element had the same cross-section dA, and the same dimension dx, and the ρ of an element, was the x of the element.

Ex. 2. The moment of inertia of an infinitely thin plane plate, with respect to an axis in the plate and thru the center of gravity of the plate.

$$I_o = \delta \int (x^2)ydxdz = \delta dz \int yx^2dx$$

Here we have an element of finite length every part of which is of the same distance from the I-axis, YY; hence ρ is x for the whole length, so we can place the constant factors δ and dz (the thickness) in front of the integral sign and sum the terms yx^2dx. Fig. 250. The value of y must be found in terms of x and the integral must be taken between proper limits. With such a plate the problem is not new; it is worth while, however, to recall the results found in a former chapter, and we must not forget to prefix the constant factors, and then use the thickness in getting the volume, and δ in getting the mass.

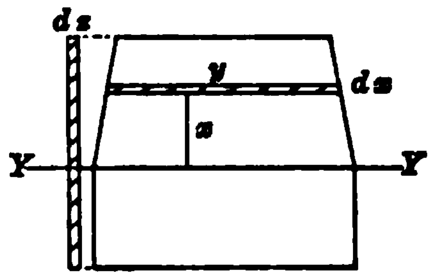


Fig. 250

248. The moment of inertia of thin plates. See **152-157**.

The *rectangle*,

I-axis, a diameter parallel to b. $\quad I_0=\delta dz\dfrac{bh^3}{12}=m\dfrac{h^2}{12}=mk_0^2$

I-axis is the base $\quad I=\delta dz\dfrac{bh^3}{3}=m\dfrac{h^2}{3}=mk^2.$

The triangle,

I-axis thru G parallel to base $\quad I_0=\delta dz\dfrac{bh^3}{36}=m\dfrac{h^2}{18}=mk_0^2$

I-axis, the base $\quad I=\delta dz\dfrac{bh^3}{12}=m\dfrac{h^2}{6}=mk^2$

I-axis thru vertex parallel to base $\quad I=\delta dz\dfrac{bh^3}{4}=m\dfrac{h^2}{2}=mk^2.$

The circle,

I-axis, a diameter $\quad I_0=\delta dz\dfrac{\pi r^4}{4}=m\dfrac{r^2}{4},\quad k^2{}_0=\dfrac{r^2}{4}$

I-axis, a *polar* axis $\quad I_0=\delta dz\dfrac{\pi r^4}{2}=m\dfrac{r^2}{2},\quad k^2{}_0=\dfrac{r^2}{2}.$

The ellipse,

I-axis, minor axis $\quad I_0=\delta dz\dfrac{\pi ba^3}{4}=m\dfrac{a^2}{4},\quad k^2{}_0=\dfrac{a^2}{4}$

I-axis, major axis $\quad I_0=\delta dz\dfrac{\pi ab^3}{4}=m\dfrac{b^2}{4},\quad k^2{}_0=\dfrac{b^2}{4}$

I-axis, polar axis $\quad I_0=\delta dz\pi ab\dfrac{a^2+b^2}{4},=m\dfrac{a^2+b^2}{4},\quad k^2=\dfrac{a^2+b^2}{4}.$

All of the above are preliminary to the finding of I for solids of three finite dimensions. Some of their uses will now be shown.

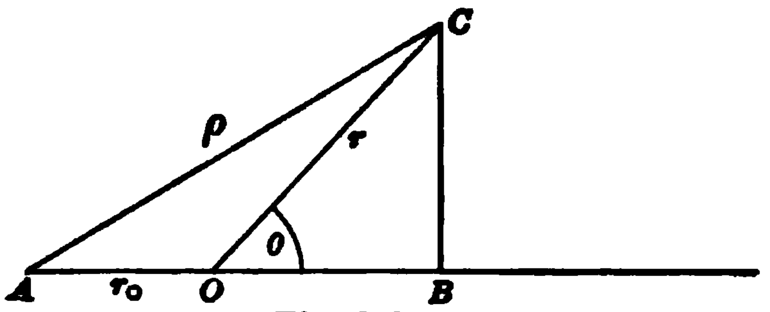


Fig. 251

249. I about a parallel axis. Having given the I_0 of a solid, find the I for a parallel axis distant r_0.

Assume the two parallel axes are perpendicular to our paper; the axis for I_0 being at O, and the axis for I at A. Let dm be at any point in a line thru C parallel to the I-axis, and ρ its distance from the I-axis. Then

$$I=\int \rho^2 dm.$$

but $\rho^2 = r_0^2 + r^2 + 2r_0 r \cos\theta$, and our integral becomes

$$I=r_0^2\int dm+\int r^2 dm+2r_0\int(r\cos\theta)\,dm=mr_0^2+I_0+2r_0\int(OB)dm$$

Now $OBdm$ is the moment of an element of mass about an axis thru G; hence $\int OBdm=0$, and our formula for a parallel axis when I_0 is known is

$$I=I_0+mr_0^2$$

Application. **A rectangular solid** with dimensions a, b, c, has an axis of inertia thru its center parallel to the edges b. Find I_y. Fig. 252. *Solution.* Pass two consecutive planes perpendicular to OX, and distant x and $x+dx$, giving the lamina whose volume is $bcdx$, and whose mass is $\delta bcdx$.

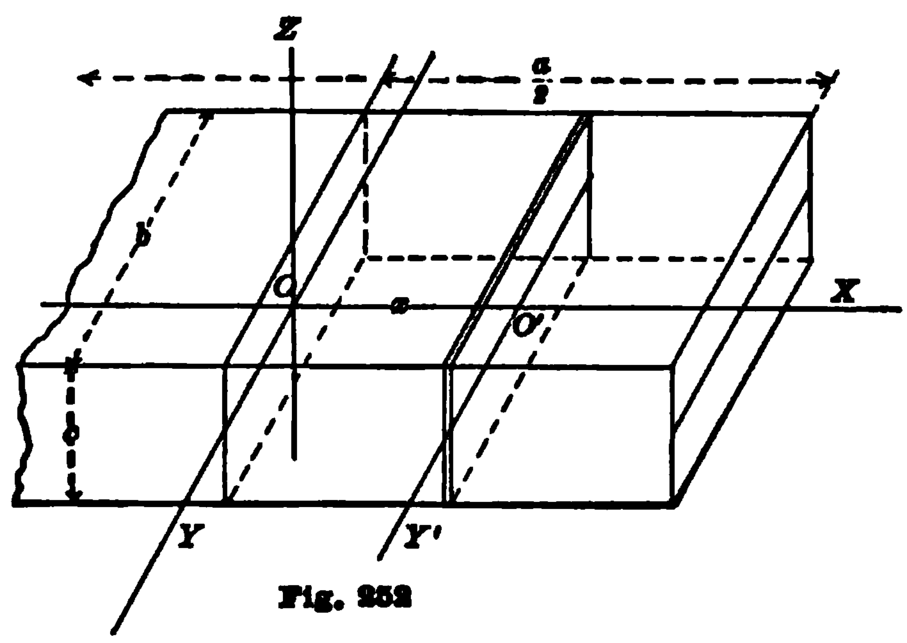


Fig. 252

The I of this element about its C. G. parallel to OY is by **248** and by **249**,

$$I_0'=\delta\,dx\frac{bc^3}{12} \qquad dI_y=\delta\frac{bc^3}{12}dx+(\delta bcdx)x^2$$

Integrating from $-\frac{a}{2}$ to $+\frac{a}{2}$, $I_y=\delta\left(abc.\frac{c_2}{12}+abc\frac{a^2}{12}\right)=m\frac{a^2+c^2}{12}$.

$$k_0^2=\frac{a^2+c^2}{12}.$$

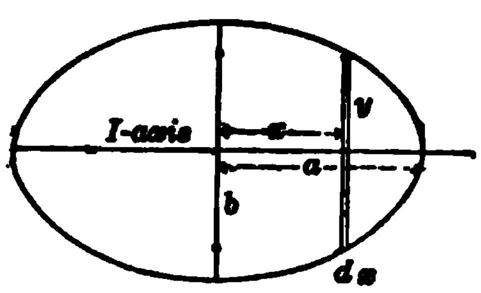


Fig. 253

250. Solids of revolution, whose axes are the I-axes. The simplest element is a thin cylinder or a thin plate whose axis is the inertia-axis. Fig. 253 shows how an element may be taken for a prolate ellipsoid.

Applications. 1. **An oblate ellipsoid.** The minor axis is the I-axis. Let the equation of the generating ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The element is $dm=\delta 4\pi yx dx$, the mass of the thin cylinder.

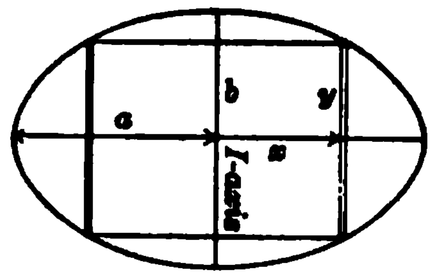


Fig. 254

$$y=\frac{b}{a}(a^2-x^2)^{\frac{1}{2}}$$

$$I=\delta 4\pi\frac{b}{a}\int_0^a (a^2-x^2)^{\frac{1}{2}}x^3dx$$

Let $x=a\sin\theta$, $dx=a\cos\theta d\theta$, $(a^2-x^2)^{\frac{1}{2}}=a\cos\theta$.

$$I_o=4\pi\delta a^4 b\int_0^{\frac{\pi}{2}}\sin^3\theta\cos^2\theta d\theta=\frac{8}{15}\pi\delta a^4 b=m.\frac{2a^2}{5},\qquad k_o^2=\frac{2}{5}a^2$$

2. **A solid sphere.** *I*-axis a diameter.

$$I_o=\frac{8}{15}\pi\delta r^5=\left(\frac{4}{3}\pi\delta r^3\right)\frac{2}{5}r^2=m.\frac{2r^2}{5},\qquad k_o^2=\frac{2}{5}r^2$$

3. **A thick cylinder** *of revolution*, length a, radii r_1 and r_2.

$$I_o=\pi\delta a\cdot\frac{r_2^4-r_1^4}{2}=\delta\pi a\,(r_2^2-r_1^2)\frac{r_2^2+r_1^2}{2}=m.\frac{r_2^2+r_1^2}{2},\qquad k_o^2=\frac{r_2^2+r_1^2}{2}.$$

4. **A solid cylinder,** length a, radius r.

$$I_o=\frac{\delta\pi a r_4}{2}=m.\frac{r^2}{2}\qquad\qquad k_o^2=\frac{r^2}{2}.$$

5. **A cone of revolution,** height $=h$, radius of base r. The element is a thin cylinder as before.

$$I_o=\delta\int 2\pi y(h-x)y^2dy=\frac{2\pi\delta h}{r}\int_0^r(r-y)y^3dy=\frac{\pi\delta h r^4}{10}=\frac{\pi\delta h r^2}{3}\cdot\frac{3r^2}{10},\quad k_o^2=\frac{3r^2}{10}$$

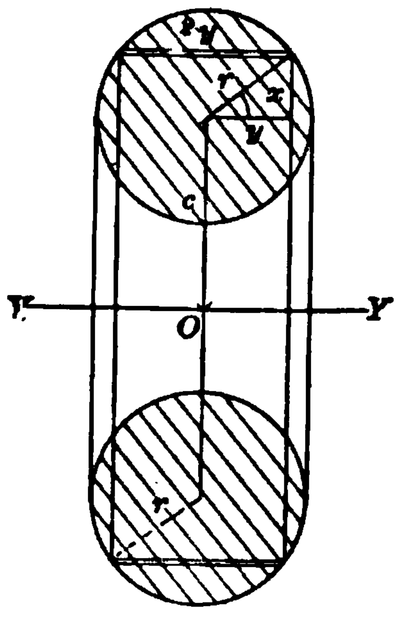


Fig. 255

6. **A torus**, Fig. 255, with a circular cross-section, radius r; distance to center of generating circle c. Element of volume = thin cylinder $= 2\pi(c+x)\ 2ydx$. The quantity c must be greater than r.

$$I_o = \delta 4\pi \int_{-r}^{+r} (r^2-x^2)^{\frac{1}{2}}(c+x)^3 dx$$

$$= \delta 4\pi r^2 \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} (c+r\sin\theta)^3 \cos^2\theta d\theta = \delta\pi^2 cr^2 \cdot \frac{3r^2+4c^2}{2}$$

By the Theorem of Pappus, see **121**, $m = 2\delta c\pi^2 r^2$;

hence $$k_o^2 = \frac{3r^2+4c^2}{4}.$$

From examples 3 and 4, it is seen that the I of a hollow body is the *difference* between the I's of two bodies; the enclosing body, as tho solid; and the other, the vacant space, as tho solid.

When the I-axis is transverse to the axis of revolution, and when sections are non-circular, formula of **249** is employed.

When the I-axis is polar to a plate element, the $I_p = I_x + I_y$, where the axes OX and OY are rectangular and in the plate. Let the plane of the paper contain the plate with the polar axis at O, Fig. 256.

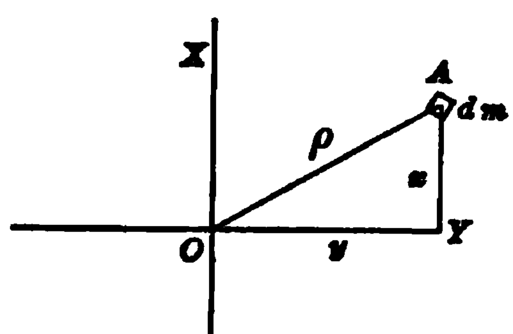


Fig. 256

$$I_p = \int \rho^2 dm = \int x^2 dm + \int y^2 dm = I_y + I_x$$

a very convenient formula.

7. Take a thin elliptical plate, where $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the equation.

$$I_a = m\frac{b^2}{4},\quad I_b = m\frac{a^2}{4},\quad \text{hence } I_p = m\frac{a^2+b^2}{4}.$$

8. **A general ellipsoid**, whose semi-axis are a, b, c. The equation of the surface is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Let the I-axis be OY (the diameter $2b$). Pass two consecutive planes perpendicular to the X-axis, distant x. The lamina is an elliptical plate whose semi-axes are y' and z'. We find y' (in terms of x) by making $z = 0$ in the above equation, and solving for y, getting

$$y' = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}$$

and by a similar process making $y=0$, we find

$$z'=\frac{c}{a}(a^2-x^2)^{\frac{1}{2}}$$

Now, using Form. **249**, we get

$$dI_0=\delta x^2.(\pi y'z')dx+\delta(\pi y'z').\frac{z'^2}{4}dx$$

$$I_0=\frac{\pi\delta bc}{a^2}\int_{-a}^{+a}(a^2-x^2)x^2dx+\frac{\pi\delta bc^3}{4a^4}\int_{-a}^{+a}(a^2-x^2)^2dx$$

$$=\frac{4\pi\delta a^3bc}{15}+\frac{4\pi\delta abc^3}{15}=\frac{4}{3}\pi\delta abc.\frac{(a^2+c^2)}{5}=mk_0^2,\qquad k_0^2=\frac{1}{5}(a^2+c^2).$$

9. **An elliptical cylinder,** length l, axes of cross-section $2a$ and $2b$; I_0-axis transverse parallel to b axis.

$$I_0=\pi\delta lab.\left(\frac{a^2}{4}+\frac{l^2}{12}\right),\qquad k_0^2=\frac{a^2}{4}+\frac{l^2}{12}.$$

Ex. 10. **Circular cylinder, transverse is I-axis,** length l, radius r.

$$I_0=\pi\delta r^2l\left(\frac{r^2}{4}+\frac{l^2}{12}\right),\qquad k_0^2=\frac{r^2}{4}+\frac{l^2}{12}$$

251. Solutions by different methods. It is evident that there are several different ways of solving the above problems. For example, the elementary part of a sphere, could have been a thin circular plate *parallel* with the I-axis, or *perpendicular* to the I-axis; or a ring with a finite radius, and an infinitesimal cross-section, in a plane perpendicular to the I-axis. The last method may be more fully shown. The cross-section of the ring is $(\rho d\theta)d\rho$; the radius is $\rho\sin\theta$; the volume $=(2\pi\rho\sin\theta)(\rho d\theta)(d\rho)$

the mass $\quad=2\pi\delta\rho^2d\rho\sin\theta d\theta$

hence $\quad I_0=2\pi\delta\int_0^r\int_0^\pi\rho^2d\rho\sin\theta d\theta\,(\rho^2\sin^2\theta)$

$$=2\pi\delta\frac{r^5}{5}\cdot\int_0^\pi\sin^3\theta d\theta$$

$$=2\pi\delta\frac{r^5}{5}\left(-\cos\theta+\frac{\cos^3\theta}{3}\right)\Big]_0^\pi$$

$$=\frac{8}{15}\pi\delta r^5=m\frac{2r^2}{5}\text{ as before.}$$

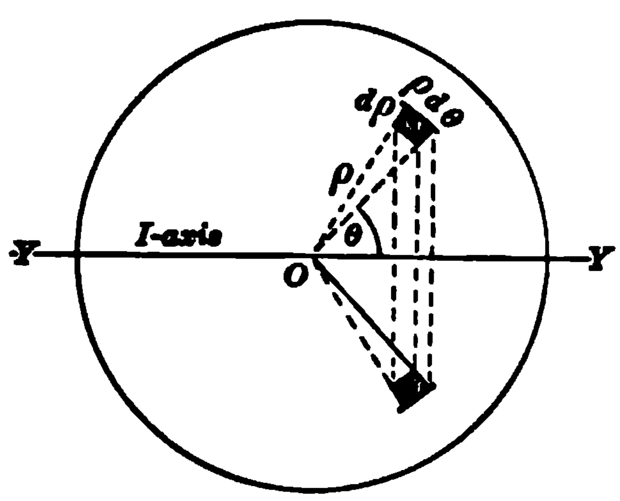


Fig. 257

The order of integration is immaterial, as the limits in one case are not functions of the variable in the other. Had we integrated for θ first we should have got

$$dI = 2\pi\delta\rho^4 d\rho \times \frac{4}{3} = \frac{8}{3}\pi\delta\rho^4 d\rho.$$

which is the Moment of Inertia of a spherical shell whose radius was ρ and whose thickness $d\rho$, and $k_0{}^2 = \frac{2}{3}\rho^2$.

Had we then integrated from $\rho = r_1$ to $\rho = r_2$, we should have had the I of a **thick spherical shell.**

$$I = \frac{8}{3}\pi\delta . \frac{r_2{}^5 - r_1{}^5}{5} = m . \frac{2}{5} . \frac{r_2{}^5 - r_1{}^5}{r_2{}^3 - r_1{}^3}$$

252. ***I* of composite bodies.** In getting the I of rotating and rolling bodies, we more often than not find them composite having a common I-axis; such as a shaft with its pulleys; a fly-wheel with its hub, spokes and rim; bodies with cavities of various shapes; and composite bodies with different values of δ.

When dimensions and density are only approximate, the results of all calculations will be equally approximate.

The following **Table** is partly for reference and partly for exercises and practice in the selection of methods.

253. The Moment of Inertia of Solids with reference to gravity axes.

Solid	I-axis	Volume	Mass	I_0	k_0^2
Slender Rod length l cross-section A	Transverse	lA	$\frac{w}{g}lA$	$\frac{\delta A l^3}{12}$	$\frac{l^2}{12}$
Rectangular Plate sides b and h thickness $= t$	Parallel to b	bht	$\frac{w}{g}bht$	$\frac{\delta tbh^3}{12}$	$\frac{h^2}{12}$
Triangular Plate b and h thickness t	Parallel to base	$\frac{tbh}{2}$	$\frac{wtbh}{2g}$	$\frac{\delta tbh^3}{36}$	$\frac{h^2}{18}$
Circular Plate radius r thickness t	Diameter	$\pi r^2 t$	$\frac{w}{g}t\pi r^2$	$\delta t\pi\frac{r^4}{4}$	$\frac{r^2}{4}$
Circular Plate radius r thickness t	Polar axis	$\pi r^2 t$	$\frac{w}{g}t\pi r^2$	$\pi\delta t\frac{r^4}{2}$	$\frac{r^2}{2}$
Elliptical Plate $2a$ and $2b$ axes thickness t	b-axis	πabt	$\frac{w}{g}\pi abt$	$\frac{\delta t\pi a^3 b}{4}$	$\frac{a^2}{4}$
Elliptical Plate $2a$ and $2b$ axes thickness t	a-axis	πabt	$\frac{w}{g}\pi abt$	$\frac{\delta t\pi ab^3}{4}$	$\frac{b^2}{4}$
Elliptical Plate $2a$ and $2b$ axes thickness t	Polar axis	πabt	$\frac{w}{g}\pi abt$	$\frac{\delta t\pi ab(a^2+b^2)}{4}$	$\frac{a^2+b^2}{4}$
Sphere radius r	Diameter	$\frac{4}{3}\pi r^3$	$\frac{4w}{3g}\pi r^3$	$\frac{8}{15}\delta\pi r^5$	$\frac{2r^2}{5}$
Thick Spherical Shell radii r_1 and r_2	Diameter	$\frac{4}{3}\pi(r_2^3-r_1^3)$	$\frac{4w}{3g}\pi(r_2^3-r_1^3)$	$\frac{8}{15}\pi\delta(r_2^5-r_1^5)$	$\frac{2}{5}\cdot\frac{r_2^5-r_1^5}{r_2^3-r_1^3}$
Thin Spherical Shell thickness t	Diameter	$4\pi r^2 t$	$\frac{4w}{g}\pi r^2 t$	$\frac{8}{3}\delta\pi r^4 t$	$\frac{2}{3}r^2$
Ellipsoid of Revolution polar axis $2b$ equat. axis $2a$	$2b$-axis	$\frac{4}{3}\pi a^2 b$	$\frac{4\pi w}{3g}a^2 b$	$\frac{8}{15}\pi\delta a^4 b$	$\frac{2}{5}a^2$
Ellipsoid axes $2a$, $2b$, $2c$	$2b$-axis	$\frac{4}{3}\pi abc$	$\frac{4\pi w}{3g}abc$	$\frac{4}{15}\pi\delta abc(a^2+c^2)$	$\frac{a^2+c^2}{5}$
Rectangular Prism a, b, l sides	b-axis	abl	$\frac{w}{g}abl$	$\frac{\delta abl}{12}(a^2+l^2)$	$\frac{a^2+l^2}{12}$
Circular Cylinder radius r length l	Transverse	$\pi r^2 l$	$\frac{w}{g}\pi r^2 l$	$\delta\pi r^2 l\left(\frac{r^2}{4}+\frac{l^2}{12}\right)$	$\frac{r^2}{4}+\frac{l^2}{12}$
Elliptical Cylinder $2a$, $2b$, l	Transverse $2b$-axis	πabl	$\frac{w}{g}\pi abl$	$\delta\pi abl\left(\frac{a^2}{4}+\frac{l^2}{12}\right)$	$\frac{a^2}{4}+\frac{l^2}{12}$
Solid Cone of Revolution alt. $= h$ radius of base r	axis h	$\frac{\pi}{3}hr^2$	$\frac{\pi w}{3g}hr^2$	$\frac{\pi\delta}{10}hr^4$	$\frac{3}{10}r^2$
Torus, Radius of Generating Circle r. From center to axis c	Polar axis	$2c\pi^2 r^2$	$\frac{2w}{g}c\pi^2 r^2$	$\delta\pi^2 cr^2\left(\frac{3r^2+4c^2}{2}\right)$	$\frac{3r^2+4c^2}{4}$

254. UNBALANCED MOMENTS. ROTATION OF SOLID BODIES, ABOUT FIXED AXES.

We saw in our study of statics that the reason why a body acted upon by two forces (from two other bodies) in such a manner as to form a couple, did not turn about an axis was that a second couple (from the actions of two other bodies) exactly balanced the former couple. We shall now discuss the motion which is caused or modified by a single couple acting upon a rigid body, **the moment being unbalanced.**

Example. Suppose a circular disk of homogeneous material is mounted on a horizontal cylindrical shaft, and that the shaft rests on smooth bearings as shown in Fig. 258. The weight of disk and shaft is exactly balanced by the upward lift of the bearings. As the support S and the weight of disk and shaft balance, these forces are not represented in the drawing. Now suppose by means of a thread or flexible cord, some other body pulls down on the disk with a constant tension F. Instantly the pressure upon the bearings is increased by the quantity F. In other words, the force acting at A is automatically and instantaneously resolved into an equal force F acting at C, and a couple whose moment is $+Fr$.*

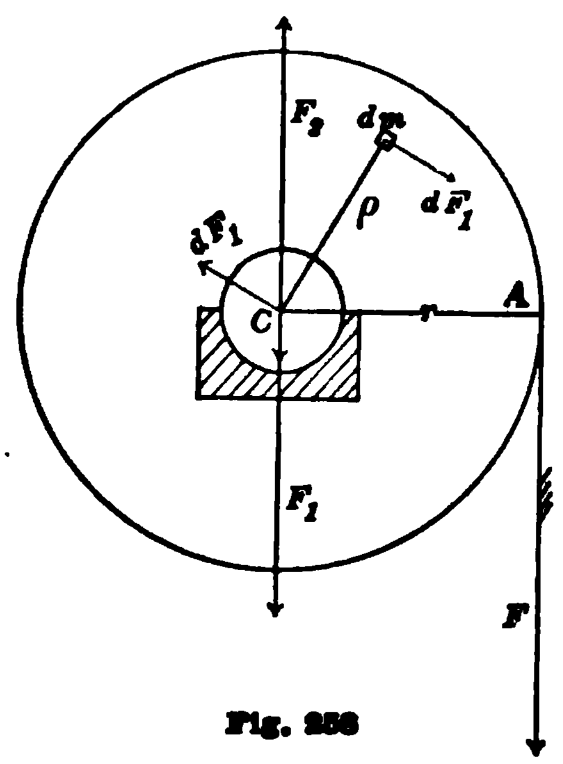


Fig. 258

The new force, F_1, is balanced by the added lift of the bearings, while the couple Fr is *unbalanced* and motion results.

255. Definitions of angular acceleration. If the *linear velocity* of a point in a rotating body be laid off on a tangent to its path, its length is seen to be $v = r\omega$, where ω is the angular velocity as defined in Physics, that is, $\omega = \frac{d\theta}{dt}$, $d\theta$ being an element of the arc (to radius unity) described in the time dt, and ds, for the radius r, being $rd\theta$. Now the angular acceleration is defined as $\alpha = \frac{d\omega}{dt}$; and since

$$v = r\omega$$

$$\frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$

$$a = r\alpha$$

* At this point the student should re-read **34-6.**

which gives the relation between the *linear* and *angular accelerations* of a point in a rotating body.

We are now to find the rate (amount per second) at which the angular velocity is generated and increased by the unbalanced *couple* aforesaid, in the case of the disk shown in Fig. 258.

The instant the couple acts, every particle, dm, of the disk and shaft feels the action and instantly responds with appropriate motion. Just how the force applied at A, reaches every other point by means of internal stresses in the rigid material, we need not now discuss. The moment Fr is at once resolved and distributed thruout the entire mass to be moved. It amounts to an automatic resolution of the couple Fr into countless component couples having the same axis, each with one force dF at C, and the other at dm with an arm ρ, so that $Fr=\int \rho dF$, since the whole is equal to the sum of its parts.* Each dF acts upon a dm of the mass, and the magnitude of dF is suited to the mass dm and to the *linear acceleration* to be produced. Let α stand for the angular acceleration we are to find; then the linear acceleration at a distance ρ is $\rho\alpha$. The magnitude of the force dF which is to produce a linear acceleration $\rho\alpha$ on the mass dm is,

$$dF=\rho\alpha dm \tag{1}$$

and the moment of the component couple is

$$\rho dF=\alpha\rho^2 dm \tag{2}$$

Since the sum of all the moments of the component couples is equal to the moment of the given couple, we have, if L stands for the unbalanced moment,

$$L=Fr=\int \rho dF=\alpha\int \rho^2 dm=\alpha I_0 \tag{3}$$

Hence

$$\alpha=\frac{L}{I_0}=\frac{Fr}{I_0} \tag{4}$$

since α must be constant for all the mass elements, and I_0 is by definition $\int \rho^2 dm$, the moment of inertia of the entire solid disk and shaft.

256. Analagous equations. The formula showing the relation between an unbalanced *moment* L and the resultant *angular* acceleration α: $L=I\alpha$, is exactly analogous to the formula showing the relation between an unbalanced *force* F, and the resultant *linear* acceleration a: $F=ma$.

The analogy extends thru all the formulas relating to translation and rotation as the following useful table shows. L stands for Fl,

* See **21** and **26'**.

the unbalanced moment, and F stands for an unbalanced force; θ stands for the total arc described during the time t, and s stands for the total distance moved during the time t.

Fundamental Formulas.

Translation.		Rotation.	
I. Uniform Motion.	$F=0$ $a=0$ $v=v_0$ $s=s_0+v_0t$	Uniform Rotation.	$L=0$ $\alpha=0$ $\omega=\omega_0$ $\theta=\theta_0+\omega_0 t$
II. Uniformly Accelerated; F constant	$a=\frac{F}{m}$ $v=v_0+at$ $s=s_0+v_0t+\frac{1}{2}at^2$	Uniformly Accelerated; L constant.	$\alpha=\frac{L}{I}$ $\omega=\omega_0+\alpha t$ $\theta=\theta_0+\omega_0 t+\frac{1}{2}\alpha t^2$
Any Acceleration. III. F variable.	$a=\frac{F}{m}=\frac{dv}{dt}=\frac{d^2s}{dt^2}$ $v=\frac{ds}{dt}$ $vdv=ads$ $v^2-v_0^2=2\int_{s_0}^{s}ads$	L variable.	$\alpha=\frac{L}{I}=\frac{d\omega}{dt}=\frac{d^2\theta}{dt^2}$ $\omega=\frac{d\theta}{dt}$ $\omega d\omega=\alpha d\theta$ $\omega^2-\omega_0^2=2\int_{\theta_0}^{\theta}\alpha d\theta$
Kinetic Energy.	$K.E.=\frac{1}{2}mv^2$		$K.E.=\frac{1}{2}I\omega^2$

For Energy see Chapter XVIII.

257. Unbalanced couples. These formulas can be used to find the relations between ω, θ and t for a rotating body when both L and I are known. It is therefore necessary to show how these two quantities are found.

The unbalanced couple is the resultant of all the couples acting upon a body rotating about a fixed axis.

Illustration. Fig. (259) represents a shaft carrying two pulleys or disks, and supported in rectangular boxes which are connected with fixed foundations. T_1 and T_2 are the tensions in the plies of a "driving" belt; and P_1 and P_2 are the tensions of a "following belt." The arrows at A and B mark the normal and tangential actions of the journal boxes *upon the shaft.* Counting normal and tangential forces separately, there are nine forces acting upon the moving combination: two T's, two P's, two p's, two fp's and W. Three of them act thru C and produce no turning moment: p_1, p_2 and W. Each of the other six is resolved into an equal parallel force thru C and a couple. Two of the couples have positive moments: T_1r_2 and P_2r_3. All the others are negative. Hence the unbalanced (or resultant) moment is:

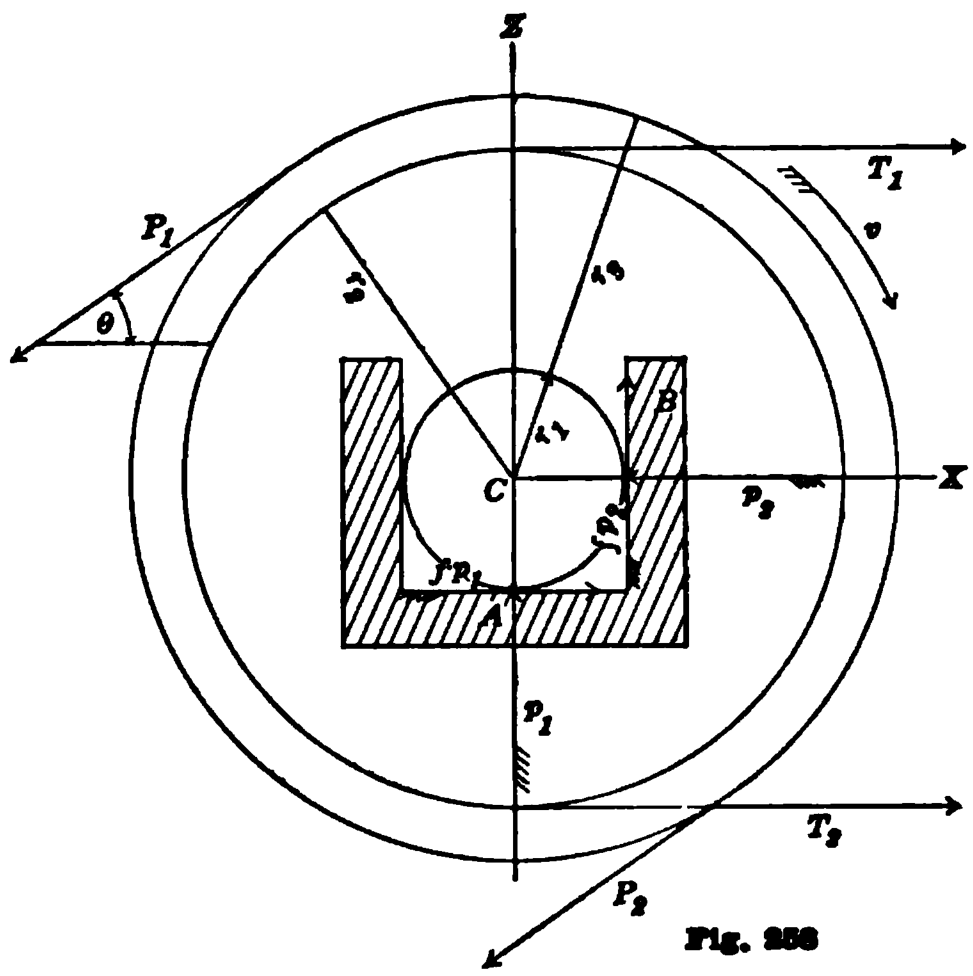


Fig. 259

$$L=(T_1-T_2)r_2+(P_2-P_1)r_3-f(p_1+p_2)r_1 \tag{1}$$

f being the co-efficient of friction. The belts are assumed to be flexible and r_2 and r_3 include half the belts' thickness. All the quantities in this value of L are assumed to be known except p_1 and p_2, so these must be found from the fact that the nine forces acting thru C balance: hence

$$\Sigma X=0, \text{ and } \Sigma Z=0$$

$$\Sigma X=T_1+T_2+fp_1-(P_1+P_2)\cos\theta-p_2=0$$

$$\Sigma Z=p_1+fp_2-W-(P_1+P_2)\sin\theta=0$$

Whence
$$p_1=\frac{W+(P_1+P_2)\sin\theta-f[(T_1+T_2)-(P_1+P_2)\cos\theta]}{1+f^2}; \tag{2}$$

$$p_2=\frac{f(W+(P_1+P_2)\sin\theta)+(T_1+T_2)-(P_1+P_2)\cos\theta}{1+f^2}. \tag{3}$$

These values substituted in (1) give us the unbalanced moment L, which is assumed to cause an acceleration a in pulleys and shaft.

The unusual shape of the journal box shown in Fig. 259 was chosen

for the purpose of showing how frictional forces are developed, and what new forces they develop. The friction on the horizontal surface at A gives a tendency on the part of the shaft to roll *to the right* against the vertical face B, thereby increasing p_2.

Again the friction on the face B gives a tendency on the part of the shaft to *climb*, thereby *diminishing* p_1.

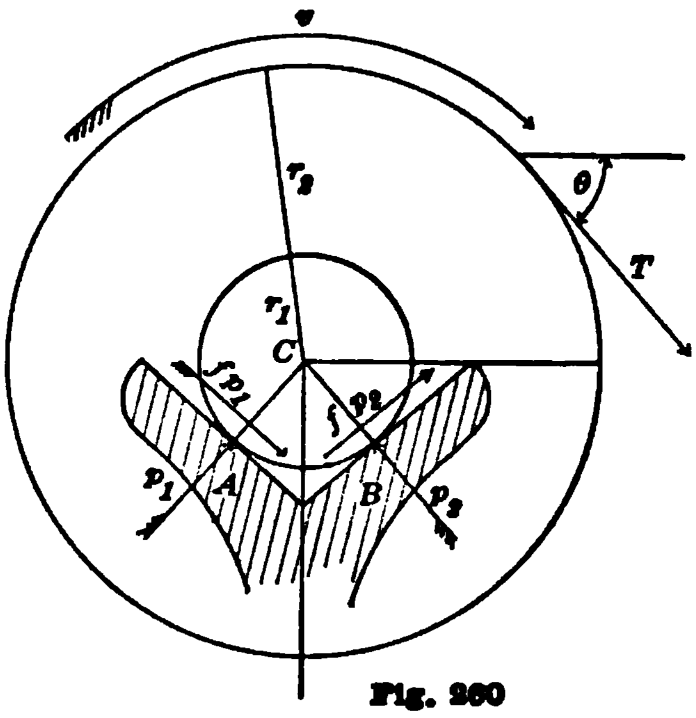


Fig. 260

258. The moments of friction. A second example will show these matters perhaps more plainly. Let Tr_2, Fig. 260, be the resultant moment of all external couples except those due to friction on the bearings. Then

$$L = Tr_2 - fr_1\,(p_1+p_2).$$

If the supporting surfaces have an inclination of 45°, the balance of all forces acting thru C gives:

$$\Sigma X = T\cos\theta - (p_2-p_1)\frac{1}{\sqrt{2}} + f(p_2+p_1)\frac{1}{\sqrt{2}} = 0$$

$$E\Sigma = (p_2+p_1)\frac{1}{\sqrt{2}} + f(p_2-p_1)\frac{1}{\sqrt{2}} - W - T\sin\theta = 0$$

whence $$p_1+p_2 = \frac{\sqrt{2}[W+T\sin\theta - fT\cos\theta]}{1+f^2}$$

and L is known.

259. In the case of a cylindrical bearing, while the action of the block is better distributed along a small arc of contact, there is a central point where the resultant acts, with both normal and tangential components. If TR represents the sum of all the moments except that due to friction, we have with the notation shown in Fig. 261.

$$L = TR - fp_nr$$

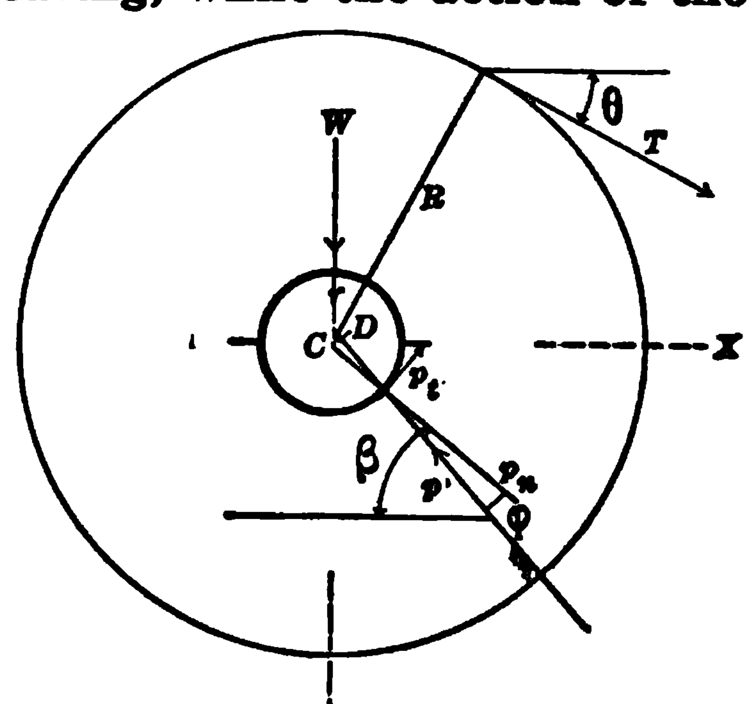


Fig. 261

Both the magnitude and position of p_n are found as follows:—Let β be the angle, at present unknown, which p_n makes with a horizontal plane. The forces acting thru C are: W, a

component of T, p_n, and a component of $fp_n = p_t$. Since C has no motion of translation, these forces balance.

Hence
$$\Sigma X = T\cos\theta + fp_n\sin\beta - p_n\cos\beta = 0$$
$$\Sigma Z = p_n\sin\beta + fp_n\cos\beta - W - T\sin\theta = 0$$

whence
$$p_n(\cos\beta - f\sin\beta) = T\cos\theta$$
$$p_n(\sin\beta + f\cos\beta) = W + T\sin\theta$$

whence
$$\frac{\sin\beta + f\cos\beta}{\cos\beta - f\sin\beta} = \frac{W + T\sin\theta}{T\cos\theta} = K.$$

Where K is used for brevity,

$$\frac{\tan\beta + f}{1 - f\tan\beta} = K, \qquad \therefore \quad \tan\beta = \frac{K - f}{1 + fK}$$

$$p_n = \frac{T\cos\theta\sqrt{(1+K^2)(1+f^2)}}{1+f^2}$$

and the value of L, the resisting moment, is found.

The student must not forget the trigonometric triangle for finding five other functions when one is known, see Fig. 262.

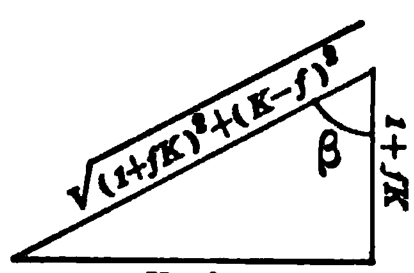


Fig. 262

Since $\tan\beta = \dfrac{K-f}{1+fK}$, $\sin\beta$, and $\cos\beta$ are readily written from the triangle.

260. A graphical analysis of the last problem. Since but three forces act upon the shaft and its pulley: gravity, the force T and the supporting journal box—two of which are fully known, the graphical solution will be simple.

Resolve T into an equal and parallel force thru C and a couple Tr.

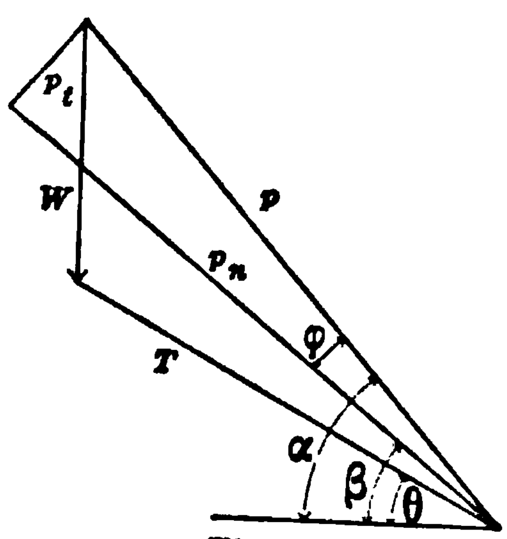


Fig. 263

Assume that the support p is also resolved in a similar way. The three forces acting at C, give a static triangle, Fig. 263, which determines p in *magnitude* and *direction*.

Whence
$$p = \sqrt{T^2 + W^2 + 2TW\sin\theta}$$

If there be no friction, P acts directly towards C.

If the co-efficient of friction is f, p is resolved into

$$p_n \text{ and } p_t = fp_n, \quad p = \sqrt{p_n^2 + (fp_n)^2}, \quad p_n = \frac{p}{\sqrt{1+f^2}}$$

Since the angle between p and the radius (normal) to the point of actual contact with the bearing, must be ϕ, the components p_n and p_t are readily drawn, and *the direction of the radius is found* as is the *position* of p_n, and the angle β.

It is evident that center line of the supporting force misses the center by the distance $r \sin \phi$, so that the *resisting* moment due to friction is $pr \sin \phi$, and the resultant *unbalanced* moment is

$$L = TR - pr \sin \phi$$
$$= TR - rf\left(\frac{T^2 + W^2 + 2TW \sin \theta}{1+f^2}\right)^{\frac{1}{2}}$$

The relative simplicity of the second solution is obvious. It is furthermore evident that the lines of action of the three external forces; T, W, and p, when in actual position, do *not* meet at a point, unless $TR = Pr \sin \phi$; that is, unless the driving moment is no greater than the moment of friction, so that $L = 0$, and the rotation is uniform.

261. A corollary with $\theta = \frac{1}{2}\pi$. Suppose the weight which is centered on the shaft is W, and that a vertical tension T is applied to a drum whose radius is r_2. The co-efficient of friction between shaft (whose radius is r) and the bearings treated as one, is $f = \tan \phi$. We are to find the unbalanced moment turning the shaft.

The magnitude of the support is $p = W + T$. The shaft *climbs* until the radius to the central point of the surface of contact makes an angle ϕ with the vertical. It is evident without a figure that the arm $x = r \sin \phi = \dfrac{rf}{\sqrt{1+f^2}}$, and that the resultant moment is

$$L = Tr_2 - (W+T)x = Tr_2 - (W+T)\frac{rf}{\sqrt{1+f^2}}$$

The angle ϕ is usually small, so small in fact that the difference between $\sin \phi$ and $\tan \phi$ is negligible. This is equivalent to saying that f is so small that its square (which is still smaller) may be neglected in the radical, so that the value of L becomes

$$L = Tr_2 - (W+T)rf \text{ very nearly.}$$

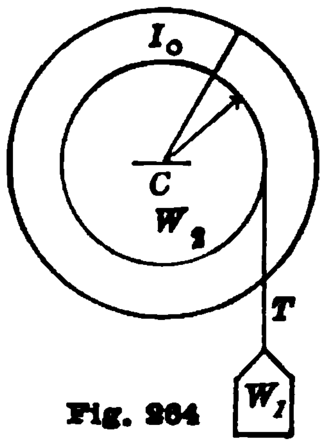


Fig. 264

262. The tension in a moving rope. Suppose by means of a weight W_1, connected to a small wire rope which is wound about a drum, a shaft with fly-wheel is made to revolve on smooth ideal bearings at C. Fig. 264. Let the Moment of Inertia of the shaft and fly-wheel be I_0. Find the tension in the wire, the linear and angular accelerations of W_1 and W_2 respectively, and find θ and S if the time t of the operation be given.

The *unbalanced moment* L acting upon W_2 is $Tr = L$.

Hence $\qquad a = \frac{Tr}{I_o}$

The unbalanced force F acting upon W_1 is $F = W_1 - T$;

hence $\qquad a = (W_1 - T)\frac{g}{W_1}$

Now by **256** $\qquad a = ra$

whence $\qquad \frac{W_1 - T}{W_1}g = \frac{Tr^2}{I_o}$

$$T = \frac{I_o}{I_o + m_1 r^2} \cdot W_1$$

It follows that $\qquad a = \frac{m_1 r g}{I_o + m_1 r^2}$

$$a = \frac{m_1 r^2 g}{I_o + m_1 r^2}$$

Since T is constant the formulas for time and distance are (**256** II),

$$s = \frac{1}{2}at^2, \text{ and } \theta = \frac{1}{2}at^2$$

$$s = \frac{1}{2}\cdot\frac{m_1 r^2 g}{I_o + m_1 r^2}\cdot t^2, \text{ and } \theta = \frac{1}{2}\cdot\frac{m_1 r g}{I_o + m_1 r^2}\cdot t^2$$

For final velocities we may integrate the formulas $vdv = ads$ and $\omega d\omega = ad\theta$ between the limits o and v_1, and o and s; and o and ω_1, and o and θ.

$v_1^2 = 2as \qquad \omega_1^2 = 2a\theta$

$v_1 = \frac{m_1 r^2 g}{I_o + m_1 r^2}\cdot t \qquad \omega_1 = \frac{m_1 r . g}{I_v + m_1 r^2}\cdot t$

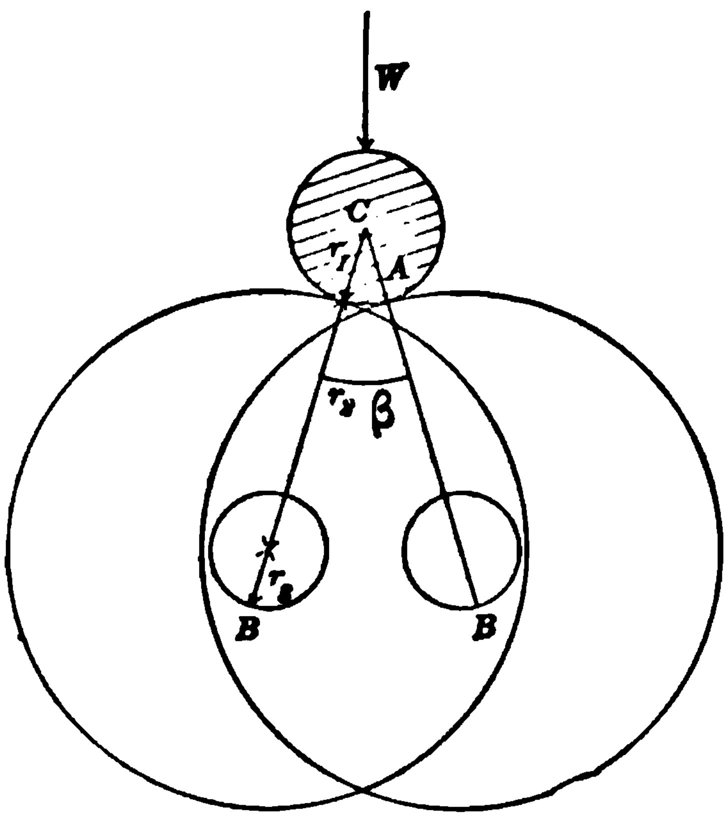


Fig. 265

263. Friction rollers and roller bearings. Aside from lubricants, which are always needed where slipping is unavoidable, the retarding (and generally wasteful) influence of friction may be greatly diminished by "friction-rollers" or balls. The resisting moment due to friction without rollers

is very nearly fWr_1, see **261.** With a pair of rollers as shown in Fig. 265; the normal pressure on the bearings at B and B_1 is for each $\frac{1}{2}W \sec \frac{\beta}{2}$, and the resisting moment to be overcome at each point B is

$$L = \frac{r_3}{2} fW \sec \frac{\beta}{2}$$

By the law of the lever, the tangential action at A is to the tangential action at B *inversely* as the radii r_2 to r_3; hence the action at A is

$$\frac{r_3}{r_2} \cdot \frac{1}{2} fW \sec \frac{\beta}{2},$$

so that the *resisting moment* at A is

$$\frac{r_1 r_3}{2r_2} \cdot fW \sec \frac{\beta}{2}.$$

As the resisting moment is the same at each point A, the total resisting moment on the shaft C is

$$L = \frac{r_1 r_3}{r_2} fW \sec \frac{\beta}{2},$$

from which we see that the moment *without* rollers has been reduced by the factor

$$\frac{r_3}{r_2} \sec \frac{\beta}{2}.$$

If the angle β is small, the secant of $\frac{1}{2}\beta$ is not much greater than one. No allowance is here made for "rolling friction" which is not due to slipping but to imperfect surfaces and the elasticity of materials.

Since the ratio $\frac{r_3}{r_2}$ is generally quite small there is a substantial saving in energy.

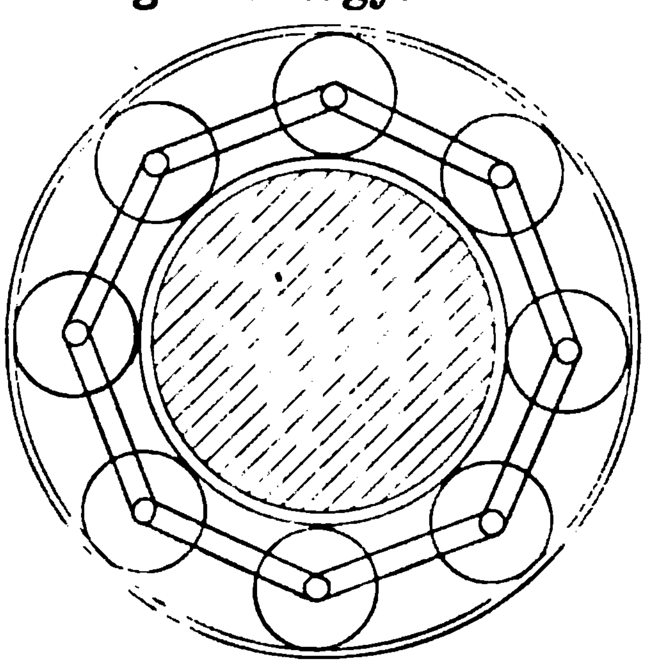


Fig. 266

Roller bearings with an endless chain of spheres or cylinders rolling within suitable chambers, eliminate nearly all friction; they need not be lubricated. Fig. 266. The friction to be overcome is purely rolling friction without slipping.

264. An extended rigid mass, otherwise free, is so acted upon by a concentrated unbalanced force that it at once acquires a linear acceleration but no angular acceleration.

The position of the line of action of the force relative to the mass is required. Fig. 266½.

Let F be the force and m the mass. Assume a set of rectangular axes, the given line of action being YY. Let dm be an element of the mass with the two co-ordinates x and z. Break up the force F into an indefinite number of co-linear forces dF, and then resolve every dF into an equal parallel force, dF, acting at a separate element dm; and a couple ρdF where ρ is the perpendicular distance of the element from the axis of Y. Next, resolve the couple ρdF into two component couples, xdF and zdF.

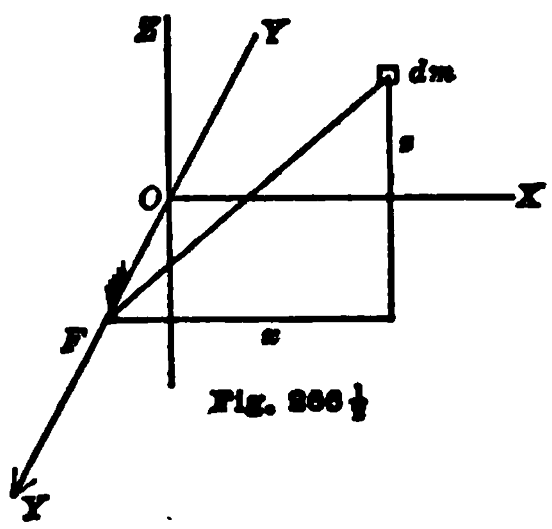


Fig. 266½

Finally, let dF be so proportioned to dm that it gives to it the common linear acceleration a parallel to YY, that is, $dF = adm$, where a is a constant. Every element in the mass is thus acted upon, and for every element there are the two couples xdF and zdF applied to the mass of the element, with a tendency to turn it about the axes of OZ and OX respectively. Summing results, we have

$$F = \int dF = a\int dm = am. \tag{1}$$

$$\left.\begin{aligned} L_z &= \int xdF = a\int xdm \\ L_x &= \int zdF = a\int zdm \end{aligned}\right\} \tag{2}$$

From equations (2) we have, by hypothesis that there is to be no rotation,

$$\int xdm = 0 \quad \text{and} \int zdm = 0 \tag{3}$$

which show that the planes ZY and XY must respectively pass thru the *center of mass* of the given body; that is:

The line of action of the given force must pass thru the center of mass of the given body in order to impart *no angular acceleration.*

Conversely, a centric unbalanced force (one whose line of action passes thru the center of mass of a body) imparts a linear acceleration but no angular acceleration; and consequently an unbalanced *eccentric force* acting upon an extended mass, imparts a linear acceleration $a = \dfrac{F}{m}$ in the direction of the force, and an angular acceleration α about an axis thru the C. G., perpendicular to the plane of the given line of action, and the C. G. of the given mass, whose value is given by the equation

$$\alpha = \frac{L}{I_o} = \frac{lF}{I_o}$$

in which l is the distance from the C. G. of the body to the given line of action of the force and $L = \sqrt{L_x^2 + L_z^2}$.

265. Rotation and translation combined. When a body is acted upon by other bodies in such a way that the line of resultant action does not pass thru the center of mass of the body whose motion we are to investigate, the action is eccentric, and the result is both translation and rotation, since the force is automatically resolved into an unbalanced force at the center of mass and an unbalanced couple.

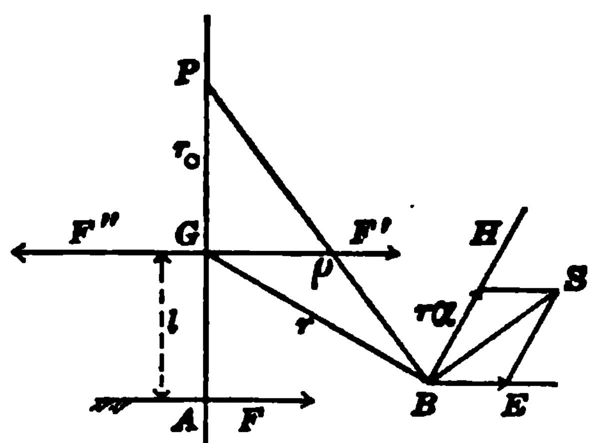


Fig. 267

Let G, Fig. 267, be the center of mass of the body, and let an unbalanced force act at A in the line shown in the figure. The component force thru G is F^1, and the component couple has a moment Fl, which in this case appears to be left-handed.

There is of course a linear acceleration, a; and an angular acceleration, α.

The force $F' = F$ produces a linear acceleration in all points of the body of $a = \dfrac{F}{m}$ in a direction parallel to F. The couple produces an angular acceleration about G (as it moves) whose value is $\alpha = \dfrac{Fl}{I_o}$.

The *resultant acceleration* of any point as B at a distance r from G is found by laying off $BE = a$, and then $BH = r\alpha$, the former being parallel to F, and the latter perpendicular to r.

It takes but a moment's consideration to make it clear that there must be a point P on the prolongation of AG thru G, where a and $r\alpha$ *would be exactly equal and opposite*, so that the resultant acceleration at that point would be zero; showing that there would be *no unbalanced force acting at P arising from the action of F at A*. Let PG be r_0; then we have $a = r_0\alpha$. Substituting for a its value $\dfrac{F}{m}$, and for α its value $\dfrac{Fl}{I_0}$, we have $\dfrac{F}{m} = \dfrac{r_0 Fl}{I_0}$.

Hence
$$r_0 = \frac{I_0}{ml} = \frac{k_0^2}{l}, \text{ and } l = \frac{k_0^2}{r_0}$$

in which k_0 is the radius of gyration of the solid body.

The point P may be called the "**Instantaneous Center.**" It is easily shown by similar triangles, that the resultant acceleration of any point, as B, is perpendicular to $PB = \rho$, and equal to $\rho\alpha$. The influence of gravity has been omitted in this discussion.

266. The center of percussion. The peculiar relation between the point P and the line of application A is shown by the following: Let a heavy rigid bar, whose length is s, Fig. 268, be hung on a smooth peg at P, and be struck by a hammer as shown at A, below G at a distance $l = \frac{k_0^2}{r_0}$, the bar will instantly begin to rotate around P with *at first* no lateral action upon the peg. The point A is then called the "*Center of Percussion,*" with reference to the axis at P. It was found (**253**) that for a slim uniform rod,

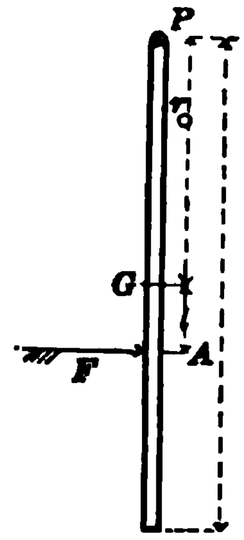


Fig. 268

$$k_0^2 = \frac{s^2}{12}, \qquad \text{hence } l = \frac{s^2}{12 r_0}.$$

$$\begin{array}{lll}
\text{If} & r_0 = \frac{s}{2}, & l = \frac{s}{6} \\
 & r_0 = \frac{2}{5}s, & l = \frac{5}{24}s \\
 & r_0 = \frac{s}{3}, & l = \frac{s}{4} \\
 & r_0 = \frac{s}{4}, & l = \frac{s}{3} \\
 & r_0 = \frac{s}{6}, & l = \frac{s}{2}
\end{array}$$

The body struck may have any shape; it will always have a center of mass G and a perpendicular can be drawn from G to the line of action of the applied force, and the length of that perpendicular can be called l. It was shown above that there is always a point P on the line l produced where the linear acceleration due to the translation of the mass, was exactly equal and opposite to the linear acceleration due to the rotation of the mass about G.

As the radius of gyration, k, is a mean proportional between r_0 and l, the graphical construction of l or r_0 is very simple.
When the radius of gyration is known, the value of r_0 can be found for any value l, and conversely. Moreover, since it is only a constant product, $r_0 l$, the letters may change places, which means that *if the force F is applied at P, the point A will be the instantaneous center*; *i. e.*, P is the center of Percussion for A.

267. Again the force F need not be a blow or stroke of a hammer. F may be as gentle and inaudible as the force of gravity. For example:

Suppose a strong thread is wound about a vertical great circle of a solid sphere, with one end fast to the sphere and the other end to a nail directly above the point P. Fig. 269. The sphere is at rest, and the thread at P is straight and vertical. Gravity is balanced by the shelf. Now let the shelf be suddenly dropped. The sphere is now acted upon by an *unbalanced* force which is the resultant of W and a certain tension in the thread; that resultant acts down according to Chapter II, at some point A beyond the center. It produces both translation and rotation, and A is the *center* of *percussion* for the *point* P, which is the instantaneous axis. We can find GA by our equations. The Table in **253** gives

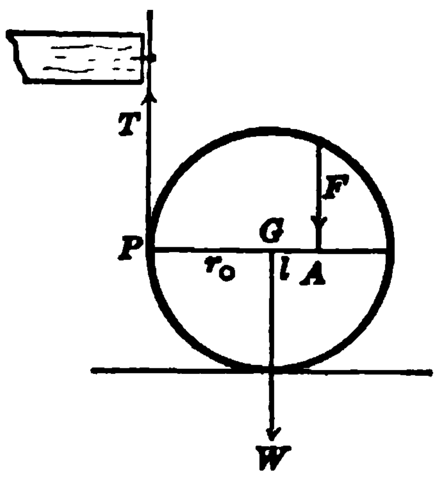


Fig. 269

$$k_o^2 = \frac{2}{5} r^2$$

$$r_o = r$$

hence
$$l = \frac{k_o^2}{r} = \frac{2}{5} r.$$

which gives us the center of percussion, as the sphere rolls down; and there is a point A for every position of the sphere on the way down, for the sphere is being constantly acted upon eccentrically by the same unbalanced force. This illustration will appear again under the head of "rolling bodies."

268. Rolling bodies. *A cylinder rolls down a rough plane,* Fig. 270. Discuss its motion. [It is hardly necessary to say that the plane is "rough" since the cylinder would not roll if it were smooth.] The forces acting are gravity and the plane, their resultant does not act thru the center, hence the body has rotation as well as translation. W is resolved into $W \sin\theta$ parallel to the plane, and $W \cos\theta$, normal; the latter is balanced by the normal action of the plane. The tangential action of the plane T combined with an *equal part of* $W \sin\theta$, forms a couple which produces an *angular* acceleration α, while the *remainder* of $W \sin\theta$, that is $(W \sin\theta - T)$, produces a linear acceleration parallel with the plane.

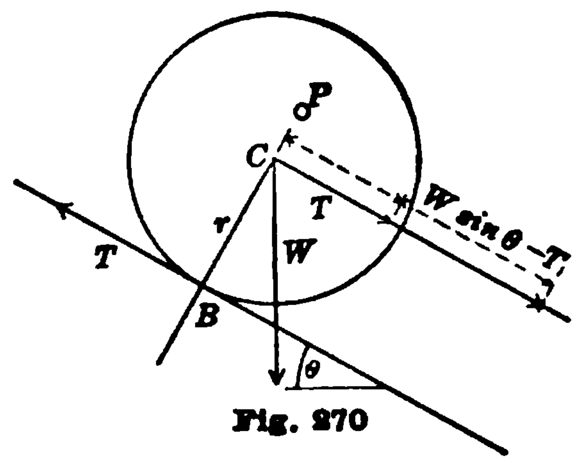


Fig. 270

Hence
$$a = \frac{(W \sin\theta - T)g}{W}, \qquad \text{and } \alpha = \frac{Tr}{I_o} \tag{1}$$

Since the point B is the instantaneous axis and has no resultant acceleration, we have
$$a = r\alpha.$$

Hence $$a = \frac{W\sin\theta - T}{m} = \frac{Tr^2}{I_o} = \frac{Tr^2}{mk_o^2} \tag{2}$$

Wherefore $$T = \frac{k_o^2}{r^2 + k_o^2} W \sin\theta \tag{3}$$

and $$a = \frac{r^2}{r^2 + k_o^2} g \sin\theta \tag{4}$$

which shows that the acceleration is independent of W.

This formula holds as long as r is the rolling radius and k_o is the radius of gyration of the rolling body, with reference to an axis thru C.

1. For a solid circular cylinder,

$$k_o^2 = \frac{r^2}{2} \text{ and } T = \frac{1}{3} W \sin\theta \tag{5}$$

2. Had the rolling body been a *solid sphere,*

$$k_o^2 = \frac{2r^2}{5} \text{ and } T = \frac{2}{7} W \sin\theta \tag{6}$$

3. Had the body been a very thin cylinder,

$$k_o^2 = r^2 \text{ and } T = \frac{1}{2} W \sin\theta \tag{7}$$

4. Had the body been a thin spherical shell,

$$k_o^2 = \frac{2}{3} r^2 \text{ and } T = \frac{2}{5} W \sin\theta \tag{8}$$

269. The reader will take notice that the force T which is *due* to friction, is *not* $fW\cos\theta$; *it is always less*; that is,

$$T < fW\cos\theta.$$

If $$T > fW\cos\theta,$$

then *friction alone* would not prevent the body from slipping as well as rolling. This will be considered in a later problem.

Again since the force producing translation alone is $F = W\sin\theta - T$, it follows that the larger T is, the smaller F is, the less is a, the less is v for any time t $\left[\text{since } s = v_0 t + \frac{1}{2}at^2\right]$, and the *longer* is t for any required velocity or space. In other words, in popular phrase: the solid sphere is the fleetest of the rollers; next the solid cylinder; next the spherical shell; and slowest of all, the thin cylinder or hoop.

270. The equations are sufficient. Having found a and α, both of which are constant as long as the conditions are unchanged, the equations given in **256** suffice for finding velocity, time, space, etc.

If the force $W \sin \theta$, acting at the center C, and the force T acting at B, be combined, the resultant, (see **39**), is a parallel force, $W \sin \theta - T$, acting at P, whose distance from C is $l = \dfrac{Tr}{W \sin \theta - T}$.

Substituting for T its value $\dfrac{k_o^2}{r^2 + k_o^2} W \sin \theta$, we have $l = \dfrac{k_o^2}{r}$, which shows that P is the center of Percussion for the point B. Hence the *resultant* of all the forces acting on a rolling body, act parallel to the plane at the center of percussion which is continually $r + l = r + \dfrac{k_o^2}{r}$ distant from the plane. The resultant acceleration of P is $(r+l)\alpha$, substituting for $\alpha = \dfrac{Tr}{I} = \dfrac{rg \sin \theta}{r^2 + k_o^2}$, and for $(r+l) = \dfrac{r^2 + k_o^2}{r}$; that is $(r+l)\alpha = g \sin \theta$.

The actual linear acceleration of the center of percussion P, $g \sin \theta$, is the acceleration which a body would have *sliding* down a *smooth* plane of the same slope. It is, of course, understood that during the rolling, the body does no crushing or smoothing of the surface. It rolls as tho compelled to do so by a ribbon wrapt around it and secured to a smooth plane at a point higher up.

Velocities and times compared. Let the student fill out the Table outlined below.

Homogeneous bodies rolling down a plane whose inclination is θ and length s.

Bodies	α		a		T		v_1		ω_1		t_1	
Inclination θ	30°	90°	30°	90°	30°	90°	30°	90°	30°	90°	30°	90°
Solid Sphere												
Solid Cylinder												
Hollow Sphere												
Thin Cylinder or Hoop												

A ribbon or belt is used when $\theta = \dfrac{\pi}{2}$ as shown in Fig. 271.

The magnitude of T is sometimes a surprise to a workman who holds the end of a rope while a piece of lumber rolls, or falls and unwinds.

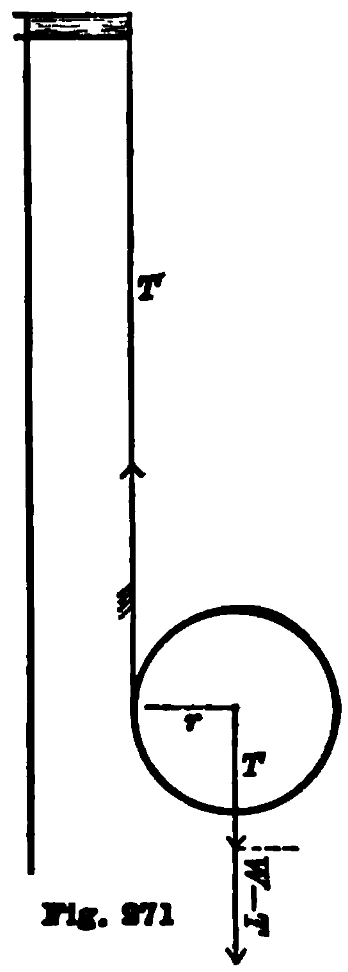


Fig. 271

271. An illustration when both a and α are negative. A heavy cylinder has [by some unknown force] been set in motion, and is rolling along by itself on a rough horizontal plane. Its axis has a linear velocity v_0. It is retarded by the roughness of the plane and comes to rest in t_1 seconds. Derive expressions for the resisting moment and the resisting force, and illustrate the actions of forces. Fig. 272.

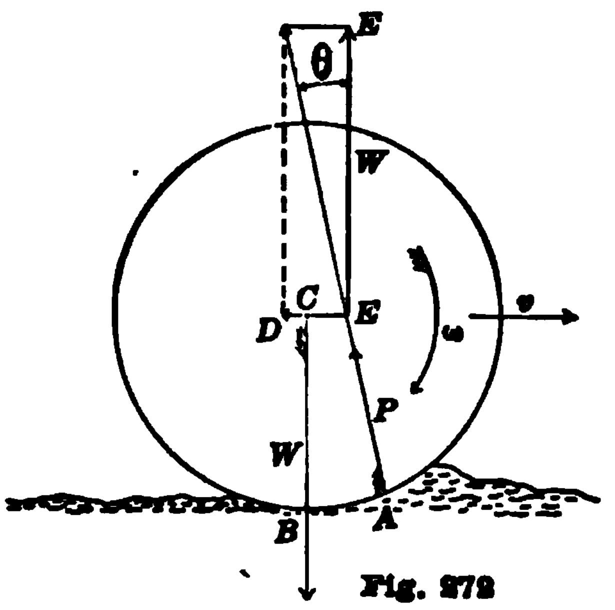


Fig. 272

Assuming that the retarding action is constant, it is evident that $a = -\frac{v_0}{t_1}$; and since it rolls with a radius r, we have $\alpha = -\frac{\omega_0}{t_1} = -\frac{v_0}{rt_1}$, since $v_0 = r\omega_0$. Hence the translation is retarded by a force $\left(\frac{-Wv_0}{gt_1}\right)$, and its rotation is retarded by a couple whose moment is $\left(-\frac{v_0}{rt_1}\cdot I\right)$.

The cylinder is acted upon by two forces: the vertical pull of the earth, and the oblique, upward, and backward action P of the plane. At the point E where the line of action of P crosses a horizontal thru C, resolve P into a horizontal and a vertical component.

The vertical component must be W; the *horizontal* component must be $\left(-\frac{Wv_0}{gt_1}\right)$. If x_0 be the distance CE, the moment of the retarding couple is $-Wx_0 = -\frac{v_0 I}{rt_1} = -\frac{Wv_0k_0^2}{grt_1}$. Hence $x_0 = \frac{v_0k_0^2}{gt_1r}$. As k_0^2 is a function of r^2, x_0 is a fractional part of r on the forward side of C. The angle θ is known from the fact that $\tan\theta = \frac{v_0}{gt_1}$ which is independent of both W and r, as it evidently should be.

Hence the resisting *moment* is

$$L = Wx_0 = \frac{v_0}{gt_1} W . \frac{k_0^2}{r}. \text{ (negative)}$$

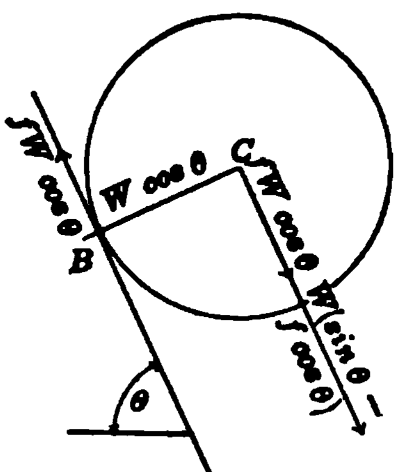


Fig. 273

and the resisting *force* is

$$F = \frac{v_0}{gt_1} W \text{ (negative).}$$

272. Rolling and sliding. This is the case when $fW \cos \theta$ *is less than the force that would cause the body to roll without slipping.* We have now the unbalanced force $W(\sin\theta - f\cos\theta)$, and the unbalanced couple $frW\cos\theta$.

Hence $$a = (\sin\theta - f\cos\theta)g,$$

and $$\alpha = \frac{frW\cos\theta}{I} = \frac{rfg\cos\theta}{k_0^2}.$$

But $\alpha r < a$, hence the *acceleration of sliding is*

$$a - r\alpha = \left(\sin\theta - f\cos\theta - \frac{fr^2\cos\theta}{k_0^2}\right)g = a_s.$$

The velocity of sliding at the time $t = ta_s$, and the whole *amount* of sliding in the time t_1 is $\frac{1}{2} a_s t_1^2$.

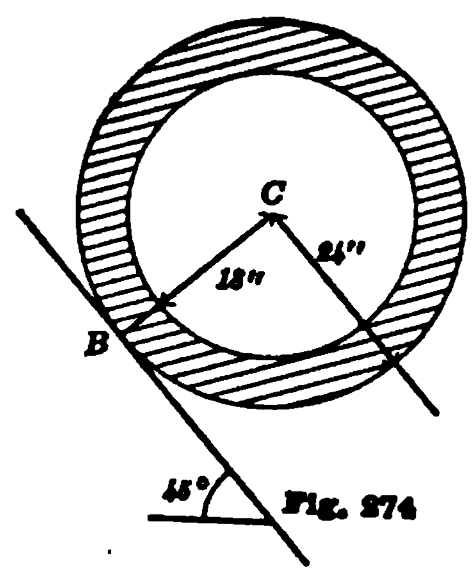


Fig. 274

A *Numerical Example,* illustrating *Rolling* and *Sliding* simultaneously. Let $s = 112\sqrt{2}$ feet, $\theta = 45°$, the radii of a thick hollow iron cylinder C whose length is two feet, are $r = 18$ inches, $R = 24$ inches, specific weight per cubic foot 490 lbs., and $f = \frac{1}{8}$. The cylinder starts from rest. It is found that $W = 1715\pi$, and $k_0^2 = \frac{r^2 + R^2}{2} = 3.125$. The linear acceleration of C is $a = 14\sqrt{2}$, (if $g = 32$), hence $t_1 = 4$ sec.

The angular acceleration is $$\alpha = \frac{g\sqrt{2}}{25}.$$

Hence the *acceleration of sliding* is

$$a_s = a - R\alpha = 14\sqrt{2} - \frac{2g\sqrt{2}}{25} = 11.44\sqrt{2}$$

The *amount* of *the sliding* in 4 sec is

$$\frac{1}{2}\left(11.44\sqrt{2}\right)16 = \frac{1}{2} a_s t_1^2 = 91.52\sqrt{2} \text{ ft.}$$

Accordingly, it rolls $20.48\sqrt{2}$ ft. and slides $91.52\sqrt{2}$.

273. A cylindrical roller is drawn up a rough inclined plane by a hanging weight W_1 with a small flexible wire passing over a "smooth" peg, as shown. Fig. 275. Find T, a_1 and α.

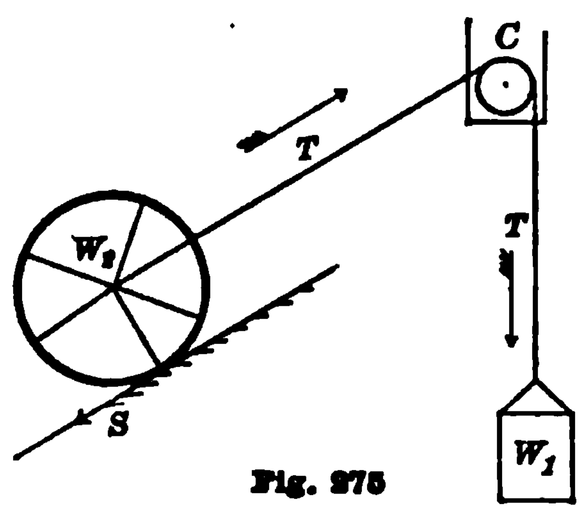


Fig. 275

1. As W_2 rolls with increasing angular velocity, we will first find an expression for the unbalanced moment causing α. Let S represent the tangential action of the plane on W_2; then the moment of the couple is Sr.

Hence
$$Sr = I\alpha = \frac{W_2 k_0^2 \alpha}{g} = \frac{W_2 k_0^2 a}{gr} \tag{1}$$

$$a = \frac{Sr^2 g}{W_2 k_0^2} \tag{2}$$

The force producing translation in W_2 is

$$F = T - W_2 \sin\theta - S = W_2 \frac{a}{g}. \tag{3}$$

Considering the forces acting on W_1 we have

$$W_1 - T = W_1 \frac{a}{g} \tag{4}$$

Adding (3) and (4) and substituting the value of S from (1)

$$W_1 - W_2 \sin\theta - \frac{W_2 k_0^2}{r^2} \cdot \frac{a}{g} = (W_1 + W_2) \cdot \frac{a}{g}$$

hence
$$\left.\begin{aligned} \frac{a}{g} &= \frac{W_1 - W_2 \sin\theta}{W_1 + W_2 + \dfrac{W_2 k_0^2}{r^2}}. \\ T &= W_1\left(1 - \frac{a}{g}\right). \\ S &= \frac{W_2 k_0^2}{r^2} \cdot \frac{a}{g}. \\ \alpha &= \frac{a}{r}. \end{aligned}\right\} \tag{5}$$

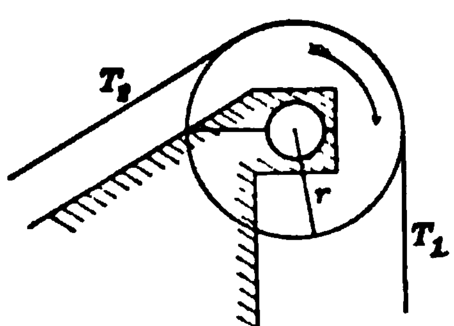


Fig. 276

2. If, Fig. 276, there had been a heavy pulley with smooth bearings at C instead of a smooth peg, we should have had two values of T: T_1 the tension resisting the descent of W_1, and T_2 accelerating the motion of W_2. In that case there would have been a new equation, arising from the necessity of having a couple to turn the pulley:

$$(T_1-T_2)r_1=I_1\alpha_1=\frac{I_1a}{r_1} \tag{6}$$

Writing T_2 for T in (3), and T_1 for T in (4), we have with (6) three equations for finding T_1, T_2 and a.

274. **Prob. A hanging weight W_1 draws a rolling solid cylinder up an inclined plane by means of a rope wrapt around it.** With the notation shown in the figure, Fig. 277, we are to find the tension in the cord, the linear and angular accelerations, and the tangential action of the plane.

Let the linear acceleration of W_1, and of the point A be a; let the angular acceleration of W_2 be α. Since the body rolls, the point B has no acceleration, the linear acceleration of C is $\frac{a}{2}$, and $\alpha=\frac{a}{2r}$.

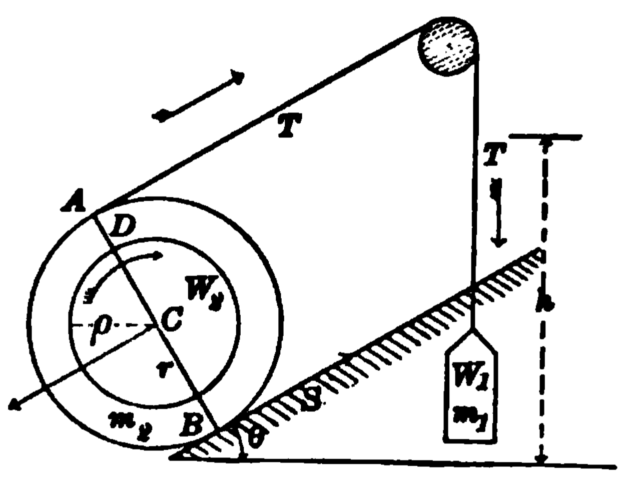


Fig. 277

Considering the motion of W_1 we have at once

$$\frac{W_1-T}{W_1}=\frac{a}{g} \tag{1}$$

Considering the translation of W_2 we have

$$T+S-W_2\sin\theta=\frac{a}{2g}W_2 \tag{2}$$

Considering the rotation of W_2 we have

$$(T-S)r=I_0\alpha=\frac{W_2}{g}\cdot\frac{r^2}{2}\cdot\alpha$$

$$T-S=W_2\frac{a}{4g}\cdot \tag{3}$$

Adding (2) and (3) we have

$$2T=W_2\left(\sin\theta+\frac{a}{2g}+\frac{a}{4g}\right)$$

Substituting the value of $\frac{a}{g}$ from (1) we have

$$T=\frac{W_1W_2(4\sin\theta+3)}{8W_1+3W_2}. \tag{4}$$

Substituting for T in (1) we have

$$\frac{a}{g}=1-\frac{T}{W_1}=1-\frac{W_2(4\sin\theta+3)}{8W_1+3W_2}=\frac{8W_1-4W_2\sin\theta}{8W_1+3W_2} \tag{5}$$

Subtracting (3) from (2) we get

$$S=\frac{W_2}{2}\left(\sin\theta+\frac{a}{4g}\right)$$

Substituting for $\frac{a}{4g}$ from (5) we get

$$S=\frac{W_2}{2}\left(\sin\theta+\frac{2W_1-W_2\sin\theta}{8W_1+3W_2}\right)$$

$$S=\frac{W_1W_2(1+4\sin\theta)+W_2^2\sin\theta}{8W_1+3W_2} \qquad (6)$$

Corollaries. 1. If $\theta=0$, the plane is horizontal, and we have

$$\left.\begin{aligned}T&=\frac{3W_1W_2}{8W_1+3W_2}\\ S&=\frac{W_1W_2}{8W_1+3W_2}=\frac{T}{3}.\\ \frac{a}{g}&=\frac{8W_1}{8W_1+3W_2}\end{aligned}\right\} \qquad (11)$$

2. If $\theta=\frac{\pi}{2}$, the plane is vertical, and S must be the tension in a ribbon, *not* the rope, up which the cylinder rolls and winds.

$$\left.\begin{aligned}T&=\frac{7W_1W_2}{8W_1+3W_2}\\ S&=\frac{5W_1W_2+W_2^2}{8W_1+3W_2}\\ \frac{a}{g}&=\frac{8W_1-4W_2}{8W_1+3W_2}\end{aligned}\right\} \qquad (12)$$

3. Show that if the rolling body be a thin hollow cylinder, the general values are

$$\left.\begin{aligned}T&=\frac{W_1W_2(1+\sin\theta)}{2W_1+W_2}\\ S&=\frac{1}{2}W_2\sin\theta\\ \frac{a}{g}&=\frac{2W_1-W_2\sin\theta}{2W_1+W_2}\end{aligned}\right\} \qquad (13)$$

4. Show that, if $W_1 = \frac{1}{2} W_2 \sin\theta$, the general values are: $\frac{a}{g}$ = zero, and either W_2 does not move or it moves with uniform velocity.

$$\left.\begin{aligned} T &= \frac{1}{2} W_2 \sin\theta = W_1 \\ S &= \frac{1}{2} W_2 \sin\theta \end{aligned}\right\} \qquad (14)$$

5. If
$$W_1 < \frac{1}{2} W_2 \sin\theta$$
$\frac{a}{g}$ becomes negative and the motion is down the plane.

The student will notice on comparing the actions of the rough plane in the last two problems, that in Fig. 275 the tangential component of the plane's action is *down*, while in Fig. 277 it is *up*, and that the change is evidently brought about by the change in the position of the rope from acting at C to acting at A. He may then infer that there *must be* a point D, between A and C, where, if the rope could act on the circumference of an inner pulley, the tension at D would cause S to become zero.

Problems.

Ex. 1. Let the student redraw Fig. 277, with the rope tangent to the interior circle at D (but still parallel to the plane), and derive new equations and find the value of CD, assuming all the time that S is zero. Let him then give to the point D its appropriate name.

Ex. 2. A four-wheeled truck is drawn up a plane with an acceleration a as the cylinder was drawn in Fig. 275. Assume W_2 for the weight of truck and wheels, and I_0 for the sum of the I_0's for the wheels. Find T, and S for each wheel.

Ex. 3. A cylindrical disk 2 ft. in diameter and six inches thick has, projecting on *each end* of its axis, a shaft 6″ in diameter and 5 ft. long including the disk. It rolls without slipping (by means of ribbons or fine teeth) down a narrow-gage track which has an inclination of 45°. The shaft and disk are homogenous. Find a and α. Fig. 278.

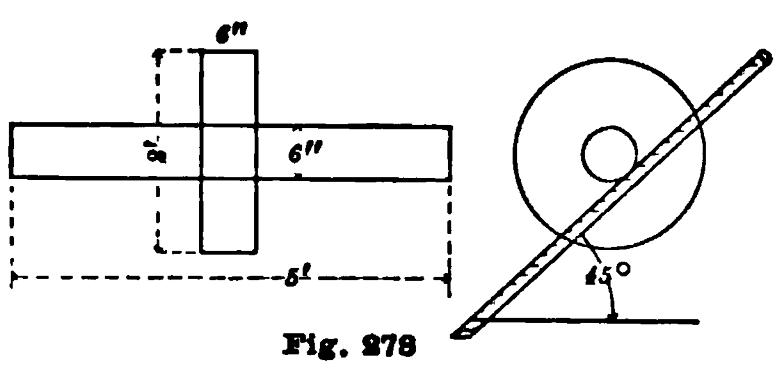


Fig. 278

Ex. 4. A solid rigid sphere rolls down a trough of rigid material, the inclination being θ, and the trough angle being 90°. Fig. 279.

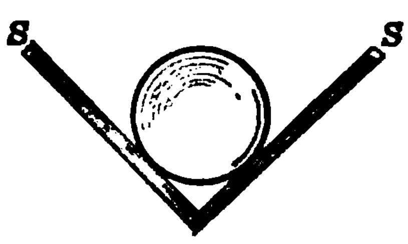


Fig. 279

In point of fact, with elastic bodies, there is a zone

of contact and there is much slipping on all sides of the ideal point of contact.

Ex. 5. (An extra ideal problem.) A heavy but perfectly flexible rope, with length l, weighing w lbs. per foot, is secured to the horizontal shaft of a fly-wheel and then wrapt around it leaving a hanging end whose length is s_1. Given I_0 for the wheel and shaft, and r for the *effective* radius (including the radius of the rope) of the shaft and rope, and assuming that there is no friction of any kind, and that the wheel, when unclamped, is free to turn, discuss the motion of the wheel up to the time when the rope has run completely off, after the wheel has been set free.

275. The compound pendulum. A rigid body mass m, whose center of mass is at G is hung upon a knife edge at O. The distance $OG = r_0$ is known. Fig. 280.

As the body swings, all parts have the same angular velocity and the same angular acceleration, but different radii. As we know from simple pendulums, particles of the body near O, tend to oscillate quickly; those remote from O oscillate slowly, but as the body is rigid all must oscillate together, the slower being helped along by the more rapid, by means of shearing stress, so that the result is a compromise movement which is in every respect the same as the movement of a certain simple pendulum whose length is l. The problem is now to find l.

Let the body be swinging as shown, and let the point P, the extremity of the unknown l, have the velocity v, so that $v = l\omega$, the angular velocity ω being common to all parts of the body. Let l (and therefore OG) make an angle θ with a vertical line thru O. Let there be an element of mass, dm, at A with a radius ρ making an angle ϕ with OG.

The linear *acceleration* at P is

$$g \sin \theta = a = l\alpha$$

Accordingly, the linear acceleration of A must be

$$\rho\alpha = \frac{\rho g}{l} \sin \theta$$

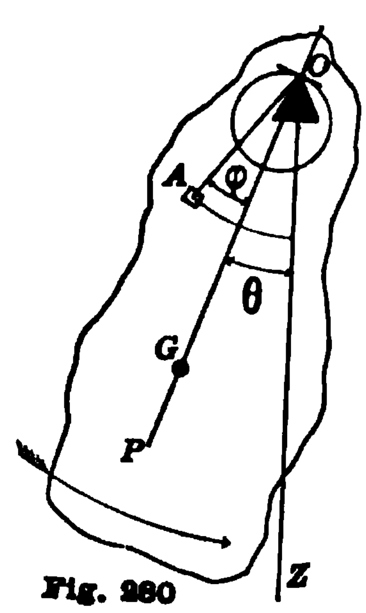


Fig. 280

This linear acceleration requires for the mass dm at A an accelerating force sufficient to give it the acceleration $\rho\alpha$. This required force is

$$dF = dm\rho\alpha = gdm \frac{\rho}{l} \sin \theta. \qquad (1)$$

The actual accelerating force at A perpendicular to OA is $gdm \sin (\phi + \theta)$ due to gravity, which is in excess of what is required, by the quantity,

$$gdm \sin (\phi + \theta) - \frac{\rho g dm}{l} \sin \theta = dF' \qquad (2)$$

This surplus force produces a moment about O.

$$\rho dF' = dL = \rho g dm \sin(\phi+\theta) - \frac{\rho^2 g dm}{l}\sin\theta.$$

The magnitude (and sign) of dL varies from point to point thruout the body, and since our hypothesis was that the body had the acceleration of P, it is evident that the integral of dL must be zero; hence

$$\int \rho\sin(\phi+\theta) g dm = \int \frac{g}{l}\rho^2 \sin\theta dm. \qquad (3)$$

During this integration θ is constant while ρ and ϕ vary. The first member of (3) is the moment of the *weight* of the *body* about the axis at O, and is therefore

$$g\int \rho\sin(\phi+\theta)dm = mgr_0\sin\theta = Wr_0\sin\theta$$

The second member of (3) is

$$\frac{g\sin\theta}{l}\int \rho^2 dm = \frac{g\sin\theta}{l}I = \frac{gm\sin\theta}{l}\cdot k^2$$

Hence, canceling the common factors, we have

$$r_0 = \frac{k^2}{l}, \text{ or } l = \frac{k^2}{r_0}.$$

It thus appears that P is the Center of Percussion for the axis O, in accordance with **267**.*

CHAPTER XVI.

Deviating Forces.

276. Component accelerations. When a body is acted upon by an unbalanced force,† the acceleration is always in the direction of the action. If the force be resolved into components, each component will produce an acceleration which is proportional to itself and in the same direction. If these component accelerations are laid off as vectors, their diagonal resultant, found as in the case of forces, will fully represent the acceleration of the unresolved force. This statement is equally true when the body acted upon is in motion or, at first, at rest.

* For an account of Capt. Kater's famous compound pendulum, see Encyclopoedia Britannica.

† Unless otherwise stated, the forces acting upon moving bodies are now assumed to be *centric;* i. e., their lines of action pass thru the centers of mass.

From the fundamental equations of motion, we had for velocity and acceleration along s,

$$v = \frac{ds}{dt}, \qquad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

In like manner,

velocity and acceleration along x,

$$v_x = \frac{dx}{dt} \qquad a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$$

velocity and acceleration along y,

$$v_y = \frac{dy}{dt} \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$$

Acceleration along the normal $= a_n$; along the tangent $= a_t$.

2. If a body A, Fig. 281, having a motion of translation with a velocity v, is acted upon by an accelerating (*i. e.*, unbalanced) force F in a direction oblique to v, its acceleration is oblique also. In this case it is best to resolve F into components, one along v and one at right angles with it. Inasmuch as the direction of v immediately changes, its path is not a straight line. Hence one component of F, $F \cos \theta$, will lie along the tangent, while $F \sin \theta$ lies along the normal. The tangential component produces an acceleration of v. The normal component is called a **Deviating Force,** because its effect is to cause the body to *deviate* from a rectilinear path and move on a curve; and its acceleration, as a vector, lies along the normal pointing towards the center of curvature. The two accelerations are generally written a_t and a_n. The deviating component does not affect the numerical value of the velocity.

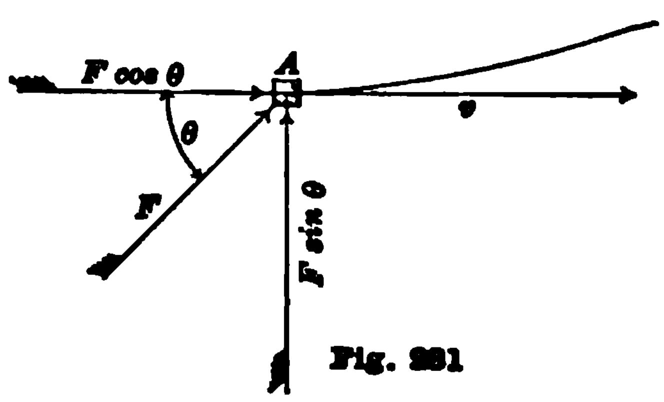


Fig. 281

In the case of a body sliding down a smooth curved guide, Chapter XIV, only the tangential component of W was considered, as the normal component was balanced (and *more than balanced*) by the action of the guide.

There is no limit to the number of problems involving deviating forces, but only a few classes can now be considered.

277. Circular paths. Where θ is constantly 90°.

Since $F \cos \theta = 0$, $a_t = 0$, and v is constant. If now F is also constant, its deviating effect will be constant and the body has a motion along a circular path.

Let Q stand for the deviating force. Like every force, it is measured by the product of the mass of the body it acts upon, by the normal acceleration which it produces: that is to say,

$$Q = ma_n = \frac{Wa_n}{g}$$

Given W $\left(\text{and therefore } \frac{W}{g}\right)$ we shall find Q if we find a_n. Hence we must now solve the *problem*: To find the normal acceleration of a heavy particle whose center of mass moves in a circle whose radius is r, with a constant velocity v.

The normal acceleration a_n lies along AO. Fig. 282. Let it be resolved into components *along* AB and AC. That is into

$$\frac{d^2x}{dt^2} \text{ and } \frac{d^2y}{dt^2}.$$

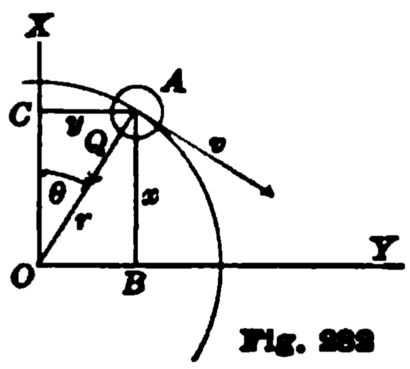


Fig. 282

From the figure

$$x = r\cos\theta$$
$$dx = -r\sin\theta d\theta$$
$$d^2x = -r\cos\theta(d\theta)^2$$

hence

$$\frac{d^2x}{dt^2} = -r\cos\theta\left(\frac{d\theta}{dt}\right)^2$$
$$= -r\cos\theta.\omega^2 \qquad (1)$$

Similarly

$$y = r\sin\theta$$
$$dy = r\cos\theta d\theta$$
$$d^2y = -r\sin\theta(d\theta)^2$$
$$\frac{d^2y}{dt^2} = -r\sin\theta.\omega^2 \qquad (2)$$

But

$$(a_n)^2 = \left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right) = r^2\omega^4$$

Hence

$$a_n = r\omega^2 = \frac{v^2}{r} \qquad (3)$$

Since

$$v = r\omega.$$

Having found a_n we have the value of $Q = ma_n$

or

$$Q = \frac{Wv^2}{gr} \qquad (4)$$

This formula is most important; it holds when F is not constant and the path is not a circle, provided r is the value of the radius of curvature.

The quantity Q is popularly (but erroneosuly) called the "centrifugal force," meaning that it is the force which tends to *make the body* "fly the center." It does no such thing; it is the force which *prevents* its flying the center. However, it does measure its *reaction* upon the external guide, or upon the deviating *pull* of a rope fastened to a pin or anchor at the center O. But we are not considering how A *acts upon some other body*; we are considering only how A **is acted upon, whereby** it moves in a circular path.

As a *deviating* force always acts towards the center of curvature, it is properly called a *centripetal* force.

278. LEMMA. **The centralization of deviating forces.** The resultant deviating force required to make all the parts of a solid body move along the arcs of concentric (or co-axial) circles with the same angular velocity, is the same as it would be if the entire mass were concentrated at its center of mass. This must be proved.

Let G be the center of mass of a solid body, Fig. 283, and OZ an axis about which the body is rotating. An element of mass dm at A has a radius ρ, which makes an angle θ with the plane thru G and the axis. The deviating force for dm is $dm\frac{v^2}{\rho} = dm\rho\omega^2$, since $v = \rho\omega$. We are to find the resultant of all such elements. This may be resolved into a component perpendicular to the plane YZ: $\rho\omega^2 \sin\theta \; dm$, and one parallel to it: $\rho\omega^2 \cos\theta \, dm$. Now the integral

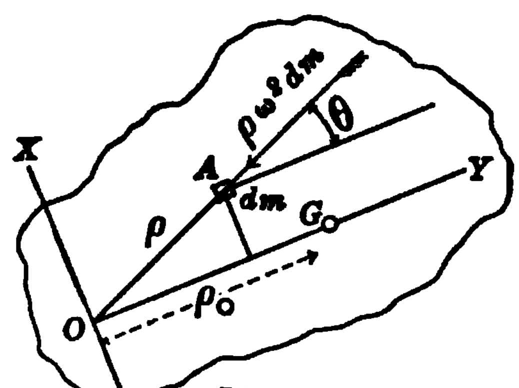


Fig. 283

$$\omega^2 \int \rho \sin\theta \; dm = 0$$

since it is ω^2 times the moment of all the elements of the body with reference to a plane which passes thru the center of mass of the body.

The integral $\omega^2 \int \rho \cos\theta \, dm$ is ω^2 times the moment of all the masses with reference to the plane ZX perpendicular to OG; *i. e.*, $\int \rho \cos\theta \; dm = mOG\cdot$

Hence the resultant
$$Q = m\rho_0\omega^2. \qquad Q.\ E.\ D.$$

279. The dangerous magnitude of Q. It is seen that Q increases as the square of the angular velocity. Its magnitude is often underestimated, as is the case when a rapidly moving locomotive or automobile enters upon the arc of a circle of small radius, and no adequate deviating force is provided. For example suppose a locomotive weighing 100 tons moves at the rate of 88 feet per second (60 $\frac{\text{m}}{\text{hr}}$) on a curve whose radius is 600 ft. (Of course, this is a frightful speed for such a radius.)

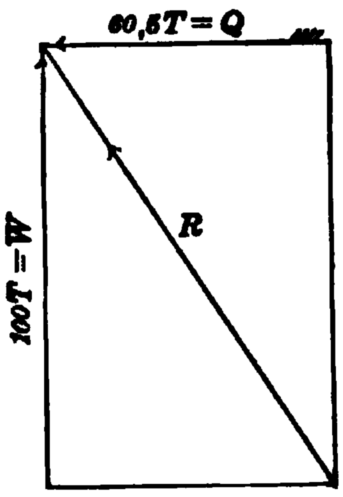


Fig. 284

$$Q = \frac{200000(11)^2}{300} = \frac{242000}{3}$$
$$= 80666.7$$
$$= 4\tfrac{1}{3} \text{ tons.}$$

Unless this deviating force is provided, the machine will not keep the curve: That is, it will roll over, or carry the track sideways, or "jump the track." To prevent this, the track must not only provide a vertical support of 100 tons, but also a horizontal deviating force of $60\frac{1}{2}$ tons, both acting thru the "center" of the locomotive.

This resultant action must come up betweent he rails or it will not come at all, and over goes the machine. If the rails are on the same level, as the deviating force comes from the lateral action of the outer rail upon the flanges of the wheels, or by the outer rail and a guard rail; the analysis is as follows, Fig. 285:

The lower Q represents the lateral force of *the rails upon the flanges* of the wheels. This force is resolved into an equal force Q_1 acting at the center of gravity of the machine, and a couple Q and Q'' whose arm is h and whose moment is Qh, which in the figure is right-handed. The moment of W about the outer rail is $\frac{1}{2}Ws$, in which s is the inside gauge of the track. If $\frac{1}{2}Ws > Qh$, the machine will not turn over tho the outer rail will carry more than half the load.

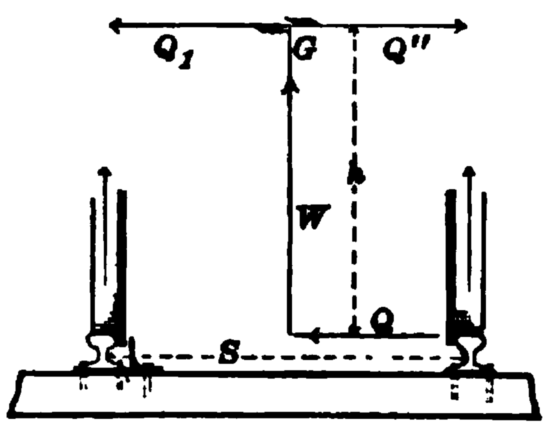


Fig. 285

If $\frac{1}{2}Ws < Qh$, and the rails do not move laterally, the locomotive climbs over the track, or turns over.

280. An elevated outer rail. If the outer rail is elevated, and if it be stipulated that each rail shall carry half the required support, then the elevation is found for the velocity v. Since the support must provide both a vertical W and a horizontal Q, we proceed as follows; Fig. 286:—

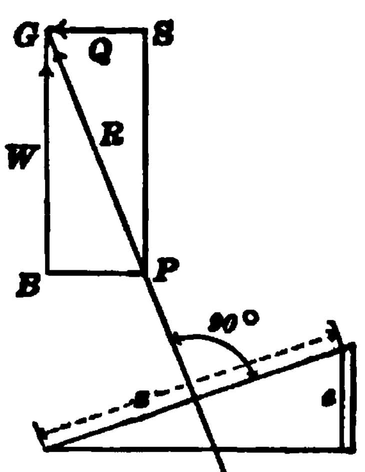


Fig. 286

If G is the center of mass, $SG = Q$ and $BG = W$, we have PG is the required line of reaction, which must bisect the "gauge" at right angles.

The small triangles, are similar; hence the elevation of the rail is

$$e = \frac{Qs}{\sqrt{W^2 + Q^2}}$$

where $Q = \dfrac{W}{g}\dfrac{v^2}{R}$. R is the radius of the curve in feet, and v = ft. per sec.

If we write $\frac{v^2}{gR} = k$, the above equation gives $\frac{e}{s} = \frac{k}{\sqrt{1+k^2}}$.

If the elevation is less, the wheels will carry unequal loads, and the outer flange must help supply Q. If the velocity is less than that for which Q was calculated, the lower flange will be called into action.

In the case of an automobile making a curve on a horizontal plane, the friction of the ground must supply the deviating force. This cannot be greater than fW. If $Q > fW$, the auto slides or "skids." If the wheels cannot "skid," and the required Q is $> \frac{1}{2}Ws/h$, it overturns. The notation is the same as in figure 286.

281. Motion along the arc of a vertical circle. This has already been discussed with a view of finding velocity and time. We take it up now to find the necessary deviating force, the motion being still a translation.

Let the body be connected with the center O (Fig. 287) by a stiff, weightless rod which fits over a smooth pin at the center *of mass of the moving body,* and over a smooth pin at O. Assume that the body has already moved from A_1 to B and has a velocity $v = \sqrt{2gh}$ so that $v^2 = 2g(r\cos\theta - r\cos\theta_1)$.

Only two forces are acting on the body at B: the pull of the earth W, and the pull of the rod T. One component of W is balanced by a part of T, and the other causes a tangential acceleration which increases v. T must not only balance $W\cos\theta$, but it must furnish the Q which is necessary to make the body keep on the circular arc. This Q is

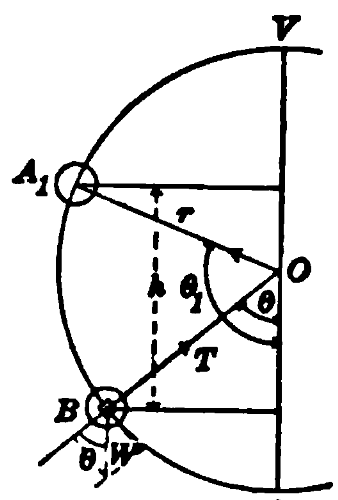


Fig. 287

$$Q = \frac{Wv^2}{gr} = 2W(\cos\theta - \cos\theta_1)$$

hence
$$T = W\cos\theta + 2W(\cos\theta - \cos\theta_1)$$

$$T = W(3\cos\theta - 2\cos\theta_1)$$

It is *very important* that the student should take note that the value of T in this formula, and in those that follow where all the motion along the arc of a vertical circle, is due to the weight and θ, **the radius r does not appear.** It follows that the results are *independent of* r.

If $\theta = 0$, the formula becomes

$$T = W(3 - 2\cos\theta_1)$$

which is the tension of the rod when the body reaches the *lowest point* A. Now turn to Fig. 288.

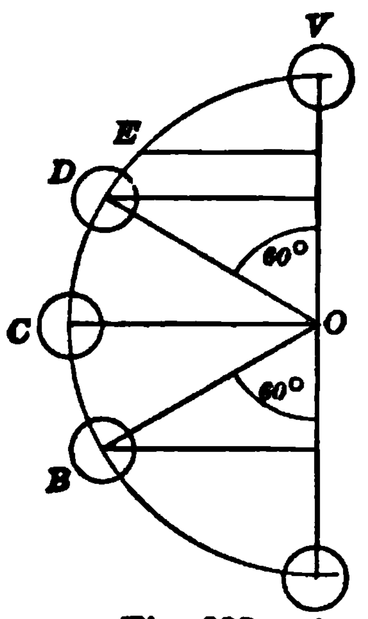


Fig. 288

The body starts at B, $\theta_1 = 60°$	$T=2W$	When A is reached	
" " " C, $\theta_1 = 90°$	$T=3W$		
" " " D, $\theta_1 = 120°$	$T=4W$		
" " " V, $\theta_1 = 180°$	$T=5W$		

This shows the tensions in the rod when the body has reached A, having swung from various heights differing by $\frac{r}{2}$.

The magnitude of T, as evidenced by breaking ropes or chains, is often a surprise to untechnical men who blindly suppose that if a rope or chain is strong enough to sustain W when at rest, it will sustain it equally well when it swings.*

Parties installing swings in playgrounds should provide for heavy people and high swings.

It may be well to note the change in the stress in the rod when the body *starts from V*. At first the tension is negative, that is to say, the rod is a strut in compression. The general formula gives

$$T = W(3\cos\theta + 2).$$

If $\phi = \pi - \theta$, this becomes $T = (2 - 3\cos\phi)$. When $\cos\phi = \frac{2}{3}$, $T = 0$, and the moving body has descended to E, Fig. 288, a vertical distance $h = \frac{r}{3}$. At that point the stress in the rod changes from compression to tension. Had the body been sliding from the top of a smooth sphere instead of being connected with a central rod, it would have moved just the same, and would have left the sphere at the point E and gone off on a tangent with a velocity $v_0 = \sqrt{2gh} = \sqrt{\frac{2}{3}gr}$. Its path during a 2nd epoch after the sphere has ceased to act on it, is in accord with the laws for projectiles investigated in **286**.

Had the tangential velocity at V, the top of the circle of revolution, due to forces which had ceased to act, been equal to (or greater than) $\sqrt{\frac{2}{3}gr}$, Q would have been equal to (or greater than) W: so there would have been no stress (or some tension) in the rod, or no pressure (or some pressure) upon an external guide.

* The writer recalls an incident of his boyhood when he set out to prove to his brothers and sisters that he could invert a bucket full of water without spilling a drop. He swung the bucket to and fro and then swung it steadily overhead, without spilling a drop—*while it was overhead;* but when it came down to the lowest point of the curve, the bail and the bucket parted company and every drop was spilled. The catastrophe was a mystery to the boy, but he learned later that "*Q*" *did it.*

Had v_0 at V been $\sqrt{\frac{2}{3}gr}$, the tension in the rod when it had swung to the lowest point A would have been $6W$. Had $v_0 > \sqrt{\frac{2}{3}gr}$, the tension at A would have been greater than $6W$.

282. "Looping the loop." Suppose a truck with light wheels is running on a "looped" track (Fig. 289), with a velocity sufficient to carry it thru V with a velocity greater than $\sqrt{\frac{2}{3}gr}$. The deviating force Q at A is *greater than* $6W$. This explains the shock which a passenger in such a truck experiences in his spinal column.

Suppose the weight of one's head, arms and body above his hips is 100 lbs., all of which is supported by his backbone; and suppose he twice passes the arc BA. When the car first reaches B, his backbone which usually supports 100 lbs. is suddenly required to support more than 500 lbs., and that increases till he reaches A, when the load on the spinal column exceeds 600 lbs. These loads are repeated as the car comes down again to the point A, but are immediately taken off as the track becomes straight. The untechnical passenger who goes off with a lame neck or back is apt to ascribe his injury to a rough track. If so, he is wrong, for what has been said here has been based on a perfectly *smooth* track. The quantity Q explains it all.

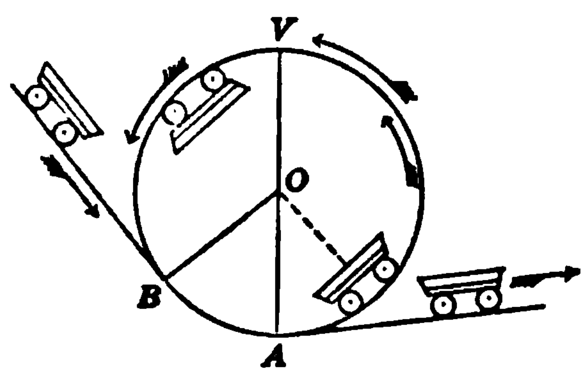


Fig. 289

283. The conical pendulum. *An ideal problem.* A body, A, Fig. 290, moves in a horizontal circle whose radius is r, constrained to keep its path by the joint action of gravity and the tension of a weightless cord attached to a point V, at a distance h above the center of the circle. Find the relations between forces, motions and time.

But two forces are acting on A: gravity and the tension of the cord.* Their resultant is $AB = W\tan\theta = \dfrac{Wr}{h}$. This resultant must be hor-

* The reader must not be mislead by such a slip as the following, which is actually in print in a reputable work on Mechanics. "The forces acting on the ball [of a conical pendulum] are:—the tension T of the string, the weight m of the particle, and *the centrifugal force.*" (The italics are ours.)

Such statements greatly confuse students. The writer is reminded of a statement in a textbook on elementary physics to the effect that a cannon ball moving thru vacant space near the earth's surface was acted upon by *two* forces: the force of gravity, and *the force of inertia that kept the ball in motion.*" The writer does not remember the author's name, but he does remember that he *burned the book.* This confusion of thought is akin to that which afflicts writers who assert that a force balances itself, or that a force and its components form a balanced system.

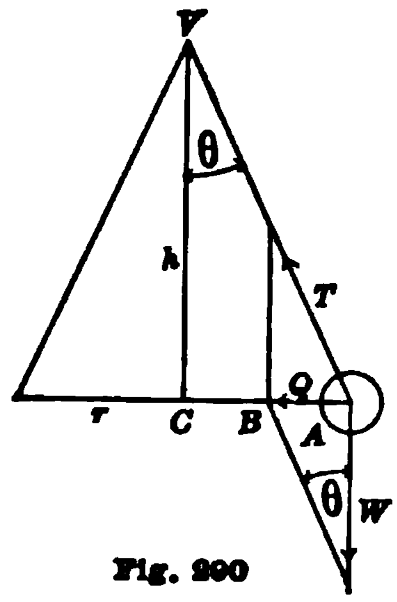


Fig. 290

izontal, or the body would not move in a horizontal plane. It is the deviating force Q. Hence

$$\frac{Wr}{h} = \frac{Wv^2}{gr} = Q \qquad (1)$$

and the general equation is

$$hv^2 = gr^2 \qquad (2)$$

hence

$$h = g\frac{r^2}{v^2} = \frac{g}{\omega^2} \qquad (3)$$

which shows that h is independent of r.

The time of a revolution is $\frac{2\pi r}{v} = \frac{2\pi}{\omega}$

Hence

$$t_1 = 2\pi\sqrt{\frac{h}{g}} \qquad (4)$$

which is independent of r, which shows that the time depends on h, and not on the length of the cord.

284. The ball governor. As ω is increased in the conical pendulum, h is diminished. This property led to the use of **the ball governor** for steam engines and certain clocks. As commonly used on an engine, there are two heavy balls hung upon a revolving shaft, Fig. 291, so connected, at C, that the arms CA and CA' revolve with the shaft, while they are free to move up and down, in vertical circles. A pair of light equal arms connects A and A', with a sleeve at B, which slides freely on the shaft carrying a lever arm which, when raised, automatically shuts off, more or less, the supply of steam which drives the engine.

If for any reason, such as the shifting of a belt so as to throw off a part of the moving machinery, or a break in a gear or shaft, the resistance to be overcome is reduced, and the speed is immediately accelerated, the angular velocity of the balls is increased, $\frac{Wv^2}{gr}$ increases beyond $W\tan\theta$, and the balls move into a larger and higher horizontal circle, making θ larger, and h less, both above and below the plane of AA'. Thus B is lifted and the supply of steam is diminished tho not wholly shut off. Thus the engine is prevented from running "away" or "racing."

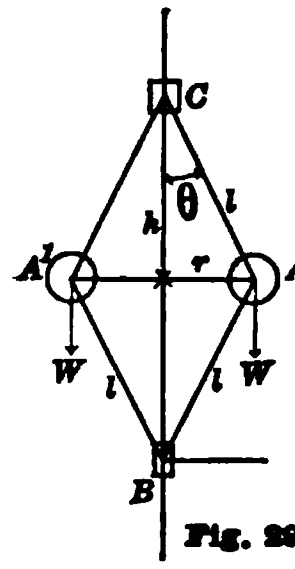


Fig. 291

It is evident that an effective "governor" must be sensitive and respond quickly, and the balls must rise without loss of time. Right

here occurred what at first seemed a paradox: It was found that if a weight was added to the sliding sleeve B, thereby pulling the swinging balls *down*, it caused them to *rise more quickly when the angular velocity was increased*. The theory of this phenomenon must now be given. The added weight W_2, including the sleeve B, hangs equally on the two arms BA and BA', and the static triangle for

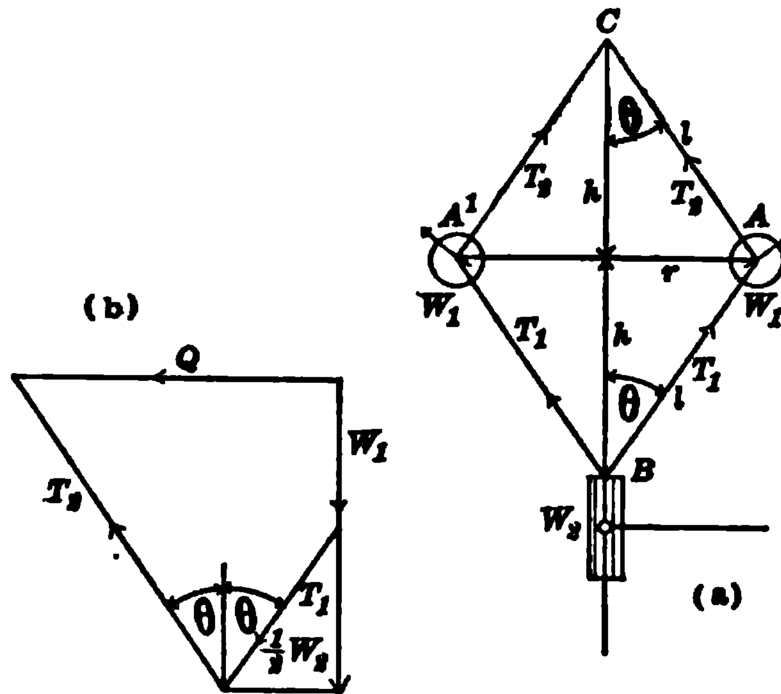


Fig. 292

$\frac{W_2}{2}$ gives, Fig. 292 (*b*), if T_1 is the tension in a BA,

$$T_1 = \frac{W_2}{2} \sec \theta. \tag{1}$$

There are now three forces acting on the ball at A: W_1, T_1 and T_2; they do not balance, but their resultant must be horizontal and equal to $Q = \frac{W_1 v^2}{gr}$. Hence the resultant polygon for steady motion, Fig. 292*b*.

It is evident that as the sum of the vertical components of forces acting on W_1 must be zero,

$$T_2 \cos \theta = W_1 + T_1 \cos \theta$$

$$= W_1 + \frac{W_2}{2}. \tag{2}$$

and Q = the sum of the horizontal components of T_1 and T_2.

$$Q = T_1 \sin \theta + T_2 \sin \theta \tag{3}$$

$$= \frac{W_2}{2} \tan \theta + \left(W_1 + \frac{W_2}{2}\right) \tan \theta$$

$$= (W_1 + W_2) \tan \theta = \frac{(W_1 + W_2) r}{h} \tag{4}$$

But also $$Q = \frac{W_1 v^2}{gr} = \frac{W_1}{g} r\omega^2$$

Hence $$\omega^2 = \frac{g}{h}\left(1 + \frac{W_2}{W_1}\right), \text{ and } h = \frac{g}{\omega^2}\left(1 + \frac{W_2}{W_1}\right) \tag{5}$$

Differentiating the last equation we get after changing signs so as to make $d\omega$ positive, indicating an increase in ω,

$$-dh = \frac{2g}{\omega^3} \cdot \left(\frac{W_1 + W_2}{W_1}\right) \cdot d\omega$$

and the slider *rises* $$2dh = \frac{4g}{\omega^3}, \left(\frac{W_1 + W_2}{W_1}\right) \cdot d\omega.$$

It is now readily seen that the *rise* of the slider is increased by the addition of W_2. This is not so paradoxical as it appeared at first, since the addition of W_2 pulled B down for a given ω, and so gave it a *better chance* to rise when ω increased.

It goes without saying that the governor is also efficient in keeping up (approximately) the angular velocity by turning on more steam, when the load on the engine is increased and there is a tendency to slow down.

285. Loss of weight due to the earth's rotation. This is not the place to discuss the full theory of the effect of latitude upon the weight of the body, but *one* of the modifying influences is now in order. In Fig. 293, let A represent the position of a mass m on a meridian circle, NES, north latitude θ. Its radius of rotation is $AB = R\cos\theta$. The deviating force required to compel the mass to move in a circle is

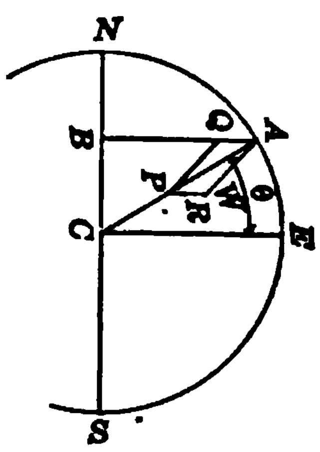


Fig. 293

$$Q = mR\omega^2 \cos\theta.$$

The value of ω, since it is constant, is found by dividing 2π by the length of a *siderial* day which is about 86,164 seconds. See **317**.

The deviating force (loss of weight) at the equator is $mR\omega^2$. At the poles, it is nothing. At latitude 60° it is $\frac{1}{2}(mR\omega^2)$ very nearly.

Assuming that the earth is a solid sphere, the resultant attraction is directed towards the center, C. Resolving that attraction into Q and W (the observed weight), it is evident that W is less than the real attraction, and also that it is *not directed towards the center of the earth.* This last fact goes to explain why the earth is *not* a perfect sphere, and why the action of W draws the masses towards the equator, thereby "deforming" it and making it an *oblate spheroid* whose resultant attraction is *not towards its own center of mass.*

286. The motion of a projectile in vacant space. The problem being ideal, it will be assumed further that all vertical lines are parallel. A ball fired from a gun has at least three epochs in its motion: Its motion in the gun from rest to the muzzle; its motion from the muzzle to a point where the escaping gases cease to act upon it; and the passage in "vacant space" with an initial velocity v_0 and an initial direction θ_0. The motion in the gun will be discussed in the next chapter.

The accelerated velocity of the 2nd epoch is difficult to discuss except experimentally. However, it is clear that in real cases, the great volume (and mass) of escaping and expanding gas acts with great

force and for some distance upon a projectile after it has left the gun.

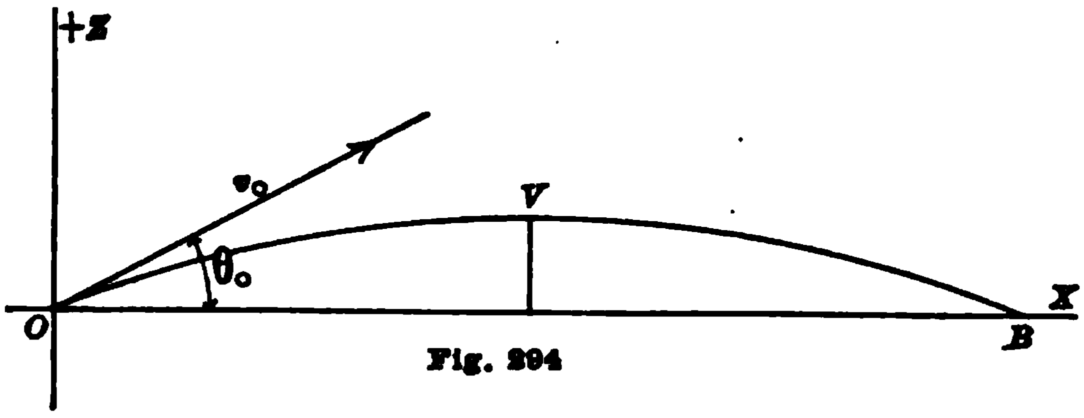


Fig. 294

Taking now the 3rd epoch (Fig. 294), after the time t the velocity is v, and the direction is θ. Only one force is acting, namely gravity, whose acceleration is vertically down, $-g$, as z is measured positive upwards.

The vertical acceleration is therefore $\dfrac{d^2z}{dt^2} = -g$ (1)

Multiplying by dt and integrating

$$\frac{dz}{dt} = -gt + C = v_o \sin\theta_o - gt. \tag{2}$$

Integrating again $\quad z = (v_o \sin\theta_o)t - \dfrac{1}{2}gt^2 + (K = o)$ (3)

Eq. (3) gives the vertical height, or ordinate, of the ball at the time t.

The horizontal acceleration is zero as there is no horizontal force acting on the ball.

Hence $\quad \dfrac{d^2x}{dt^2} = 0$ (4)

Integrating $\quad \dfrac{dx}{dt} = C_1 = v_o \cos\theta_o$ (5)

Continuing $\quad x = (v_o \cos\theta_o)t + (H = 0)$ (6)

Equation (6) gives the horizontal co-ordinate of the ball after the time t.

From the last equation $t = \dfrac{x}{v_o \cos\theta_o}$ which substituted in (3) gives

$$z = x\tan\theta_o - \frac{gx^2}{2v_o^2\cos^2\theta_o} \tag{7}$$

which is the equation of the path—a parabola passing thru O, and having a vertical axis pointing down. The ball will reach its highest point (the vertex of the parabola) when $\dfrac{dz}{dt} = 0$, that is when

$$t = \frac{v_o \sin\theta_o}{g}.$$

At that time $\quad z_1 = \dfrac{v_o^2 \sin^2\theta_o}{g} - \dfrac{1}{2}g.\dfrac{v_o^2\sin^2\theta_o}{g^2}$

$$\left.\begin{aligned} z_1 &= \frac{v_o^2 \sin^2\theta_o}{2g}. \\ \text{and} \qquad x &= \frac{v_o^2 \sin\theta_o \cos\theta_o}{g} \end{aligned}\right\} \tag{8}$$

The "range" of the projectile is the value of x when z is again zero. Making $z=0$ in (7) and solving for the x we have

$$\text{Range } OB = \frac{2v_o^2 \sin\theta_o \cos\theta_o}{g} = \frac{v_o^2}{g}\sin 2\theta \tag{9}$$

which is twice the abscessa of the vertex, as should have been expected.

The range will be greatest when $\sin 2\theta_o$ is the greatest, that is, when

$$\theta_o = \frac{\pi}{4} = 45°.$$

In that case the range is $\frac{v_o^2}{g}$. (10)

It is interesting to note that if θ_o were 90°, the vertical height to which the ball would go with the same velocity v_o, is the distance to the *Directrix* of the parabola described when $\theta = 45°$ and is one-half of the Maximum Range. Again, the velocity at any point in the parabola is found from the equation

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$= v_o^2 \cos^2\theta_o + (v_o \sin\theta - gt)^2$$

which by (3) becomes

$$v^2 = v_o^2 - 2gz \tag{11}$$

which shows that at equal heights before and after passing the vertex the velocity is the same, and that when the ball reaches the horizontal plane thru O (the end of the range), the velocity is again v_o. Finally, since the $\sin\left(\frac{\pi}{2} - 2\phi\right) = \sin\left(\frac{\pi}{2} + 2\phi\right)$ the range is the same for $\theta_o = 45° - \phi$, as for $\theta_o = 45° + \phi$. This fact was formerly much used in war for determining the elevation of bomb shells.

287. The deviating component of W. One component of the weight of the ball is always a deviating force which has the general value

$$Q = W\cos\theta = m\frac{v^2}{\rho}$$

Where ρ is the radius of a curvature. Since the horizontal velocity of the ball is constant,

$$\cos\theta = \frac{v_o}{v}\cos\theta_o$$

so that
$$Q=\frac{Wv_0\cos\theta_0}{\sqrt{v_0^2-2gz}}$$

At the vertex of the parabola $v=v_0\cos\theta_0$ and $Q=W$, hence
$$\rho=\frac{v_0^2\cos^2\theta_0}{g}.$$

288. The problem of 281. A smooth spherical ball, radius r, rests on a rough horizontal plane in a vacuum. A heavy particle slides from the topmost point, Fig. 295, and finally reaches the given plane at T. What is the position of the point E where the particle leaves the sphere, and the distance TA?

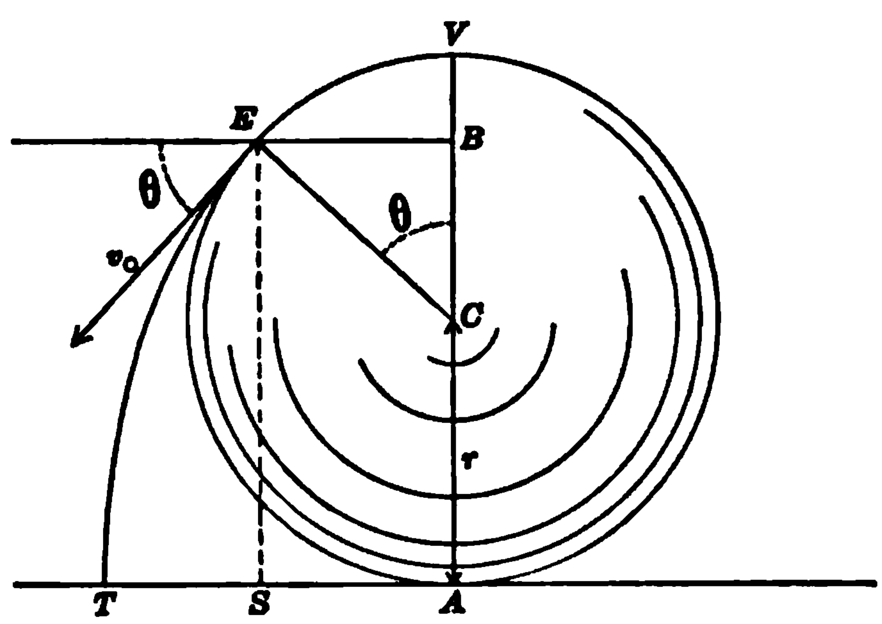


Fig. 295

It is assumed that the sphere is not moved by the action of the particle.

This problem was broached in **281** but the reader was at that time (perhaps) unprepared to solve it. It was however found that the point E was $\frac{1}{3}r$ below the level of V, and that Q was then equal to the normal component of W. Accordingly the particle left the sphere at that point. Hence it began the second epoch with a velocity $v_0=\sqrt{\frac{2}{3}gr}$, and an angle of *depression* θ, the angle being determined by the equation $\cos\theta=\frac{2}{3}$. The horizontal component of v is *constant* and equal to $v_0\cos\theta=\frac{2}{3}\left(\frac{2}{3}gr\right)^{\frac{1}{2}}$, since the motion is ideal and in vacant space. The *vertical* component of v_0 at E is $v_0\sin\theta$, and its *vertical* acceleration is henceforth g; hence its *vertical velocity*, t sec. after leaving E, is
$$v_z=\left(\frac{10}{27}gr\right)^{\frac{1}{2}}+gt.$$

so that the *vertical* distance fallen at the time t is
$$s=\left(\frac{10}{27}gr\right)^{\frac{1}{2}}t+\frac{1}{2}gt^2.$$

The vertical height of E above the plane AT is $\frac{5}{3}r$; hence if we make

$s=\frac{5}{3}r$ in the last equation and solve for t, we have the time of the fall:

$$\frac{5}{3}r=\left(\frac{10gr}{27}\right)^{\frac{1}{2}}t+\frac{1}{2}gt^2$$

$$t=\left(\frac{10r}{27g}\right)^{\frac{1}{2}}\sqrt{(10}-1)$$

Multiplying the constant horizontal $v_0\cos\theta$ by t, we have the distance ST, viz.:

$$v_0t\cos\theta=\left(\frac{8gr}{27}\right)^{\frac{1}{2}}\left(\frac{10r}{27g}\right)^{\frac{1}{2}}(\sqrt{10}-1)=\frac{4r}{27}(\sqrt{50}-\sqrt{5})=ST$$

so that the required distance is

$$AT=\frac{5}{27}(4\sqrt{2}+\sqrt{5})r. \qquad \text{Ans.}$$

289. The resisting effect of atmospheric air. All the above results apply only to *ideal* conditions, for projectiles are never fired, nor do bodies fall, *in vacuo*. There is always resistance from the atmosphere whether still or in motion. In the case of heavy and dense bodies (especially if pointed and moving with a sharp point forward) having low velocities, the resistance is small and may be neglected without serious error.

When Galileo, in order to prove that the weight of a body did not affect its velocity when falling, dropped two balls, one large and one small, from the top of the leaning tower of Pisa to the ground, the verdict of all the spectators was, that they *reached the ground at exactly the same instant*. Such would have been the fact had they fallen in a vacuum, but in the air they did not fall in *exactly* the same time, tho the difference was not noticeable. If of equal density, and spherical, the larger encountered less resistance than the other *in proportion to its weight*, so that its acceleration was greater than that of the smaller ball. This is readily proved.

When the velocity was v, let k be the resistance offered by the air to the motion of the smaller ball whose weight was W_1. Suppose the diameter of the larger ball was twice that of the smaller. Then it suffered a resistance of approximately $4k$; but the pull of the earth was $8W_1=W_2$. The *unbalanced* force acting on the smaller was W_1-k, and its acceleration, $a_1=\frac{W_1-k}{W_1}g=g-\frac{gk}{W_1}$. The unbalanced force acting on the larger was $8W_1-4k$, and its $a_2=\frac{8W_1-4k}{8W_1}g=g-\frac{gk}{2W_1}$. It is now easily seen that $a_2>a_1$, tho the difference is only $\frac{gk}{2W_1}$, which was

so small that its effect upon the *time* t, escaped notice in the crude experiment.

290. Projectiles in a resisting medium. This is a difficult subject for two reasons. In the first place it is difficult to determine with any accuracy what the resistance offered to a given body actually is, and how that resistance changes with a change in the velocity. A common assumption is that it varies as the square of the velocity; again, as the cube of the velocity, particularly in the case of a vessel moving thru the water. But no one really thinks that the exponent of v is a whole number. In the second place, when a rational assumption is made, that the resistance is kv^n, the resulting equations are exceedingly difficult to handle. In military and naval academies, extensive experiments have been made, with careful measurements of elevations, initial velocities, distances and times; and empiracal equations have been devised to suit the graphs obtained from plotting the results.

Professor Philip R. Alger, of the U. S. Naval Academy, gives the modified equation for the trajectories of the shells of big guns in the navy, as follows:

$$z = x \tan \theta_0 - \frac{gx^2}{2v_0^2 \cos^2 \theta_0}\left(1 + \frac{2kx}{3}\right)$$

in which k is the co-efficient of v^2, when kv^2 represents the atmospheric resistance. The quantity k is of course a function of the projectile itself, viz.:

$$k = A\frac{D^2}{W},$$

D being the diameter of the projectile in inches; W, its weight in lbs.; and A is a constant determined by their experiments. Prof. Alger gives the value of k, when $D = 12$ and $v_0 = 2800\frac{\text{ft}}{\text{sec}}$, as

$$k = \frac{21534}{(10)^9}.$$

The makers of long range guns and rifles adjust the "sights" in accord with the results of experiments; just as every pitcher of baseball learns by experience the proper elevation and the judicious initial velocity. For a long throw, the initial elevation is rarely more than 30°.

291. The effect of initial rotation. The rotation of the projectile in a resisting medium has much to do with the curve described. A rotating golf-ball drags the air around with it and causes unequal pressures upon the top and bottom, or sides, the resultant of which acts as a deviating force to deflect the ball from its normal path.

It is evident that the *direction* of the *stroke*, the *bevel* of the *club*, and the *condition* of the *surfaces* in *contact* have much to do with the direction taken by the ball, and with its angular velocity.

With a *perfectly smooth* club and a *perfectly smooth* elastic ball, there can be no rotation, and the direction taken must be *very nearly* normal to the face of the club. With perfectly rough club and ball, the initial direction must be very nearly in the direction of the stroke, regardless of the bevel; and if the stroke is eccentric, there must be rotation. Hence a rusty club and a rough dirty ball under a stroke with a horizontal tangent gives very little rise; while with a polished and well-oiled club, a smooth ball is readily "lifted" by a horizontal stroke.

The bevel of the face of the club shown by the tangent line, Fig. 296, gives the normal action in the direction of BC. But there is more or less friction in the direction of the tangent at B which gives the ball angular velocity and the tangent force combines with the normal force giving a resultant force which produces during the time of action the velocity v_0.

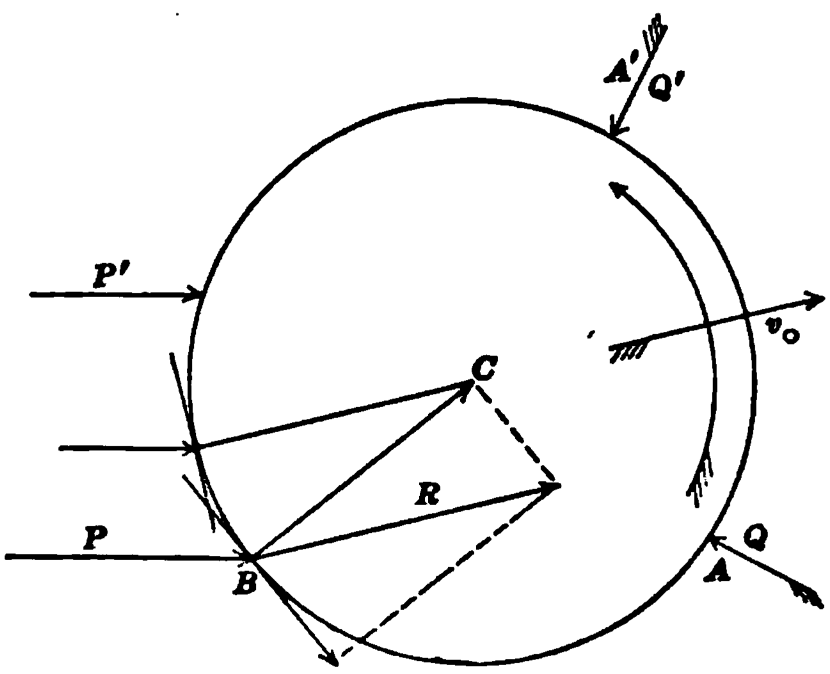


Fig. 296

Had both ball and club face been *perfectly smooth*, the initial direction of the balls' motion would have been normal to the face of the club during the contact, and there would have been no rotation of the ball.

If there were no initial slipping, the direction of the ball would have been parallel to the direction of the blow.

Let C be a vertical section of a golf ball, struck at the point B in the direction of the arrow, which is in the vertical plane of the section. The ball moves off in the direction shown by the velocity line v_0. It has, in consequence of the eccentric blow, a rotation about a horizontal axis, as indicated by the curved arrow.

The air in front of the ball from A to A' is congested and the result is a resisting pressure. Meanwhile the rotating rough ball increases the congestion along the region near A, thereby increasing the pressure shown by the arrow Q. Along the region near A', the rotation of the ball facilitates the escape of the congested air, thereby reducing the pressure at that point. The result of these actions at A and A' is to lift the ball and to counteract (or balance) a part of the deviating action of the ball's weight and causes the trajectory to be abnormally flat. In

other words, the ball is made to rise above the normal trajectory and thereby increase its range.

On the other hand, if the ball be struck at P' in the direction of the arrow shown, the rotation will be in the opposite direction, and the resultant air pressure will have a downward direction (as compared with v_0) so that the deviating force of W will be increased, and the trajectory will have a sharper curvature than the normal curve, and the ball will seem to "drop," and have a short range.

If in consequence of *not swinging the club in the vertical plane*, which passes thru the center of the ball at the instant of contact, the direction of the resultant action of the club is oblique to that plane, there will be rotation about a vertical axis, and the ball will curve to the right or the left according as the rotation is right-handed or left-handed (to one looking down).

A rapidly revolving baseball does not describe a plane curve, unless, like the golf-ball, its axis of revolution be horizontal. If the axis is vertical, and the rotation is right-handed (to one looking down) the trajectory will curve to the right on account of the congested condition of the air on the left-hand half of the front.

The rate of deviation in any dircetion depends upon the roughness of the surface, the average density of the ball, and the angular velocity.

291½. Moments of deviation, or the moments of centrifugal forces. *Examples* in which the axis of rotation passes thru the center of mass, but is not an axis of symmetry.

1. Two concentrated masses, m_1 and m_2, are mounted on the ends of a rod which intersects the axis of rotation at right angles. Fig. 296½(*a*). If the intersection O is the "center of mass" of the two masses, and the combination be given an angular velocity ω, prove that the tensions in the two ends of the rod AB will be equal.

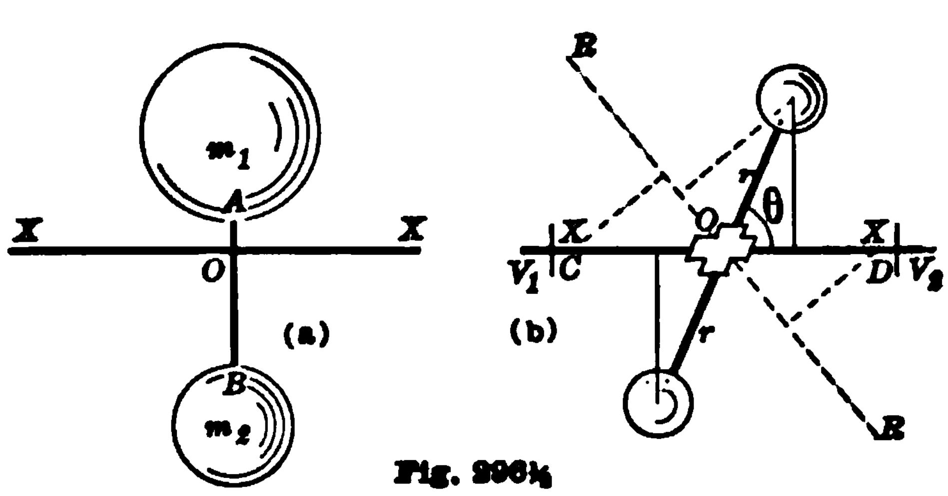


Fig. 296½

2. Two *equal masses*, Fig. 296½(*b*), are mounted upon a rigid rod which is bisected by the axis of rotation at an oblique angle θ. When there is no angular velocity, the combination is in equilibrium for all positions of the connecting rod, and the supports at the bearings, V_1 and V_2, are equal. Find the supports V_1 and V_2 when the angular velocity is ω, and the two mass centers are in the vertical plane thru the shaft as shown in the figure. Let $CD = s$, and W the total weight.

Solution. The centrifugal force applied to the shaft by each of the revolving masses is $Q = mr \sin\theta.\omega^2$. Hence the **moment of deviation** is

$$M = 2m\omega^2r^2 \sin\theta \cos\theta$$
$$= mr^2\omega^2 \sin 2\theta$$

If the deviating couple be transformed into an *equivalent couple* with a force F and an arm s, we have the force

$$F = \frac{mr^2\omega^2 \sin 2\theta}{s}$$

acting at each bearing, always in the plane of the rod and shaft, and vertical when the neighboring mass is above the shaft. Accordingly the support at D, which was $\frac{W}{2}$ when there was no rotation, is changed to $\frac{W}{2} - F$. So that

$$V_2' = \frac{W}{2} - \frac{mr^2\omega^2 \sin 2\theta}{s}$$

If $F > \frac{W}{2}$, there will be a tendency on the part of the shaft to *jump out of its bearing* every time the mass goes over. At the other end the support is at the same time

$$V_1' = \frac{W}{2} + \frac{mr^2\omega^2 \sin 2\theta'}{s}$$

If the shaft is suddenly freed from the restraint of bearings, the revolving masses would compel it to revolve with them around a *new axis* in space lying in such a position, RR, that the moment of deviation due to the given masses would be exactly balanced by the moment of deviation due to the mass of the shaft.

The obvious tendency of the masses is to bend the rod so as to make $\theta = \frac{\pi}{2}$, at which value F becomes zero.

The tenacity with which a rapidly revolving wheel preserves the direction of its axis, and the parallelism of its plane of rotation, is a matter of common observations; and the *danger from a jumping shaft* due to the moment of deviation of a wheel not properly balanced is learned by experience if not by theory.

3. **Labaratory exercise.** Suspend a slim cylinder, solid or hollow, by a stout cord or wire accurately attached to a loop at the end of the cylinder, and connected above with the end of a vertical shaft which can be rapidly turned. It will be found that, starting from rest, the cylinder can be given quite a high angular velocity without

developing a resultant deviating couple. If, however, the cylinder be disturbed by a gentle blow against one end, the cylinder will rise and rotate in a horizontal plane about a vertical axis which is now transverse. The cord should be several times as long as the cylinder.

A few links from a heavy chain can be used instead of a cylinder.

4. A steel ring 2 feet in diameter, with a section $\frac{1}{4}$ of a square inch, is given, by means of a chuck or mandrill, 24,000 revolutions per minute. Assuming that $w = \dfrac{490}{1728}$ pounds per cub. inch, find the tension per square inch in the ring.

5. If a steel rod 2 feet long, with a section $\frac{1}{4}$ square inch, be rotated, in a vacuum, about a *transverse* axis thru its *C. G.* with a speed of 18,000 revolutions per min.; what is the tension per square inch at the center?

CHAPTER XVII.

Kinematics.

A Study of the Motions of Rigid Bodies, Without Reference to their Causes.

292. The motion of three points. Given the consistent velocities (magnitude and directions) of three points of a rigid body, we are now to show how such velocities may be explained by simple rotation, or by rotation and translation combined.

By *consistent* it is meant that the given velocities must be possible for points in a rigid body. The component velocities of A and B along the line AB must always be the same.

Let A, B and C be any three points of a rigid body; and the lines AA', BB' and CC' represent their respective velocities at the given instant. From any point O in space, Fig. 297, draw three lines equal and parallel to the three velocity lines: $op = AA'$, $oq = BB'$, $os = CC'$. Thru p, q, and s, pass a plane, and drop the perpendicular on upon it, and connect n with p, q, and s. The three velocities are now resolved into a common component on, and components perpendicular to it and parallel to the same plane. The component on is their common velocity of *translation*, and the components np, nq, ns are their component velocities due to a rotation about a common axis.

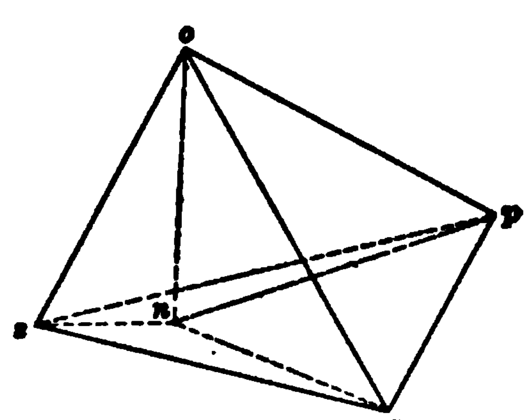


Fig. 297

A plane passed thru A perpendicular to np, and another thru B perpendicular to nq, will intersect in a line which is parallel to on. This

line is the *axis of rotation.* Consistency is assured by the following facts: first, that a plane passed thru C perpendicular to ns, will pass thru the axis already found; and second, that the three components np, nq and ns, are proportional to the distances from A, B, and C, to their common axis. Thus the given velocities are explained, and may be produced by a helical motion of the rigid body. The common component on is v_1, the velocity of translation; and the angular velocity is equal to the $\frac{np}{r_1} = \frac{nq}{r_2} = \frac{ns}{r_3} = \omega$, if r_1, r_2 and r_3 are the respective radii of rotation of the points A, B and C.

298. Helical motion. The student should have no difficulty in picturing the three points and their differing velocities in space. Let him imagine a large screw moving in a stationary nut, and then select the three general points, in, on or *connected with* the screw, and the mental picture will be clear.

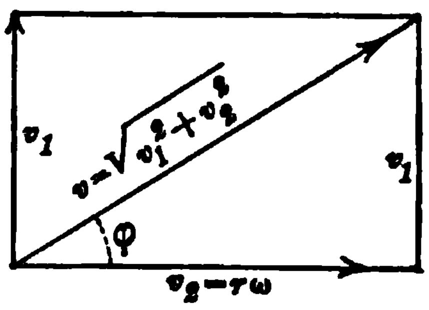


Fig. 298

The most general case is that in which *velocities* (not paths) are explained by a combination of translation, and rotation, the axis of rotation being parallel to the direction of translation. Such motion is called **Helical Motion.**

The velocity of a point on a screw results from a velocity of translation v_1, and a velocity due to rotation $v_2 = r\omega$; $v = \sqrt{v_1^2 + v_2^2}$, Fig. 298, in which r is the semi-diameter of the enclosing cylinder of the screw. The *pitch* of the screw or helix is the *advance* during the time of one revolution. The time of one revolution is $\frac{2\pi}{\omega}$, hence

$$pitch = p = \frac{2\pi v_1}{\omega}$$

and

$$v = \omega\sqrt{\frac{p^2}{4\pi^2} + r^2}$$

$$\tan\phi = \frac{v_1}{v_2} = \frac{p}{2\pi r}$$

Every point connected rigidly with the screw describes a helix of the same *pitch*, but the inclination ϕ varies, being 90° if r is zero; and approaches zero, as r approaches infinity. There may be n independent threads on the same screw, all having the same pitch. In what is called screw gearing, the number of threads is very great, like the number of teeth on a spur-gear wheel.

Instead of a *forward* motion and a right-handed turning thru a stationary nut, the screw may turn right-handed and have a *backward* velocity of translation. It thus becomes a "left-handed screw."

294. Differential screws, and rifled guns. If a screw carries threads of different pitches for separate nuts, at least one of the nuts must move when the screw is turned; and there will be relative motion between the nuts, depending upon the magnitudes of the pitches and the character of the threads, right or left-handed. If there be two nuts corresponding to the two threads of pitches p_1 and p_2, both right- (or both left-) handed, the *relative motion* of the nuts will be proportional to (p_1-p_2) *per revolution* (which can be very small in delicate apparatus). If one be right-handed and the other left-handed, their velocity of approach or separation is proportional to (p_1+p_2). See Fig. 299.

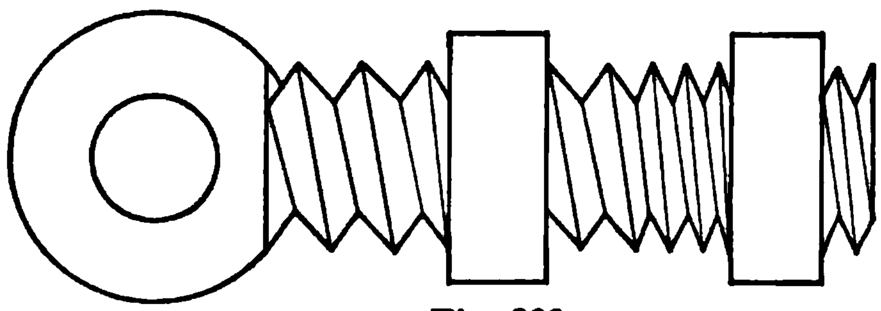


Fig. 299

In a rifled gun there are many threads with a large pitch. In such a gun, the projectile is the screw and the gun is the nut. In heavy ordinance, groves or threads, with a decreasing pitch, are used, the crude threads or lugs connected with the shell being made of pliable alloys. In a U.S. 10-inch gun, that is, the shell or solid shot (a long cylinder) is 10 inches in diameter, the number of groves or threads is 40, and the pitch at the muzzle of the gun is about 250 inches, or 21 feet. If v_1 is 2,000 feet per sec, the angular velocity $\omega=\dfrac{2\pi v_1}{p}=\dfrac{44}{7}\cdot\dfrac{2000}{21}=600$ nearly, and the number of complete turns per second is $\dfrac{\omega}{2\pi}=\dfrac{2000}{21}=95+$.

295. The closing tubes in the St. Louis Bridge. The steel screw which unites the two parts of a closing tube in the steel ribs of the (so-called) "Eads Bridge" at St. Louis, is itself a tube 17.2 inches long, $15\frac{5}{16}$ inches in exterior diameter, the high grade steel being 2″ thick. One end has a right-handed thread, the other a left-handed one. The thread is square (the bearing surface being a right-helicoid) and the *pitch* is $\frac{3}{8}$ of an inch. The screw was turned by an immense steel lever applied to a 2″ steel rod fitting holes in the central belt of the screw. After the closing tube, made initially short upon the interior screw, was inserted between the two segments of the unfinished rib during the erection, the screw was turned, forcing the ends apart $\frac{3}{4}$ of an inch at every revolution of the screw, until the proper length of the closing tube was reached, and all parts were coupled

securely.* The lever-wrench, made of railroad steel and angle-irons, is shown in Fig. 300.

WRENCH FOR TURNING THE RIGHT AND LEFT SCREW USED IN THE ADJUSTABLE CLOSING TUBES OF THE ST. LOUIS BRIDGE. THE WRENCH IN ACTUAL USE GAVE A THRUST OF SEVERAL MILLION POUNDS.

296. Instantaneous axes. Before leaving the matter of helical motion, the student must be careful to distinguish *velocity* from the *path* of a moving point. It was shown (**292**) that the velocities of points in a moving body *at any instant* could be explained in general, by a linear velocity in some direction, and an angular velocity about an axis parallel to the direction of the linear velocity. The *paths* followed in general are *not helices*, and the axis of rotation is such only for an instant, and hence it is called an "Instantaneous Axis."

It was shown in Chap. XV that if a body moves from one position to another quite different, while a single point of the body, or rigidly connected to it, retains its position in space, it is possible to find an axis passing thru the fixed point, upon which the body *can* be actually turned into the second position by simple rotation. In that case, there was no mention of velocity, or of instantaneous axis.

Returning now to Fig. 297, which may be called the "triangular pyramid of velocities," we see that *on* may be zero; that is, the three given velocities are parallel to the same plane. There is then no common component velocity like v_1, but there is, for at least *an instant*, an axis of rotation. This, of course, is a line perpendicular to the plane which contained the base of our pyramid. In fact, the pyramid of velocities is nothing but a base.

SPECIAL CASES.

297. 1. **Suppose two of the given velocities are parallel and unequal**; that is, *op* and *oq* must lie along the same line as in Fig. 301. Since the plane of the base of the pyramid must pass thru the points *p*, *q* and *s*, it must pass thru *O* and $on = v_1 = 0$, and there is no translation component.

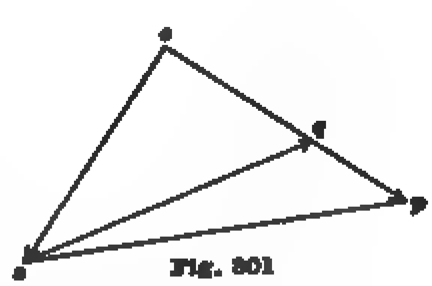


Fig. 301

2. **Suppose two of the given velocities are**

* The details of the methods used in screwing out the tubes to their normal length are given in the History of the St. Louis Bridge, referred to on page 35. The drawings of an adjustable tube and the double screw are shown on Plate XXXVIII, of the History.

parallel and equal. Since p and q must coincide, the plane of the base is indeterminate, but the simplest solution is to take a plane passing thru o as well as the points p (and q) and s. Hence the velocities are due to rotation only. Fig. 302. All other solutions involve both translation and rotation.

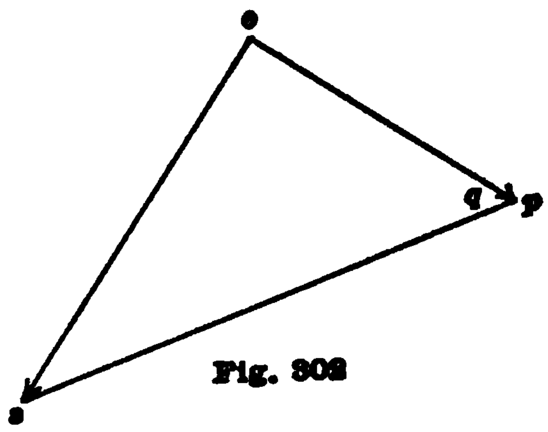


Fig. 302

3. Suppose all three velocities are parallel and equal. The simplest solution is that it is wholly a case of translation.

4. If all three are parallel and unequal, it is a case of rotation, and rigidity requires that their common direction must be perpendicular to the plane in which the points A, B and C lie; and the instantaneous axis must lie in that plane.

298. The position of the instantaneous axis. When it is known that the velocity lines of three points are parallel to the same plane, only two of those lines are necessary for finding the axis. Let all velocity lines lie in the plane of the paper. Fig. 303. Let A and B be two points with velocities as shown. (Their components along AB *must* be equal.) Pass a plane thru A perpendicular to v_a; the instantaneous axis must lie in that plane whose trace is AO. Pass a second plane thru B perpendicular to v_b; its trace is BO'. The two planes intersect in the required axis which is perpendicular to the paper at I. If ω be the common angular velocity, we shall have $v_a = r_1\omega$, and $v_b = r_2\omega$.

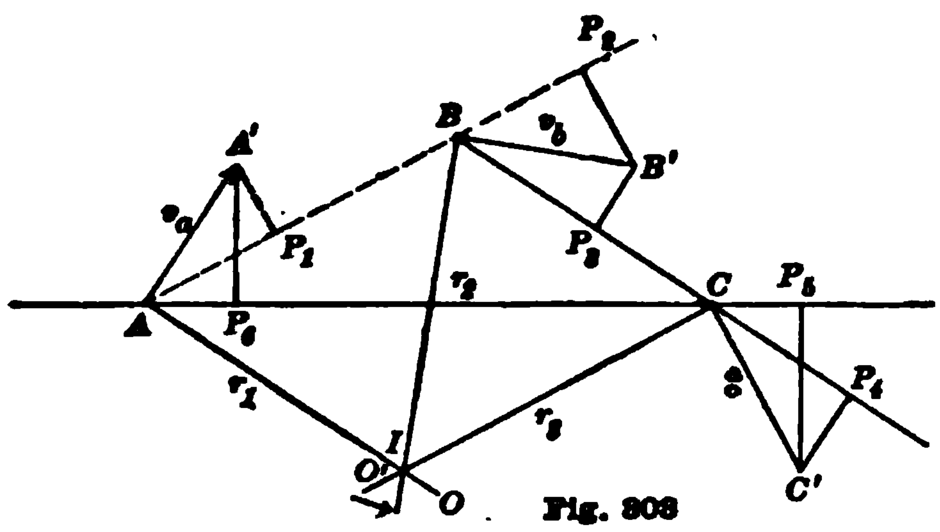


Fig. 303

The third point C will be found to have a velocity

$$v_c = r_3\omega = r_3 . \frac{v_a}{r_1} = r_3 \cdot \frac{v_b}{r_2} \cdot$$

It is readily seen that the component velocity of C along the line AC is equal to the component velocity of A along the same line. In all cases the linear velocities are proportional to their radii. If v_a and v_b are parallel and unequal, the planes AO and BO' must coincide; and as $\frac{v_a}{v_b} = \frac{r_1}{r_2}$, and as the velocities *must* be perpendicular to

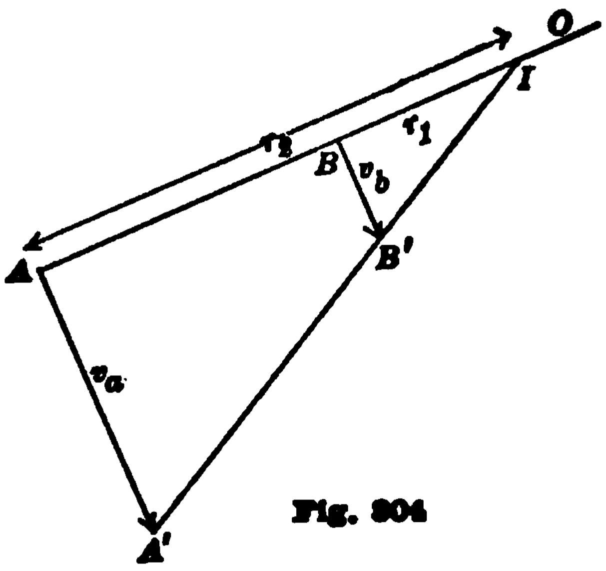


Fig. 304

AB, I is found as shown in Fig. 304, by the intersection of AB and $A'B'$.

299. Definition. A **"plane of rotation"** is any plane perpendicular to the axis of rotation. In a rigid body the component velocity of any point A in a straight line in a plane of rotation, along that line, is equal to the entire velocity of D along the same line, D being the foot of a perpendicular from the axis upon the given straight line. Fig. 305.

Let O be an axis perpendicular to the plane of the paper and AB a line in the paper of a rigid body revolving about O with an angular velocity $\omega=\frac{5}{3}$. Take at random any two points A and B on that line and drop $OD=r_0$ perpendicular to AB. Since v_1 is perpendicular to r_1, and r_0 is perpendicular to AB, we have

$$AC=r_1\omega\cos\theta_1=r_0\omega$$

and similarly $\quad BC'=r_2\omega\cos\theta_2=r_0\omega;$

that is, the component velocity of every point in AB along AB, is the same and equal to the full linear velocity of the point D. This proposition will be made use of in discussing the outlines of teeth of gears if correctly made.

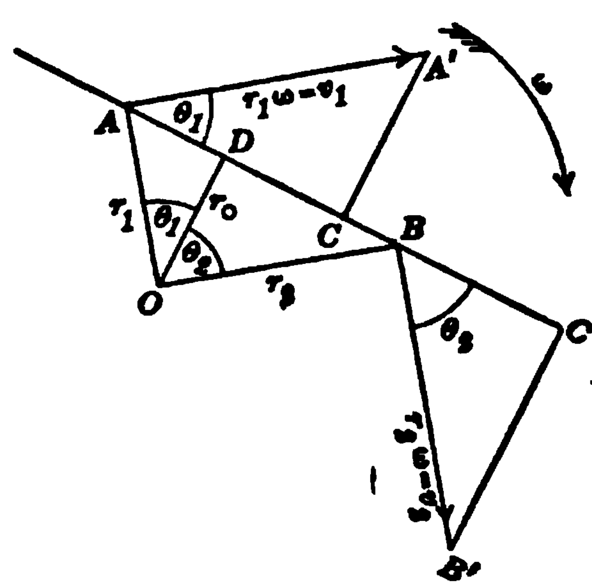


Fig. 305

300. Rotation combined with translation in a direction perpendicular to the axis.

Let a body rotate about an axis O perpendicular to the paper, Fig. 306, with an angular velocity ω; and at the same time let the axis (and body with it) have a velocity of translation v_1 in a constant direction perpendicular to that axis.

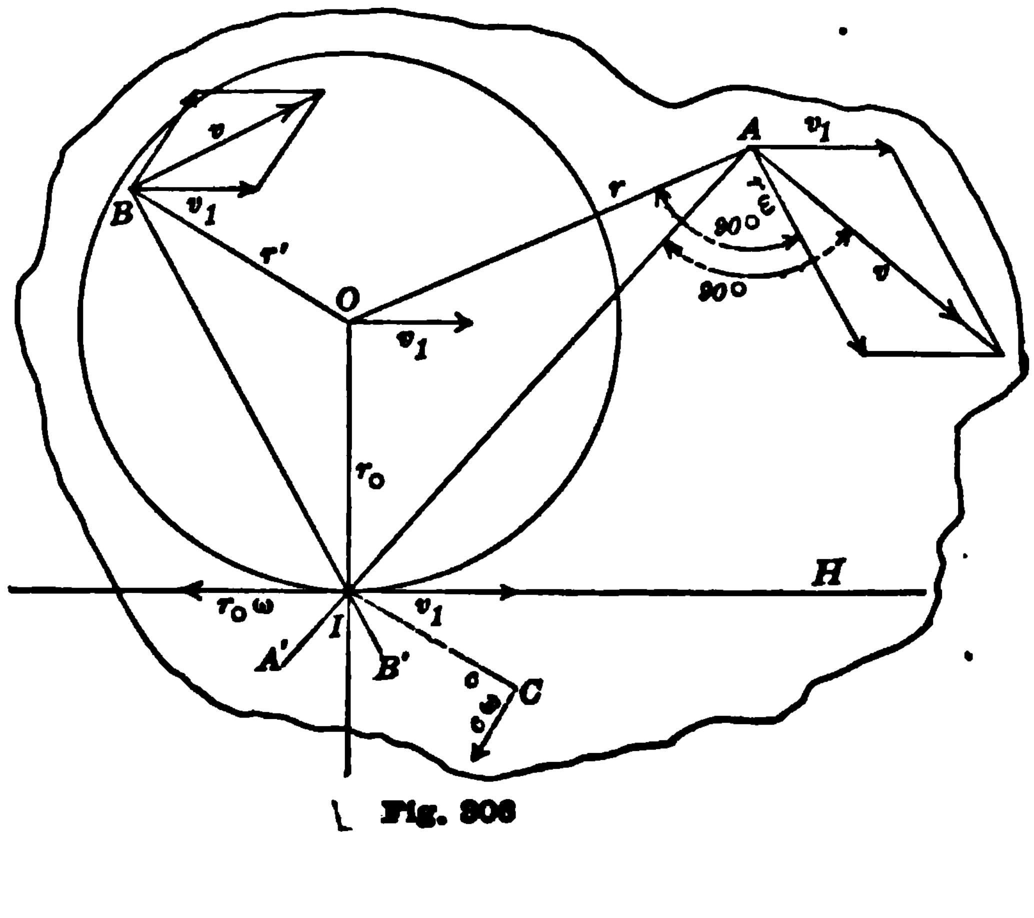


Fig. 306

1. Let A be any point in the body. The velocity of A is the resultant of v_1 and $r\omega$. The instantaneous axis is in a plane whose trace is AA'. The velocity of B, another point, is the

resultant of v_1 and $r'\omega$. The axis is in the plane whose trace is BB'. Therefore the axis itself must be at the intersection of the two planes, I.

2. The point O could have been taken instead of B, and OI been drawn at once as of necessity containing the axis.

3. Moreover, it was only required to find the point whose resultant velocity was zero; that is, $r_0\omega$ must be equal and opposite to v_1.

Hence $$r_0\omega = v_1, \text{ and } r_0 = \frac{v_1}{\omega} \tag{1}$$

4. The anguiar velocity of the body about I is the same as the given angular velocity of the body about O; that is, ω. This is shown easily: The point O is for the instant rotating about I. If x is the unknown angular velocity, the linear velocity of O is r_0x; but we know that the linear velocity of O is v_1, hence $r_0x = v_1$ and $x = \frac{v_1}{r_0}$, which was seen to be the value of ω. Hence the actual velocity of any point C in the body is found to be $c\omega$, in which c is its distance from the point to I.

301. Rolling curves. For an instant the axis I have no velocity. The next instant another line of the body, distant r_0 from O, will come to the line IH and have no velocity when there. It thus appears that the *locus* of all lines which in turn *become* instantaneous axes, form the surface of a cylinder which has O for its axis, and r_0 for its radius. Moreover, the *locus* of all *positions* of the instantaneous axes is a plane parallel to the plane in which the axis O moves, and distant from it by the radius r_0. The motion of the body is therefore as though it were connected with an ideal (or real) cylinder, which is rolling upon (or along) an ideal (or real) plane.

The path of a point connected with a cylinder thus rolling upon a plane may be:—

(*a*), A straight line;
(*b*), A waving trochoid;
(*c*), A cycloid;
(*d*), A looped trochoid.

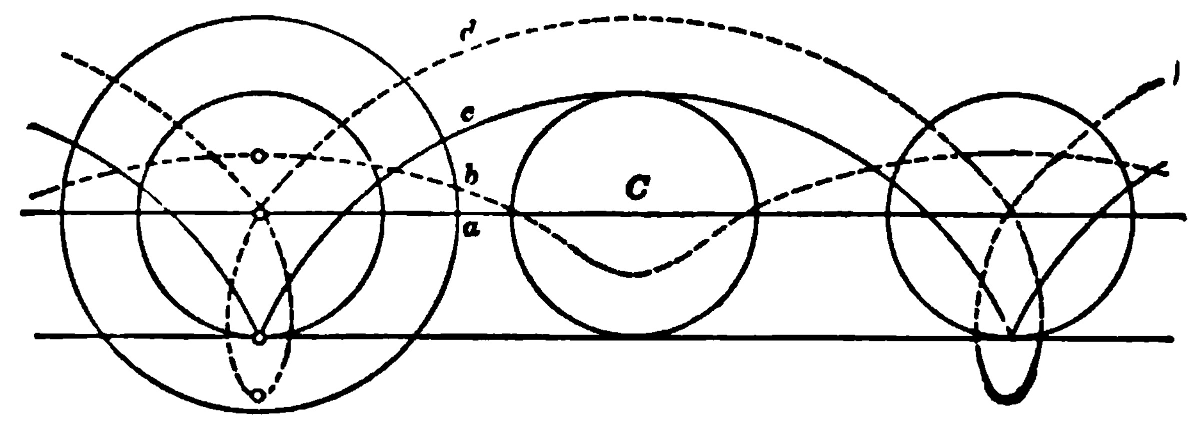


Fig. 307

The equations of these curves will be given later on.

Some oft-mooted questions may be answered by a figure like Fig. 308. I is the instantaneous axis as the circle rolls to the right. Consider the five points: the *top* C, O, I, and two points B and A, equally distant from the point of tangency I.

The line $C'I$ limits the velocity arrows. The velocity of C is seen to be $2v_1$, and the velocities of A and B are of equal magnitude, but opposite in directions. All points below I are moving towards the left; all above I are moving forward. Only I has no velocity; that is, I is the zero point, where v changes sign.

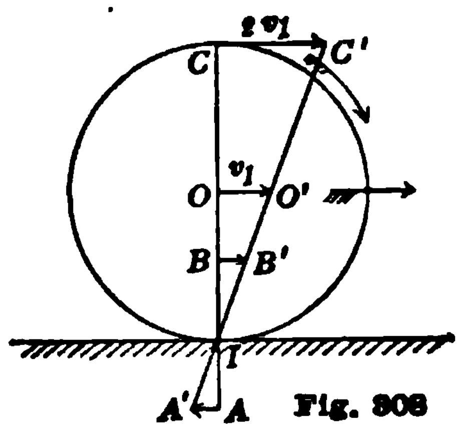


Fig. 308

302. Spirals. A second method of combining rotation and translation parallel to the same plane is as follows: Fig. 309.

Suppose a plane frame, $OPQST-O$, to be revolving with constant angular velocity ω about a fixed axis perpendicular to its plane at O. Connected with the frame are guides upon which a rigid body has a motion of translation with uniform velocity v_1, along the guides, parallel with the plane of the frame and always *perpendicular to the central revolving radius OR.* A point A has therefore a motion of rotation about O, and a motion of translation perpendicular to the line OR. The resultant velocity of A is AB, constructed as in the last problem, and the instantaneous axis is found to be in the plane whose trace is AA'. The axis must also be in OR just as, in the last problem, it was shown that it was necessary for the axis to be in the plane whose trace was OI. Hence the **instantaneous axis** is at I, and its distance from O is

$$r_0 = \frac{v_1}{\omega}.$$

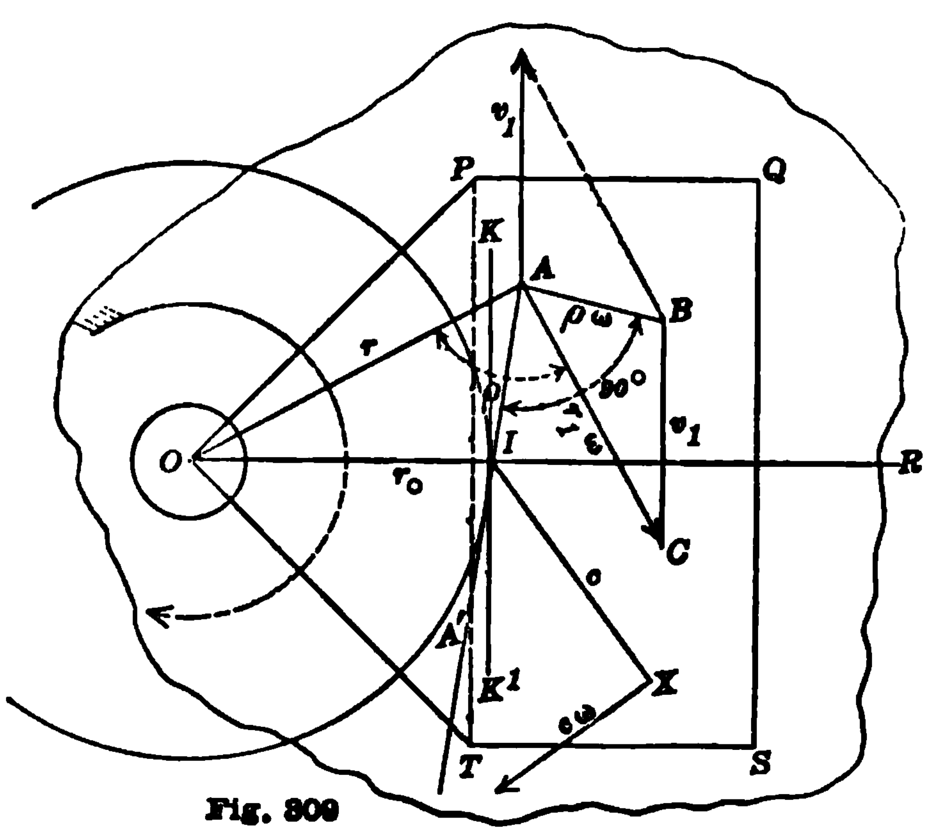


Fig. 309

It is evident that the axis I is always at the same distance from O, as the plane OR revolves right-handed. The *locus* of I is then a cylinder, and the lines which *become axes* are in the plane KK' which always touches the cylinder, so that the element of tangency is the instantaneous axis. The motion then is the same as tho a *plane rolled on a fixed cylinder,* carrying a rigid body along with it. The angular velocity of all points about I is ω, as in the former case, and the actual velocity of a point X at any instant is equal to ω times its *instantaneous radius* $IX=c$; that is, $v=c\omega$.

303. Classes of spirals. The paths of the points of the body are spirals of different classes:—

(a) Concavo-convex spirals which never touch the fixed cylinder, for all points on the side of the plane KK' opposite to the fixed axis O.

(b) Cusped-spirals (the involutes of the fixed circle) for all points in the plane KK'.

(c) Looped spirals for all points between the plane KK' and a parallel plane thru O.

(d) Spirals of Archimedes, for all points in the plane thru O parallel to KK'.

(e) More concavo-convex spirals for all points to the left of the last plane mentioned thru O, or on the side opposite to I.

The involutes (class b) are much used for the outlines of the teeth of wheels. For equations of these paths see **311**; see also Fig. 319.

304. Simultaneous rotation about parallel axes. Let O be a fixed axis, and OC be the trace of an axial plane which carries a parallel axis C. A rigid body is attached to the axis C, and has an angular velocity ω_2 about it, reckoned from the plane OC. Meanwhile, the plane OC itself has an angular velocity ω_1 about O. All paths and all velocity lines must be parallel to a plane which is perpendicular to the fixed axis, and hence there must be for every instant an instantaneous axis, and the motion must have a simple explanation. We are now to find it.

The point C is a point of the body, and has no velocity due to ω_2; hence its resultant velocity line is $CC' = OC.\omega_1$ perpendicular to OC; this shows that the instantaneous axis I is on the line OC. Let $\omega_1 = \frac{1}{2}$ and $\omega_2 = \frac{3}{4}$. Fig. 310.

Let A be any point of the body, distant r from O, and r' from C. The velocity due to rotation about O is $r\omega_1 = AP$. The velocity due to rotation about C is $r'\omega_2 = AQ$. The resultant velocity is AR. The line AI, perpendicular to AR, intersects OC in I, which is the instantaneous axis.

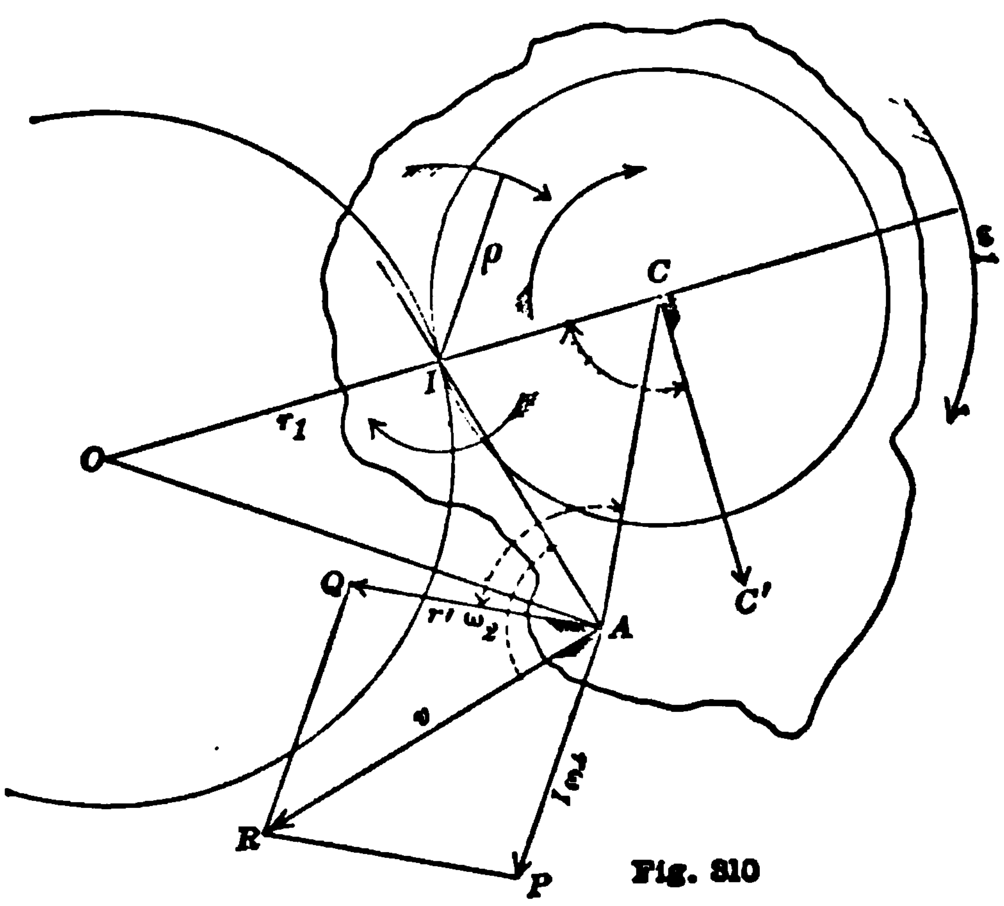


Fig. 310

Let OI be r_1, and CI be r_2.

The position of I could have been found by recalling the fact that the two velocities $r_1\omega_1$ and $r_2\omega_2$ must have been equal and directly opposite, as the velocity of I is zero.

Then
$$r_1\omega_1 = r_2\omega_2$$
$$\frac{r_1}{r_2} = \frac{\omega_2}{\omega_1}$$
$$\frac{r_1+r_2}{r_1} = \frac{\omega_1+\omega_2}{\omega_2} = \frac{OC}{r_1}$$
hence
$$OI = r_1 = \frac{\omega_2}{\omega_1+\omega_2} \cdot OC$$

It is evident from the last equation that, as the plane OC turns about O, the axis I is always at the same distance from O, hence the *locus* of I is the surface of a cylinder whose axis is O, and whose radius is r_1. Again, I is always at the same distance from C, and as the lines which in succession become I have no velocity at the instant, they lie on the surface of a cylinder, which has C for its axis, r_2 for its radius, and which rolls on the fixed cylinder about O.

We may, therefore, think of the motion as tho the body, whose motion we are studying, were rigidly connected with one cylinder as it rolls (without slipping) upon the surface of a parallel cylinder.

305. The resultant angular velocity. Let the angular velocity of all points about I be ω_3; then the resultant velocity of any point X is $r_o\omega_3$, r_o being the instantaneous radius XI. The value of ω_3 is found most easily by considering the motion of the point C. Its velocity is $(r_1+r_2)\omega_1$; it is also $r_2\omega_3$. Hence
$$r_2\omega_3 = (r_1+r_2)\omega_1$$
$$\omega_3 = \left(\frac{r_1}{r_2}+1\right)\omega_1 = \omega_1+\omega_2.$$

The above equations give the triple proportion:—
$$\omega_1 : \omega_2 : \omega_3 = r_2 : r_1 : r_1+r_2$$
which is a convenient form embodying three simple proportions.

There are several cases according to the signs and magnitudes of ω_1 and ω_2.

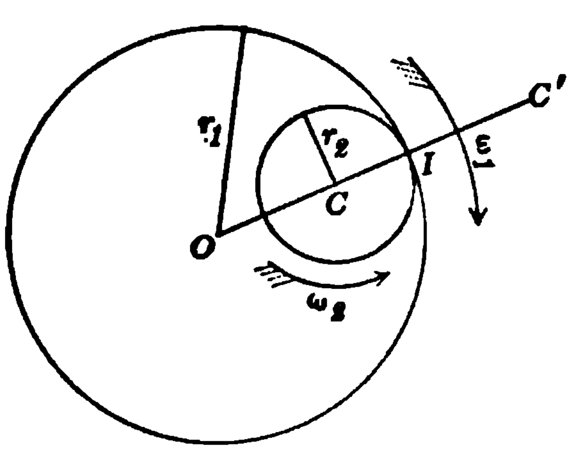


Fig. 311

(1) If ω_1 and ω_2 are both positive (or both negative) the axis I is between O and C, and one cylinder rolls upon an exterior cylinder as in Fig. 310.

(2) If ω_2 is negative and numerically greater than ω_1, I is beyond C in the line OC produced, and a small cylinder rolls within a large one. Fig. 311.

(3) If ω_2 is negative and numerically *less* than ω_1, I is beyond O, and **a large cylinder rolls upon a small one within it.** Fig. 312.

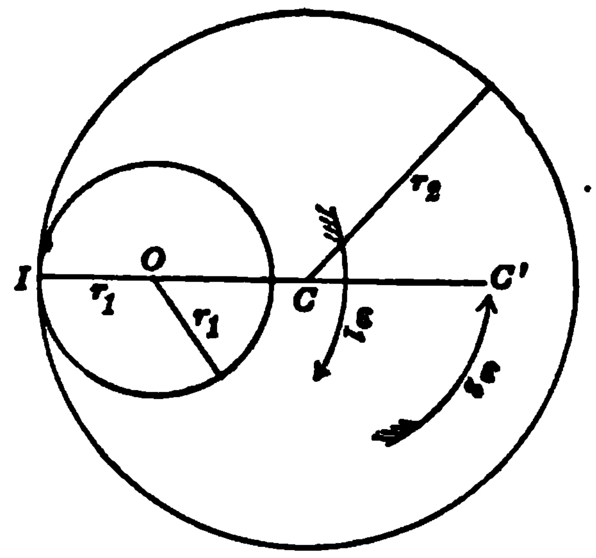


Fig. 312

In each one of these cases the student should find I graphically as in Fig. 310. In *every* case the cylinder O is stationary.

306. Simultaneous rotation about intersecting axis. In this case it will be convenient to let the plane of the axes be the plane of the paper we draw upon. Let OA be a fixed axis, Fig. 313, and OC an intersecting axis revolving around OA making with it a constant angle θ. Of course, the line OC generates the surface of a cone of revolution. Let a rigid body of any shape be so connected with this moving axis that it maintains a constant angular velocity *about it.* Let ω_1 be the angular velocity of OC about OA, and ω_2 the angular velocity of the body about OC, relative to the plane AOC. Both rotations appear right-handed to an observer at O, looking towards A and C. We are to give a simple explanation of the motion and find the position of the instantaneous axis OI. The axis must pass thru O for that point has no velocity.

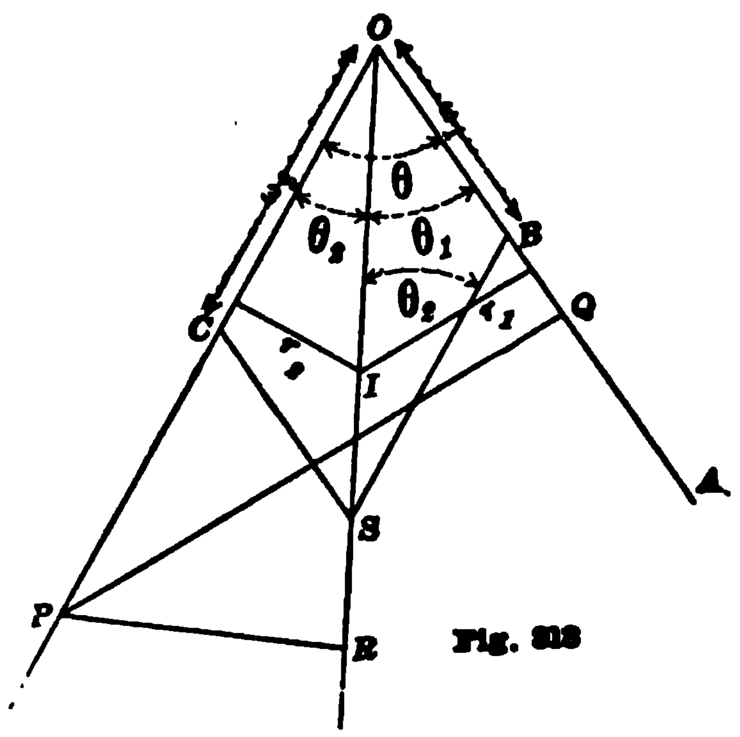


Fig. 313

If ω_1 and ω_2 have the same sign, the instantaneous axis must lie between OA and OC, and in their plane, since the component velocities must balance, one being up and the other down. If I be a point which has no resultant velocity,

$$r_1\omega_1 = r_2\omega_2.$$

But from the figure

$$\frac{r_1}{r_2} = \frac{\sin\theta_1}{\sin\theta_2}$$

hence

$$\frac{\omega_1}{\omega_2} = \frac{\sin\theta_2}{\sin\theta_1}$$

This equation suggests a graphical construction for determining θ_1 and θ_2: Lay off from O on OA a length ω_1, and on OC the length ω_2, and complete the parallelogram, and S must be on the line OI, which is the instantaneous axis.

It is evident that the locus of OI is the surface of a cone having OA for its axis, and a half-vertical angle θ_1. Moreover as θ_2 is constant as

OC rotates about OA, it is evident that all the elements of a cone with OC as its axis and a half-vertical angle θ_2, are successively touching the fixed cone and becoming instantaneous axes. Hence the motion is due to **a cone rolling upon another cone** having the same point as vertex.

The **resultant angular velocity** is found by considering the velocity of P in the axis OC, first as about OA; and second, as about OI; the results are equal. Hence

$$\omega_1 QP = \omega_3 . RP$$

but
$$\left.\begin{aligned} \frac{QP}{RP} &= \frac{\sin\theta}{\sin\theta_2} = \frac{\omega_3}{\omega_1} \\ \text{similarly, for a point in } OA, \quad \frac{\sin\theta_1}{\sin\theta} &= \frac{\omega_2}{\omega_1} \end{aligned}\right\}$$

Hence the triple proportion $\omega_1 : \omega_2 : \omega_3 = \sin\theta_2 : \sin\theta_1 : \sin\theta$.

Since $\omega_3 = \frac{\sin\theta}{\sin\theta_2} \cdot \omega_1$, its numerical value is exactly represented by the diagonal OS. Thus we see that θ_1, θ_2 and ω_3 were all determined graphically in the simplest possible way: *Lay off on the fixed axis* ω_1; *on the moving axis* ω_2; *and complete the parallolgram,* and everything is found.

The value of ω_3 by trigonometry is $\omega_3 = \sqrt{\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \cos\theta}$.

If ω_2 is negative, its value must be laid off on CO produced and the parallelogram constructed with OI outside. The motion then is either that of an acute cone rolling within a larger one, or a larger cone *rolling upon* a sharper one within it.

If θ_1 is 90°, a cone rolls upon a plane. If θ_2 is 90°, a plane rolls upon a cone.

The paths of the points are spherical curves, since the distance of a point from O is unchanged and they are variously named spherical trochoids, spherical cycloids, etc.

Just as epicycloidal, hypocycloidal and involute outlines are used for the teeth of *spur* wheels, so corresponding spherical curves are used for the outlines of teeth for bevel wheels. Modern methods of gear-cutting reach a high standard of accuracy in the forms of teeth, and unless badly worn, well-made gears are almost perfect, and nearly noiseless.

307. Skew-bevel wheels. Simultaneous rotation about two axes which are neither parallel nor intersecting is explained by the rolling and lateral slipping of two hyperloids of revolution which are tangent to each other along a common element. Segments of such surfaces were formerly used as the pitch surfaces of *Skew-bevel wheels.*

In modern machinery two such axes are so readily connected by an additional axis with two sets of bevel gears, that their theory and lack of economy are not discussed in this book.

308. The equations of plane rolling curves. CASE 1. A circle rolls upon a circle. Fig. 314.

Let the circle C roll upon the fixed circle O, and let P on the radius CA' be a tracing point of the path we are to study. Let A be the initial point of contact of the two circles, and OA the axis of X. The figure shows that the "line of centers," OC, has already turned left-handed thru the angle θ. Let $PC=mr$, in which m is a co-efficient which may have any positive value.

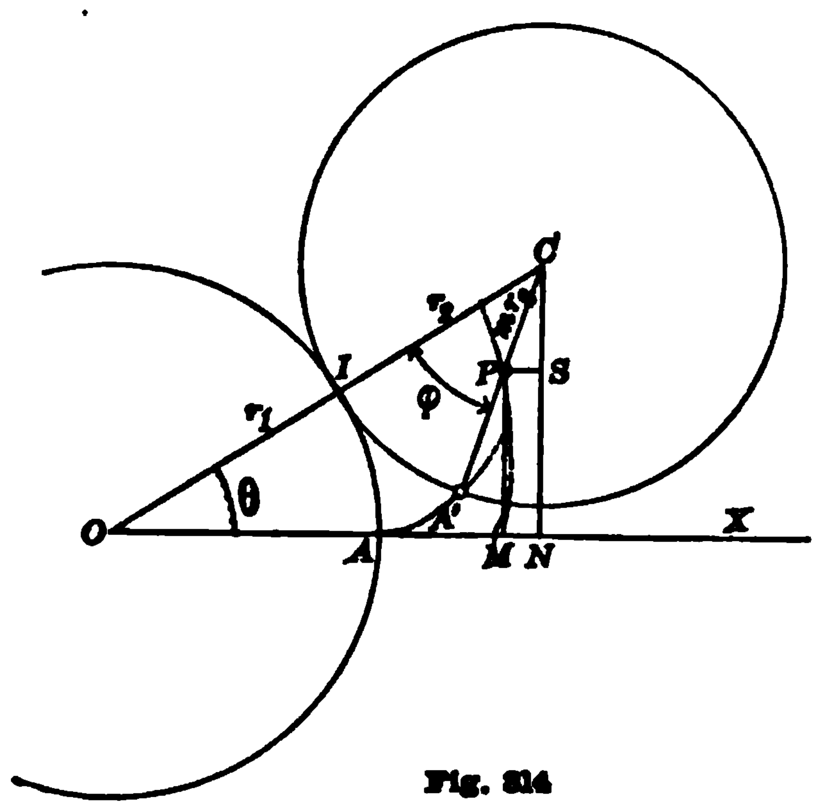


Fig. 314

The co-ordinates of P are:

$$x=OM=ON-PS$$

$$y=MP=CN-CS.$$

$ON=(r_1+r_2)\cos\theta$; $CN=(r_1+r_2)\sin\theta$.

Since the arc IA must equal the arc IA',

$$r_1\theta=r_2\phi, \qquad \phi=\frac{r_1}{r_2}\theta.$$

The angle $\quad PCS=(90°-\theta)-\phi=90°-\left(\frac{r_1+r_2}{r_2}\right)\theta$

Hence $\quad PS=mr_2\sin PCS=mr_2\cos\left(\frac{r_1+r_2}{r_2}\right)\theta$

and $\quad CS=mr_2\cos PCS=mr_2\sin\left(\frac{r_1+r_2}{r_2}\right)\theta$

Substituting, we have

$$\left.\begin{aligned} x&=(r_1+r_2)\cos\theta-mr_2\cos\left(\frac{r_1+r_2}{r_2}\right)\theta \\ y&=(r_1+r_2)\sin\theta-mr_2\sin\left(\frac{r_1+r_2}{r_2}\right)\theta \end{aligned}\right\} \qquad (1)$$

These are the general equations of the path of P. Only in a few special cases can the angle θ be eliminated. The curves are classified as follows:

(a) If $m=0$ the path is a circle whose radius is (r_1+r_2).

(b) If m is a proper fraction, the path is a waving epi-trochoid,

which returns into itself sooner or later, according to the magnitude of the common factor in r_1 and r_2.

(c) If $m=1$, the path is an epi-cycloid, a curve extensively used in forming the outlines of teeth of wheels. P describes a cusp every time it touches the circumference of the circle O.

(d) If $m=1$ and $r_1=r_2$, there is but one cusp, which is at A, and the curve is known as the Cardioid, or Heart-shaped.

(e) If $m>1$, we have a looped epi-trochoid, the loops being partly within the fixed circle if m is less than $1+\frac{2r_1}{r_2}$.

(f) If r_2 is negative and less than r_1, the circle C rolls within the fixed circle and the general name for the path of P is "hypotrochoid," and the equations become

$$\left.\begin{aligned} x&=(r_1-r_2)\cos\theta+mr_2\cos\left(\frac{r_1-r_2}{r_2}\right)\theta \\ y&=(r_1-r_2)\sin\theta-mr_2\sin\left(\frac{r_1-r_2}{r_2}\right)\theta \end{aligned}\right\} \qquad (2)$$

since $\cos\left(\frac{r_1-r_2}{-r_2}\right)\theta=\cos\frac{r_1-r_2}{+r_2}\theta$, and $\sin\frac{r_1-r_2}{-r_2}\theta=-\sin\frac{r_1-r_2}{r_2}\theta$.

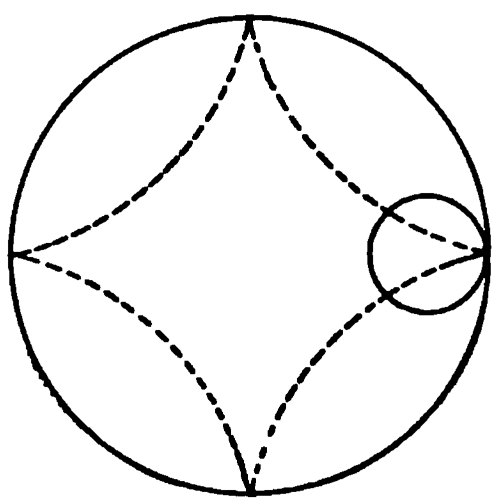


Fig. 315

(g) If r_2 is still negative and $m=1$ the curve is a hypocycloid.

(h) If $r_2=-\frac{r_1}{4}$, and $m=1$, the curve is the 4-cusped hypocycloid usually discussed in the calculus The angle θ is readily eliminated.

The equations of x and y in this case reduce to

$$x=\tfrac{1}{4}r_1(3\cos\theta+\cos 3\theta)=r_1\cos^3\theta.$$

$$y=\tfrac{1}{4}r_1(3\sin\theta-\sin 3\theta)=r_1\sin^3\theta.$$

Then $$\cos^2\theta=\left(\frac{x}{r}\right)^{\frac{2}{3}}, \text{ and } \sin^2\theta=\left(\frac{y}{r}\right)^{\frac{2}{3}}$$

and adding and reducing

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=r_1^{\frac{2}{3}} \quad \text{See Fig. 315.} \qquad (3)$$

(i) If $r_2=-\frac{r_1}{2}$, the equations become

$$\left.\begin{aligned} x&=\frac{r_1}{2}(\cos\theta+m\cos\theta) \\ y&=\frac{r_1}{2}(\sin\theta-m\sin\theta) \end{aligned}\right\} \qquad (4)$$

$$\cos^2\theta+\sin^2\theta=\frac{4x^2}{r_1^2(1+m)^2}+\frac{4y^2}{r_1^2(1-m)^2}=1 \qquad (5)$$

which is the equation of an ellipse whose principal axes are $r_2(1+m)$ and $r_2(1-m)$. It is a circle if $m=0$.

(j) If $r_2=-\frac{r_1}{2}$ and $m=1$, the path is a straight line, a diameter of the fixed circle.

309. Case 2. **The equations of a trochoid.** A circle rolls upon a straight line. The equations of the last section were not in form for substituting $r_1=\infty$, which would mean that the circle C rolled upon a plane. Accordingly, we will derive an independent set of equations. Let the tracing point P be mr distant from C. Fig. 316. If C has already rolled an arc $r\theta$, lay back from the instantaneous axis I the distance $IO=r\theta$, and take O as the origin and the line OI as the axis of X. Then we have

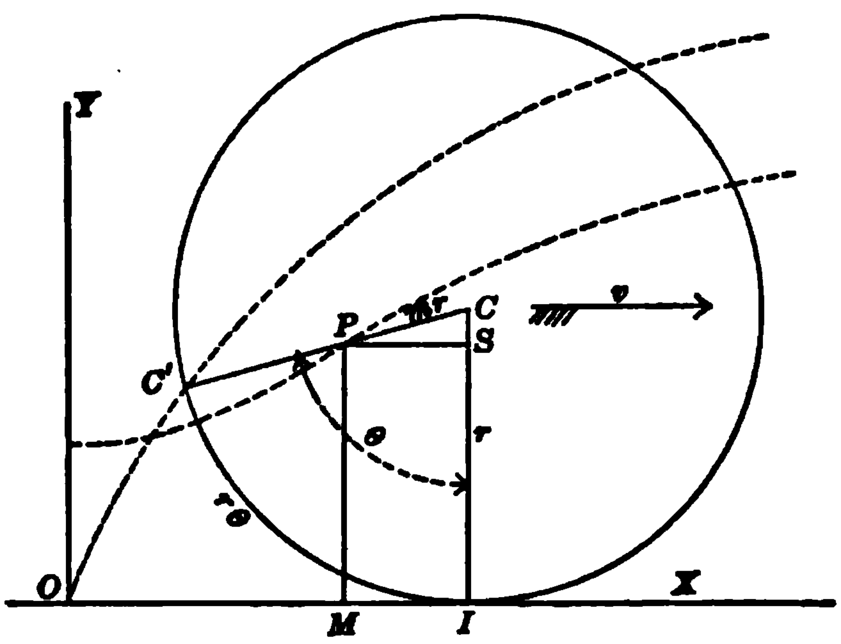


Fig. 316

$$\left.\begin{aligned} x&=OM=OI-PS=r\theta-mr\sin\theta\\ y&=MP=CI-CS=r-mr\cos\theta \end{aligned}\right\} \qquad (1)$$

If m is greater than 1, we have the equations of a looped trochoid, see Fig. 307.

If $m=1$, we have the common **equations of the cycloid.**

$$\left.\begin{aligned} x&=r(\theta-\sin\theta)\\ y&=r(1-\cos\theta) \end{aligned}\right\}$$

The cycloid is the correct curve for the outline of a tooth of a "rack."

310. Case 3. **The general equations of spirals.** *A straight line rolls upon a circle,* carrying the point P, which is distant c from the rolling line. Fig. 317. The axis of X is taken thru O and the foot of the perpendicular from P when that foot was at the point I_1 on the circumference of the circle. [The fig. shows IT too short].

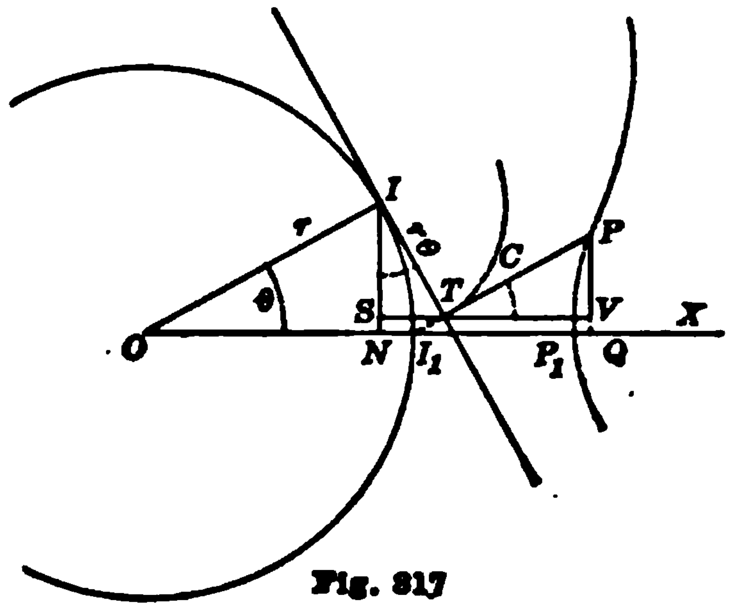


Fig. 317

From the figure we get directly, since $x=OQ$, and $y=QP$:

$$x=r\cos\theta+r\theta\sin\theta+c\cos\theta$$
$$y=r\sin\theta-r\theta\cos\theta+c\sin\theta$$
$$\left.\begin{aligned} x&=(r+c)\cos\theta+r\theta\sin\theta\\ y&=(r+c)\sin\theta-r\theta\cos\theta \end{aligned}\right\} \qquad (1)$$

These are the **general equations of the spirals.** There are SPECIAL CASES.

(a) If $c=0$, we have the equations of the *involute* of the circle O, perhaps the best possible curve for outlining the teeth of wheels.

$$\left.\begin{aligned} x&=r(\cos\theta+\theta\sin\theta)\\ y&=r(\sin\theta-\theta\cos\theta)\end{aligned}\right\} \tag{2}$$

(b) If $c=-r$, we have the equations of the Spiral of Archimedes.

$$\left.\begin{aligned} x&=r\theta\sin\theta\\ y&=-r\theta\cos\theta\end{aligned}\right\}$$

Squaring and adding as they stand, we have

$$x^2+y^2=\rho^2=r^2\theta^2$$

hence
$$\rho=r\theta \tag{3}$$

the ordinary form of the equation.

310½. The Paucellier Cell for transforming simultaneous rotation about two axes into a straight line motion. An interesting and valuable case of simultaneous rotation about parallel axes is found in the device for drawing circular arcs having long radii, in which the combination of links known as the "Paucellier Cell" is used. A simple form of the device is shown in Fig. 317½. The two parallel axes are at O and D, thru which points the axis of X is drawn. The "Cell" consists of the six bars: 2 equal a's, and 4 equal b's. They are connected by pin joints, the bars a and a being pinned by the fixed axis at O. The point P is the tracing point as the cell revolves about O.

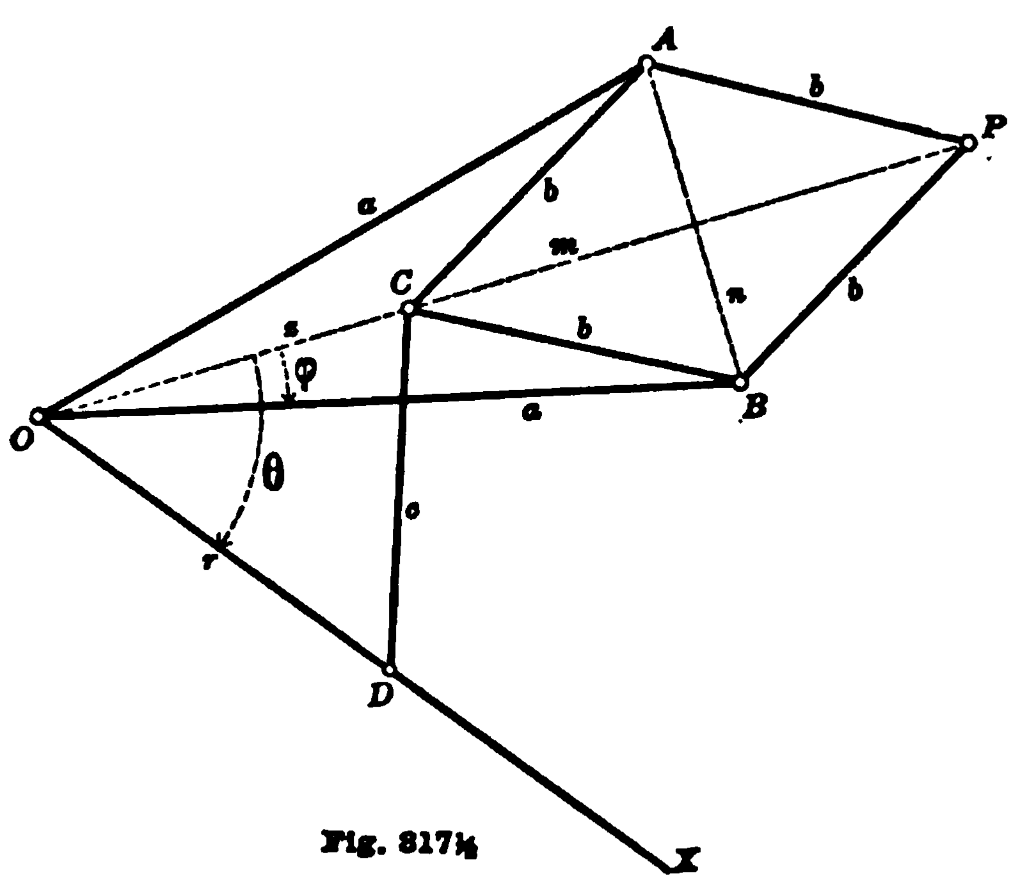


Fig. 317½

The bar c, pinned at the second fixed axis D and the joint C of the cell, controls the motion of C, which moves in a circular arc about D as the tracing point P is moved. The equation of the *locus* of P is found as follows:—

From the figure

$$n=a\sin\phi$$
$$z+m=a\cos\phi$$

Squaring and adding
$$z(z+2m)+b^2=a^2$$

But $z+2m$ is the radius vector of the tracing point, hence
$$\rho z=a^2-b^2.$$

Taking now the triangle OCD, we have
$$c^2=z^2+r^2-2zr\cos\theta$$

Substituting the value of
$$z=\frac{a^2-b^2}{\rho},$$
$$c^2=\frac{(a^2-b^2)^2}{\rho^2}+r^2-\frac{a^2-b^2}{\rho}\cdot 2r\cos\theta$$

Clearing of fractions and substituting for ρ, in terms of x and y, we have
$$(c^2-r^2)(x^2+y^2)+2(a^2-b^2)rx=(a^2-b^2)^2.$$

This is the equation of a circle whose *radius* is
$$R=\frac{a^2-b^2}{c^2-r^2}\cdot c;$$

and whose center is on OX distant $\frac{a^2-b^2}{r^2-c^2}\cdot r$ from O.

If we make $r=c$, the equation becomes
$$x=\frac{a^2-b^2}{2r}$$

which is the equation of a straight line perpendicular to OX.

If r is greater (or less) than c the circular arc is convex (or concave) towards O. The device is used instead of guides with $r=c$ to guide the outer end of the piston-rod of a steam engine in a right line. The distance $OD=r$ can be adjusted for any desired value of R by the equation
$$r^2=c^2-\frac{c}{R}(a^2-b^2)$$

Upon the Paucellier Cell as a basis extensive investigations in Linkwork have been made by mathematicians, especially Professor Sylvester, recently of Johns Hopkins University.

311. The radii of curvature of rolling curves. In drawing arcs thru a determined point, it is useful not only to know the direction of the tangent at the point, but the radius of curvature. Accordingly

a general and several special forms of equations giving values of the radius of curvature will be derived.

Let OC, Fig. 318, be the line of centers of two circles, one of which, C, rolls on the other. I is the instantaneous axis, and A a general tracing point, r the instantaneous radius of A, and θ its angle with the line of centers. The center of curvature of the path at A is on the line AI (produced in the figure).

In the time dt, A moves to A', a distance $AA' = r\omega_3 dt$, and the instantaneous axis changes to I' a distance $II' = r_1\omega_1 dt$.

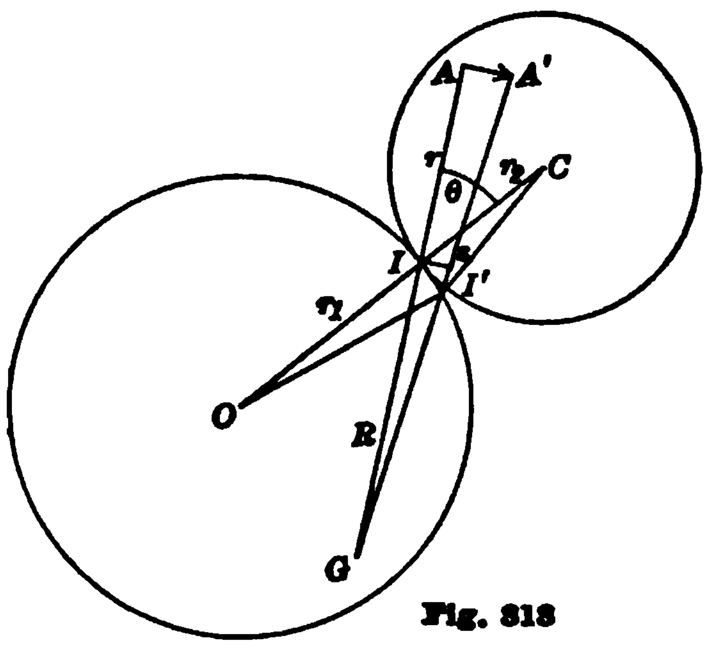


Fig. 318

The new normal $A'I'$ intersects the former normal at G, so that $AG = R$, the radius of curvature at A. From I erect a perpendicular to AG to S. The angle SII' is θ, and $IS = r_1\omega_1 \cos\theta dt$. By similar triangles

$$\frac{R}{R-r} = \frac{\omega_3 r dt}{\omega_1 r_1 \cos\theta dt}$$

Hence by proportional division

$$\frac{R}{r} = \frac{\omega_3 r}{\omega_3 r - \omega_1 r_1 \cos\theta}$$

Multiplying by r, and substituting from **305**, $\omega_1 = \dfrac{r_2\omega_2}{r_1+r_2}$, we have, after reducing,

$$R = \frac{(r_1+r_2)r}{r_1+r_2-\dfrac{r_1 r_2}{r}\cos\theta} \tag{1}$$

This gives **the general value of** R for any point determined by r and θ

Corollaries.

(1) If A be on the circumference of the rolling circle, $r = 2r_2\cos\theta$, and we have

$$R = 2r_2\cos\theta . \frac{r_1+r_2}{\dfrac{1}{2}r_1+r_2} \tag{2}$$

which is the **radius of curvature of an epi-cycloid.**

(2) If $r_1 = \infty$, the circle C rolls on a straight line. Dividing the terms of the fraction in (1) by r_1 and then making r_1 infinite, we have

$$R = \frac{r}{1-\dfrac{r_2}{r}\cos\theta} \tag{3}$$

which is the **radius of curvature of a trochoid.**

(3) If a circle rolls on a straight line and A is on the circumference, we have $r=2r_2\cos\theta$ and $r_1=\infty$, and R becomes

$$R=4r_2\cos\theta=2r, \tag{4}$$

showing that the instantaneous radius is just one-half of the **radius of curvature of the cycloid.**

(4) If $r_2=\infty$

$$R=\frac{r}{1-\frac{r_1}{r}\cos\theta} \tag{5}$$

which is the **radius of curvature of a spiral** in general, as the plane rolls on a cylinder.

(5) If $r_2=\infty$, and $\theta=90°$, the tracing point A is in the rolling line and we have the **radius of curvature of the Involute of the circle.**

$$R=r \tag{6}$$

This shows that the instantaneous radius is always the radius of curvature. See the curve *Iab*. Fig. 319.

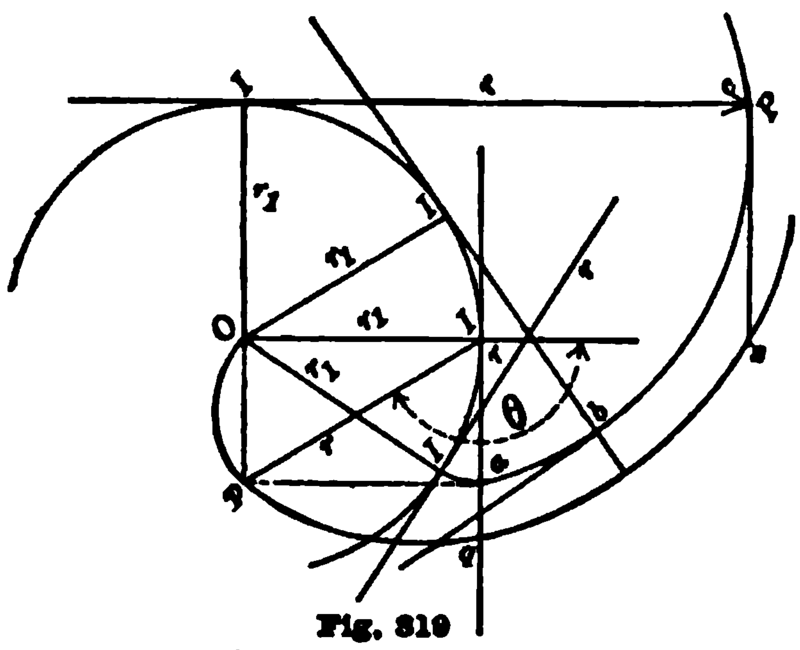


Fig. 319

(6) When a straight line rolls on a circle, and A is r_1 distant from the rolling line, on the circle side,

$\cos\theta=-\frac{r_1}{r}$, and we have

$$R=\frac{r^3}{r^2+r_1^2} \tag{7}$$

which is the **radius of curvature of the spiral of Archimedes,** *opqs*. Fig. 319.*

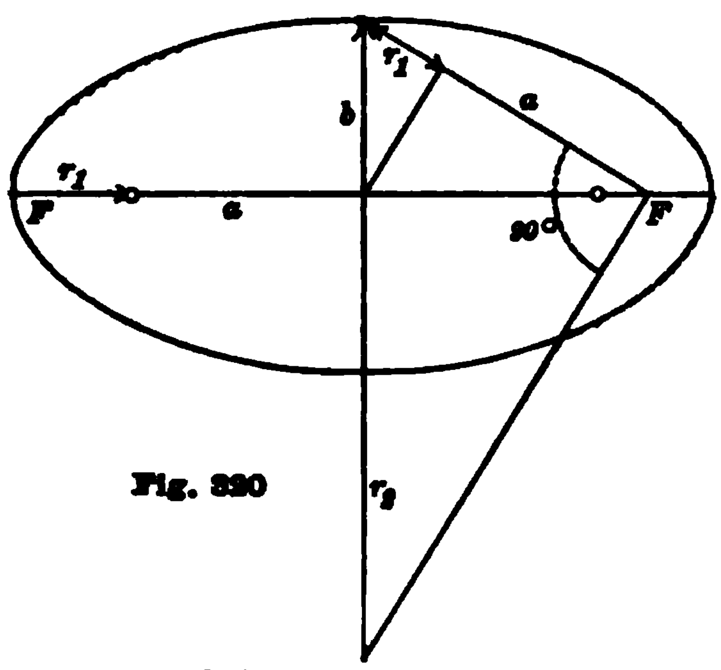


Fig. 320

312. Relative motions of rolling and sliding bodies. In all that has been said and shown in this chapter, a moving cylinder or plane has been rolling in geometrical contact with a stationary cylinder or plane, and the position of the element of contact (the

* The utility of using radii of curvature in approximate drawings may well be illustrated by a drawing of an ellipse. The elliptical effect depends much upon correct curvature at the vertices. See Fig. 320.

$r_1=\frac{b^2}{a}$ at main vertices.

$r_2=\frac{a^2}{b}$ at conjugate vertices.

instantaneous axis) was constantly changing, and all the curves or paths described were drawn on stationary planes.

We shall now assume that, in the case of two cylindres, *both rotate,* one right-handed and the other left, with equal circumferencial velocities, so that the *position* of the element of contact, I, is *unchanged.* Moreover, each cylinder (or circle) carries a plane of rotation of indefinite extent along with it. These planes or disks are ideal, and they overlap as shown by the dotted outlines. Fig. 321. A tracing point connected with one circle or disk may be supposed to trace a curve on the rotating disk of the other circle, while both axes retain their positions. Since the circumferences have the same velocity, we have

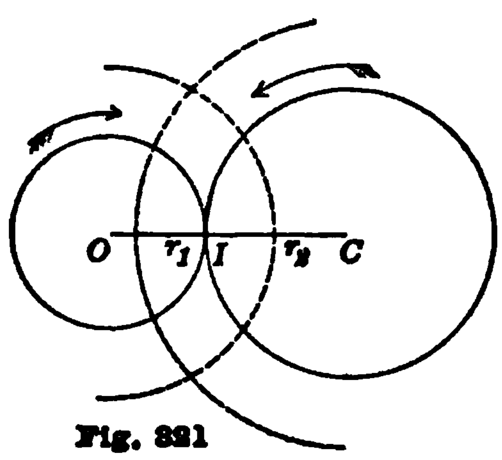


Fig. 321

$$r_1\omega_1 = r_2\omega_2$$

$$\frac{r_1}{r_2} = \frac{\omega_2}{\omega_1}$$

as formerly, and the cylinders are engaged in rolling contact like "friction wheels." The *relative motion* of the two circles (or cylinders) is the same as if one stood still and the other rolled around it, and the curves traced by the points of one upon the disk of the other, are just the rolling curves we have been studying in the last section.

Rough surfaces on cylindrical wheels are used in mechanism to transmit motion and force, by means of friction, but there is always more or less slipping between friction wheels, and when slipping cannot be allowed, "teeth" are commonly used, but the **teeth must be so constructed that the ratio $\frac{\omega_1}{\omega_2}$ or $\frac{\omega_2}{\omega_1}$ is constant.** This condition determines the positions of the engaging teeth on the two wheels, and the outlines of their surfaces of action.

More than a brief statement of the *Theory of Gearing* would be out of place in this book, but the student must be prepared to take up a book on gearing with a clear idea of its nature and an appreciation of its importance.

Suppose O and C, Fig. 322, are the parallel axes of two spur (cylindrical) wheels each supplied with teeth by means of which one "drives" the other. Let O be the "Driver" and C the "Follower." Two teeth T and T' are shown in action. The *direction* of that action is along the common normal at their point of contact. While their *action* is along the normal, their *velocity* lines are different

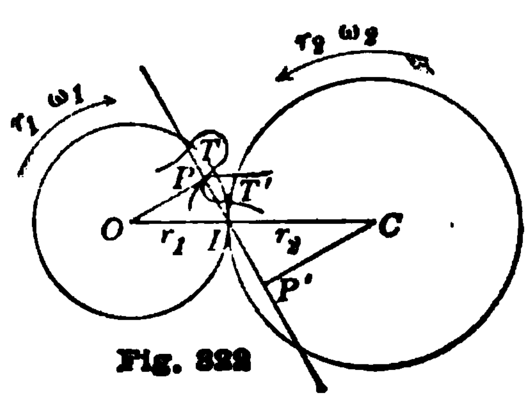


Fig. 322

and neither is along the normal, but as *one is to push the other* **their component velocities along the normal must be equal.**

Let OP and CP' be perpendiculars upon the common normal. (The point P is *not* the point of tangency of the teeth.) Now by **299** the component velocity of T along PP' equals the actual velocity of P, viz., $\omega_1 OP$.

Similarly the component velocity of the point T' (in contact with T) along PP' is equal to the actual velocity P', that is $CP'.\omega_2$. As stated above, these component velocities are equal, hence

$$OP\ \omega_1 = CP'\omega_2$$

$$\frac{\omega_1}{\omega_2} = \frac{CP'}{OP}$$

The ratio $\frac{\omega_1}{\omega_2}$, is known as the "Velocity Ratio" of the two wheels.

The velocity ratio $\frac{\omega_1}{\omega_2}$ of the ideal rolling cylinders was $\frac{r_2}{r_1}$, and that ratio *must be maintained by the teeth*; *hence we must have*

$$\frac{CP'}{OP} = \frac{r_2}{r_1}$$

This proportion is not true unless the *line of action passes thru I.* Hence the condition of correct outlines of two engaging teeth:—*their common normal, at all times, must pass thru the pitch point, i. e., the point of contact* of the ideal *rolling cylinders.* The ideal circles O and C are called the "pitch circles," and their point of contact I is called the "*pitch point.*" If the line PP' varies from I, the ratio $\frac{\omega_1}{\omega_2}$ varies; hence while ω_1 is uniform, ω_2 *is not uniform.*

The problem then is: How shall the working outlines of teeth be correctly determined? The answer is: By means of *rolling curves,* drawn as shown in previous sections.

Taking again our two "pitch circles" with their accompanying disks (shown with dotted outlines), Fig. 323, we take a *third* small cylinder B (or anything which will *roll*) on the *inner* side of the circle O, and let it roll as O and C rotate, being *always in geometrical tangency with both at I.* If T be a tracing point on the circumference of B, it will trace upon the disk belonging to C, the arc of an epi-cycloid; simultaneously

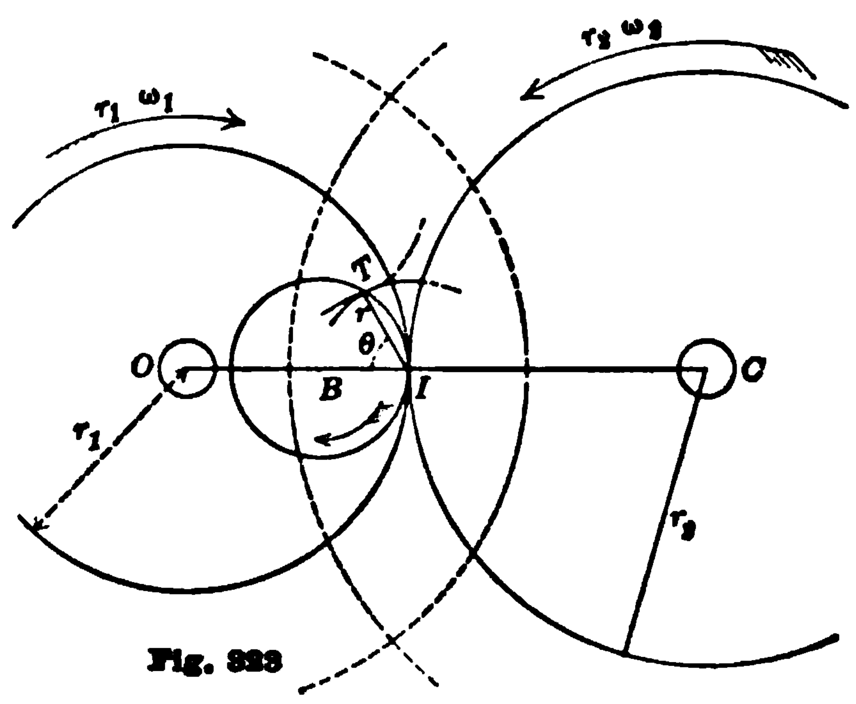


Fig. 323

it will trace upon the disk belonging to O, the arc of a hypo-cycloid. These two arcs are necessarily tangent to each other at T, so that they have a common normal; moreover as I is (so far as the disks are concerned) the instantaneous axis for both curves, IT is the *instantaneous radius* for each curve, and as it coincides with the normal, the latter *passes thru the "pitch point."* The two arcs are, therefore, correct for that part of the tooth of O which is *inside* the pitch circle O, and for that portion of the tooth on C which lies *outside* the pitch circle C. Thus as T "approaches" I, the hypocloidal arc on O engages with the epicycloidal arc on C, and the ratio $\frac{\omega_1}{\omega_2}$ is kept constant.

The outlines of the teeth are completed by using the circle B (or some other small circle) on the *inside* of the pitch circle C.

The hypocycloidal arcs are called the "flanks," and the epicycloidal arcs are called the "faces" of the teeth. It is evident that the arcs may be drawn separately, either with great accuracy by the use of models, or approximately by the use of circular arcs with the *mean radius* of curvature, or the radius of curvature for the *mean value of* θ.

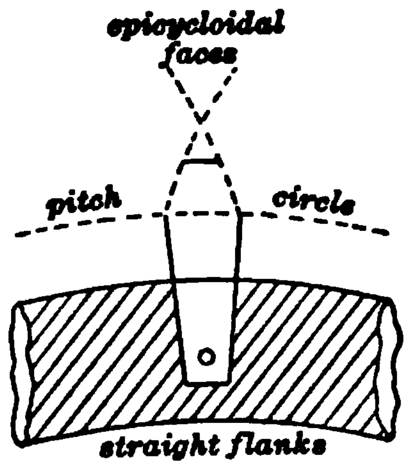


Fig. 324

Formerly, when wooden teeth were in common use (being inserted into the rim of a circle of less diameter than the pitch circle) the radius of B, r_o, was taken equal to $\frac{1}{2}$ the radius of the pitch circle. The consequence was that, in accord with **308** (j), the "flanks" were straight lines or plane surfaces, and the teeth tapered towards the rim. When great pressures are required, such teeth are weak, and metallic teeth with strong bases are used. See Fig. 324.

313. The outlines of involute teeth. It was intimated above, that non-circular curves could be used in place of the small circle B. If *a logarithmic spiral be rolled upon the pitch circles* with the pole as a tracing point, it will describe on each disk the involute of a certain *base circle* concentric with the pitch-circle.

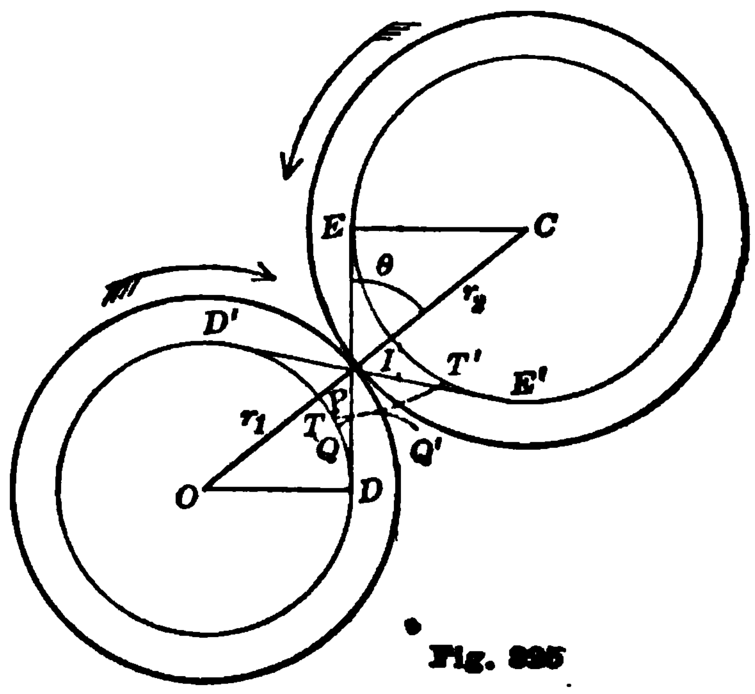


Fig. 325

The two involute arcs can be drawn without the spiral by a neat device equivalent to a crossed belt. Let O and C be the centers of pitch circles with a pitch point at I. Fig. 325. Thru I draw a "line of action" ED making the angle θ (best) equal to

75°, and draw the two "base" circles concentric with O and C and tangent to ED. Draw also the other tangent $E' D'$ and we have the appearance of a cross-belt connecting the two base circles, and we may assume that the mounted wheels, with base circles attached, form a pair, one driving the other by means of the belt. It should now be clearly seen that if the belt does the driving without slipping, the pitch circles will roll without slipping. Let the point P in the belt be a tracing point as it passes from E to D. It traces on the disk of C the involute $T'T$ of the base circle EE'. At the same time P traces on the disk of O the involute $Q'Q$ of the base circle DD'. These two involutes are always tangent at the point P, and their common normal is the line ED. Hence the curves can serve as the working outlines of two teeth which work together correctly.

A **"Rack"** is a segment of a wheel of infinite radius; in other words, it is a straight bar with teeth on it suited to engage with the teeth of a spur wheel. If involute teeth are used on the wheel, the faces of the teeth on the rack are plane surfaces perpendicular to the line of action. Fig. 326. The "base-circle" is infinite if the pitch circle is, and the involute of an infinite circle is a straight line.

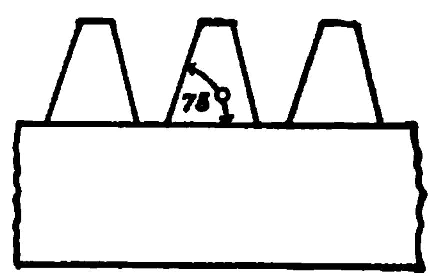


Fig. 326

314. The sliding of teeth is a very important consideration, since it involves a loss of energy. It was stated above that the *action* of two engaging teeth was *along their common normal.* That was true only on the supposition that the engaging surfaces were *smooth.* It might perhaps have been better to say that the actual *driving* was along the normal while the teeth were slipping to and from. The direction of the *real action* is deflected by the "Angle of Friction."* All teeth both roll and slide, more or less; hence their working surfaces should always be clean and lubricated. Involute teeth possess at least *three advantages* over epicycloidal teeth when well made:

(1) They are stronger, being largest at the base.

(2) They involve *less sliding.*

(3) They work perfectly if the distance between centers is not quite the calculated amount. This last is not true of other kinds of teeth.

The **velocity of sliding** is readily shown if we take the two teeth shown in part in Fig. 325. They are in contact at P. Call the tooth of the wheel O, P_1, and the other tooth P_2. The velocity of P_1 is $OP\omega_1 = v_1$ in a direction perpendicular to OP. The velocity of P_2 is

* See the author's paper upon "The Efficiency of Gearing under Friction." [Transactions of the Academy of Science of St. Louis, Vol. VIII, No. 6.]

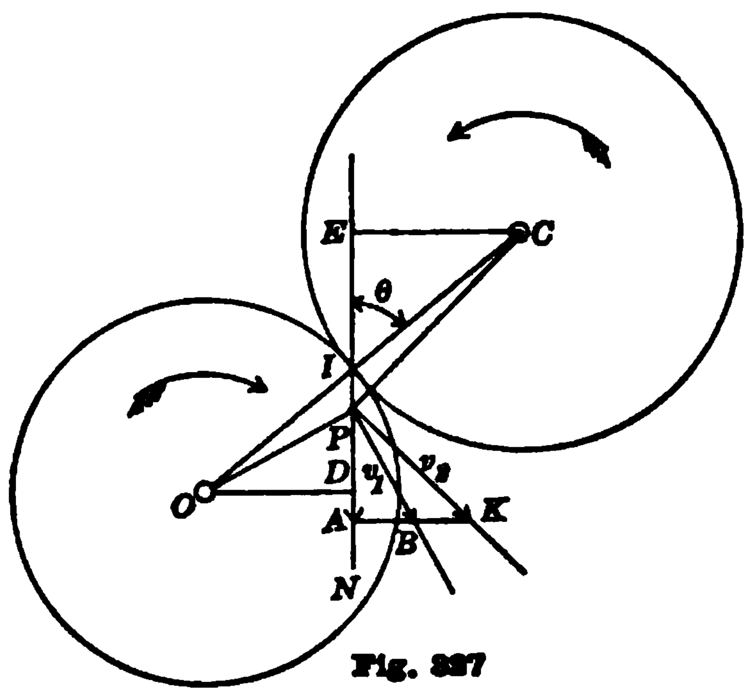


Fig. 327

$CP\omega_2 = v_2$ in a direction perpendicular to CP. See Fig. 327. The common component along the normal PN is $OD\omega_1 = EC\omega_2 = PA$. The component velocity of P_1 *perpendicular* to the normal is

$$AB = \sqrt{v_1^2 - PA^2}$$
$$= \omega_1\sqrt{OP^2 - OD^2}$$
$$= \omega_1 PD$$

The component velocity of P_2 *perpendicular* to the normal is

$$AK = \sqrt{v_2^2 - PA^2}$$
$$= \omega_2 PE$$

Hence the *velocity of sliding* is

$$AK - AB = \omega_2 PE - \omega_1 PD$$

This velocity changes with the position of P. It is only at I that the sliding becomes zero, since $IE\omega_2 = ID\omega_1$.

For further information in regard to the teeth of both spur wheels and bevel wheels, and for numerous practical details, the reader is referred to works on "Gearing." Enough has been given to show the relation of Kinematics to the Theory of Gearing.

315. The universal joint. Two shafts intersecting at O make with each other an angle θ. Each shaft carries a fork, and to the ends of the forks, an equi-armed rectangular cross is connected by pin joints. (A disk with four projecting radial pins would serve even better than a cross.) Fig. 328.

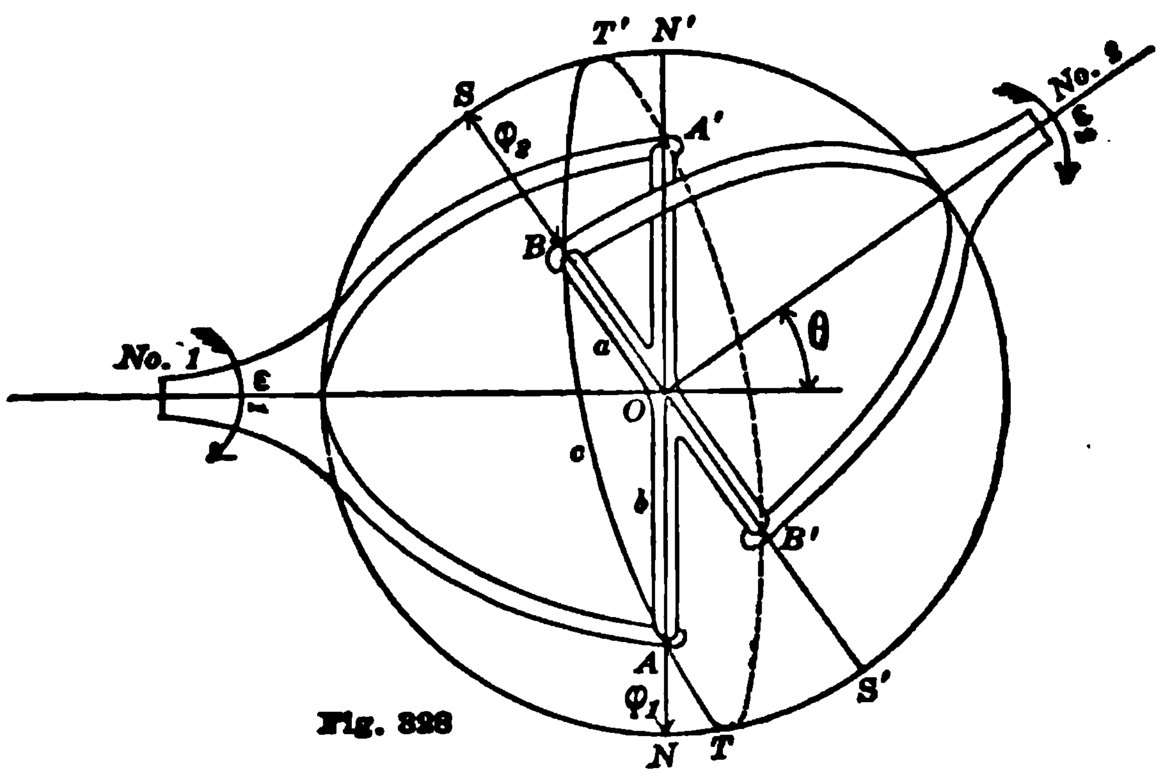


Fig. 328

Let the paper be the plane of the axes. Imagine a sphere with radius unity, on whose surface are the forks and pin joints. Let O be the *topmost point* on the surface of the sphere; that is, the pole of the great circle $SNS'N'$. During rotation all the pins describe arcs of great circles which are pro-

jected on the plane of the paper in straight lines NN' and SS'. Let A, a fork-end of No. 1, and B, a fork-end of No. 2, be above the plane of the paper. Let the arc $AN=\phi_1$, and the arc $BS=\phi_2$. The plane of the cross cuts the spherical surface in a great circle, which is projected into an ellipse, $TABT'A'B'$.

Consider now the *spherical* triangle ABO. We have

$$\begin{aligned}&\text{the arc } AO=90°-\phi_1\\&\text{the arc } BO=90°-\phi_2\\&\text{the arc } AB=90°\\&\text{the angle } AOB=180°-\theta\end{aligned}$$

By spherical trigonometry.

$$\cos AB=\cos AO\cos BO+\sin AO\sin BO\cos AOB.$$

$$0=\sin\phi_1\sin\phi_2-\cos\phi_1\cos\phi_2\cos\theta.$$

Hence $$\cos\theta=\tan\phi_1\tan\phi_2.$$

Differentiating, we have

$$0=\tan\phi_1\sec^2\phi_2\,d\phi_2+\sec^2\phi_1\tan\phi_2\,d\phi_1$$

$$\frac{+d\phi_2}{-d\phi_1}=\frac{\tan\phi_2\sec^2\phi_1}{\sec^2\phi_2\tan\phi_1}=\frac{\tan\phi_1+\cot\phi_1}{\tan\phi_2+\cot\phi_2}$$

For positive rotation $d\phi_1$ is negative while $d\phi_2$ is positive. Moreover the arcs $-d\phi_1$ and $d\phi_2$ are proportional to their respective angular velocities, hence

$$\frac{\omega_2}{\omega_1}=\frac{\tan\phi_1+\cot\phi_1}{\tan\phi_2+\cot\phi_2}=\frac{1}{\cos\theta}\cdot\frac{\cos^2\theta+\tan^2\phi_2}{1+\tan^2\phi_2}$$

$$\omega_2=\frac{\cos\theta(1+\tan^2\phi_1)}{\cos^2\theta+\tan^2\phi_1}\,\omega_1$$

From which we see that ω_2 is not constant for a constant value of ω_1, but fluctuates between two extremes according to the positions of the forks.

When $\phi_1=0$, Fork No. 1 is in the plane of the axes, and Fork No. 2 is in a plane perpendicular to that plane, so that $\phi_2=90°$ and

$$\omega_2=\frac{\omega_1}{\cos\theta}$$

which is the *greatest* value of ω_2.

When $\phi_1=90°$, $\phi_2=0$, and Fork No. 2 is in the plane of the axes and

$$\omega_2=\omega_1\cos\theta$$

which is the *least* value of ω_2.

When $\phi_1 = \phi_2 = \text{arc tan } \sqrt{\cos\theta}$

$$\omega_2 = \omega_1$$

This occurs at two points for every revolution.

The greater the angle θ, the greater the variations.

316. The effect of one or more intermediate shafts. This constantly changing angular velocity in one or both shafts is a bar to the use of the universal joint for heavy machinery. If, however, the angle θ is *bisected* by an additional shaft, with a fork at each end, *with arms in the same plane,* the ratio of $\frac{\omega_3}{\omega_1}$ is constant. See Fig. 329.

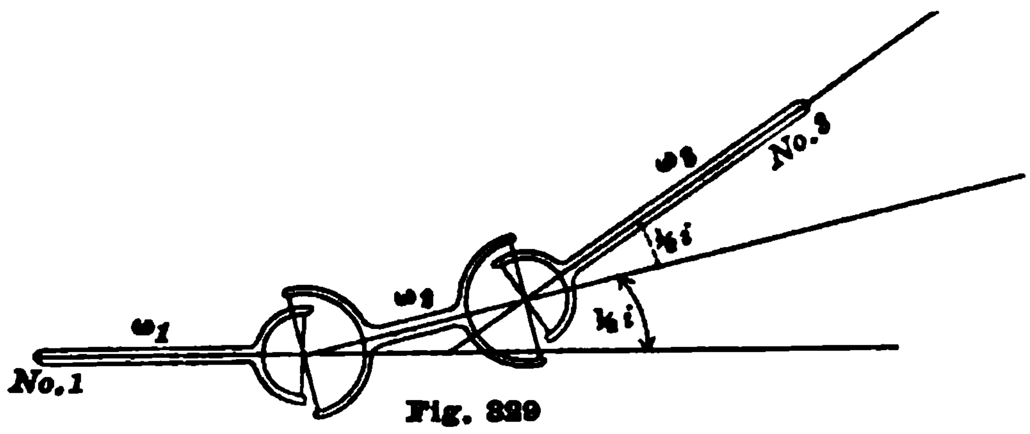


Fig. 329

The truth of this statement is recognized if one reflects that since the two forks of No. 2 are in the same plane, and the deflecting angle is $\frac{1}{2}\theta$ at each end, the position of Fork No. 3 is, when seen from the central shaft, identical with that of No. 1, so that $\phi_3 = \phi_1$ for all parts of a revolution. Hence No. 3 turns with the constant angular velocity of No. 1. The analytic proof of this result is very simple.

If a device requires an oscillating value of ω_3 the amplitude of such changing value may be *increased,* by placing the forks of No. 2 in *different* planes. If they are perpendicular to each other, the ratio $\frac{\omega_3}{\omega_1}$ will vary from $\cos^2 \frac{1}{2}\theta$ to $\frac{1}{\cos^2 \frac{1}{2}\theta}$. As a Universal joint involves no special loss of energy, it has, or may have, many applications in special machines. It is often known as "Hooke's Joint."

Devices for increasing the amplitude of an oscillating angular velocity, by the use of "Hooke's Joints." In Fig. 330, the forks of shaft No. 2 are in planes perpendicular to each other. The ratio $\frac{\omega_3}{\omega_1}$ varies from $\cos^2\beta$ to $\sec^2\beta$.

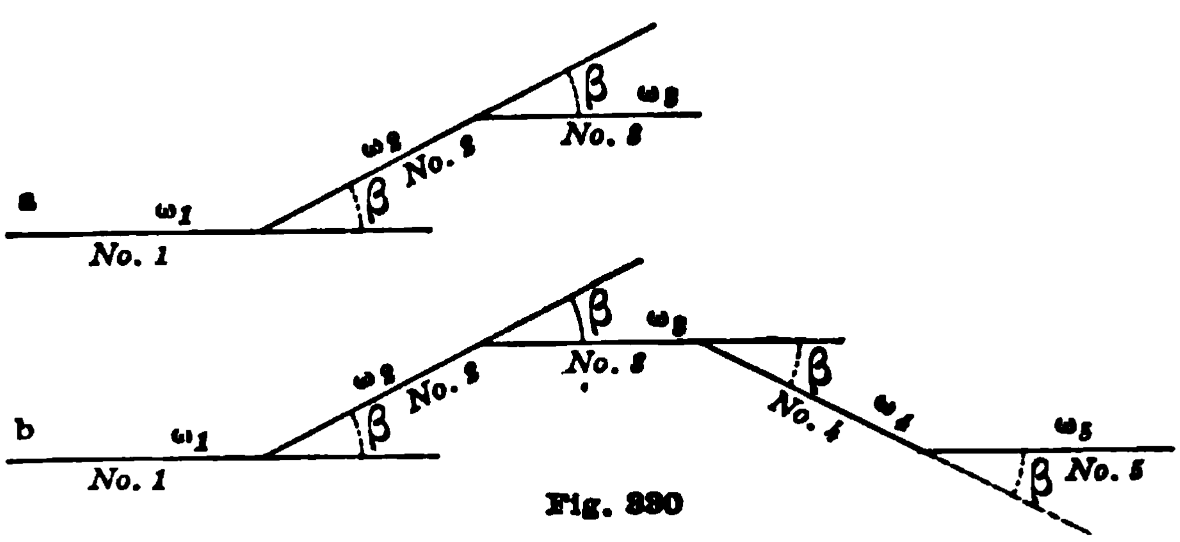


Fig. 330

In (b) where the forks on every intermediate shaft are in planes at right angles to each other, the result is that the velocity ratio $\frac{\omega_5}{\omega_1}$ varies from $\cos^5\beta$ to $\sec^5\beta$.

317. The sidereal day. The promise made in **285** is now to be kept. We shall avoid unnecessary complications, if we assume that the earth's center moves in a circular path with its polar axis perpendicular to the plane of the ecliptic. Think of a plane thru the earth's axis and the sun's center turning about the sun, one revolution a year carrying the earth's axis with it. Call its angular velocity ω_1, which is left-handed seen from the North Pole. Meanwhile the earth revolves about its axis, with *respect to the swinging plane*, one revolution every solar day; let its angular velocity be ω_2. Hence by **304** the resultant angular velocity of the earth is

$$\omega_3 = \omega_1 + \omega_2.$$

And we have the earth connected with an imaginary cylinder which rolls upon a fixed imaginary cylinder whose axis is thru the Sun. The instantaneous axis I is on the line of centers, distant r_1 and r_2 from Sun and earth respectively. Hence

$$\omega_1 : \omega_2 : \omega_3 = r_2 : r_1 : R$$

in which R is the mean distance from earth's center to the Sun's center, *i. e.*, 93,108,000 miles. Now the angular velocities are to each other as the numbers of turns in a given time, say one year. Hence

$$\omega_1 : \omega_2 : \omega_3 = N_1 : N_2 : N_3$$

Now $\quad N_1 = 1; \quad N_2 = 365\tfrac{1}{4}$ nearly; $\quad N_3 = N_1 + N_2 = 366\tfrac{1}{4}$.

Hence, the number of absolute complete revolutions of the earth in one year is $366\tfrac{1}{4}$. Each complete turn (according to a fixed star, *sidus, not* according to the Sun, *sol*) is called a *sidereal* day, while a day according to the Sun is called a *mean solar day*. Of course the sidereal day is less than 24 hours of solar time. Its actual length is given by the eq.:

is: $$\text{Sidereal day} = \frac{365.25 \times 24 \times 60 \times 60}{366.25} \text{ seconds,}$$

$$= 86164 \text{ sec.} = 23 \text{ hr. } 56\text{m. } 4 \text{ sec. nearly.}$$

The paths described by points on the earth are epitrochoids, modified by the eccentricity of the sun, and the obliquity of the earth's axis.

The student may prove that the radius of the rolling cylinder is 254,220 miles, and the radius of the fixed ideal cylinder on which the

earth's cylinder rolls is 92,853,780 miles. Of course it is the value of ω_3 which diminishes the observed weight of bodies near the equator.*

318. Trains of wheel work. By means of teeth, gear-wheels may maintain an unchanging velocity ratio, $\frac{\omega_1}{\omega_2}$; the *slipping* or *sliding* between the teeth spoken of in **314** does not affect the angular velocities as it is always perpendicular to the normal line of motion. It is quite different with *friction wheels* and with belts, since the slipping is, in these cases, always in the direction of the resultant motion. Hence toothed wheels are used where exact values of the ratio are required, as for clocks, watches, screw and gear cutting machines, and many others.

A train of wheel-work is essentially at every instant a *compound lever*, with this supreme advantage that *motion* does not *dislocate* the levers; the individual levers are always in position to act advantageously. Consider the train shown in Fig. 331. Only the "*pitch circles*" are shown, the teeth in every case are partly within and partly without the *pitch* circles. Let the axles be No. 1, 2, 3, 4, and the angular velocities be ω_1, ω_2, ω_3, ω_4. Required the value of $\frac{\omega_1}{\omega_4}$ in terms of the radii, or of the numbers of teeth, n and N, if teeth of the same pitch are used in every combination, *i. e.*, n is in every case proportional to r. Let the large wheels be drivers and the pinions (small gears) be the followers.

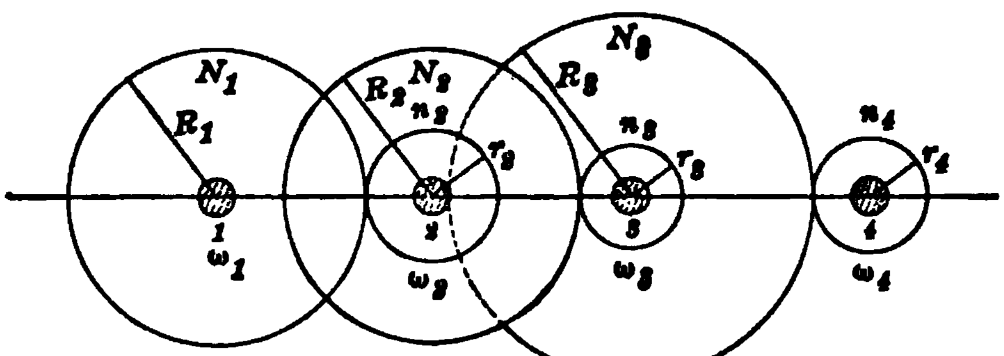


Fig. 331

By **312**

$$\frac{\omega_1}{\omega_2} = \frac{r_2}{R_1}; \quad \frac{\omega_2}{\omega_3} = \frac{r_3}{R_2}; \quad \frac{\omega_3}{\omega_4} = \frac{r_4}{R_3}.$$

Multiplying the three equations

$$\frac{\omega_1}{\omega_4} = \frac{r_2 r_3 r_4}{R_1 R_2 R_3} = \frac{n_2 n_3 n_4}{N_1 N_2 N_3}$$

If the individual ratios are $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$ the required ratio is $\frac{1}{60}$.

* When weights are measured by corresponding standard "weights" placed in a balancing pan, no loss results from the rotation of the earth, as the contents in the pans are affected equally.

If each pinion has 12 teeth, the numbers of teeth in the *wheels* will be 36, 48, 60, so that the total number of teeth will be

$$\Sigma n + \Sigma N = 180.$$

The ratio $\frac{1}{60}$ could have been effected by a single combination, with a pinion of 12 teeth and a wheel with 720 teeth, and a corresponding radius. The economy of the *Train* is evident in spite of two additional axles. The axles need not be in a plane; they may be arranged as one may wish in two or three planes. A gear wheel may act as both follower and driver; in that case it does not change the numerical value of ω, but it does change its sign. Hence it is called an **idle wheel.**

Problem.

Design a train of wheel work such that while the first shaft revolves 3600 times, the last shaft shall revolve but *once*, and in a direction opposite to that of the first shaft. Use no velocity ratio greater than 9 or less than 3.

CHAPTER XVIII.

Work and Energy.

319. When a body *moves* against the resistance offered by another body, thereby overcoming it, *i. e.*, actually *moves* in spite of the resistance, it is said *to do "work."* The amount of work done is measured by the product of the resistance, in units of force, multiplied by the *distance moved* against the resistance. The work may be done by pushing or by pulling. A locomotive may push a snow plow thru deep snow; a horse may drag a stone along a rough surface, or thru a layer of earth; a man may lift a box from the floor to the table.

1. If the resistance is constant, as it is in the case of overcoming the attractive pull of the earth upon the box, the work's measure is easily found, since W is the same for every element of the distance actually raised, so that we have

$$\text{Work} = Wh$$

In the case of the snow plow and the dragged stone, the resistance was not strictly uniform but it had an *average* value which may be approximated and used, so that

$$\text{Work} = Rs$$

It will be convenient to let the letter U represent the numerical value of "*Work done.*"

Work done in stretching a spiral spring.

2. If an elastic spring (Fig. 332) is drawn out from its neutral position a distance s, the resistance is *not* constant, but it is uniformly varying. It began at zero, and ended at sp, if p is the "force" of the spring. See **230**. The *average* resistance offered in this case is $\frac{1}{2}sp$, so that the "work" is

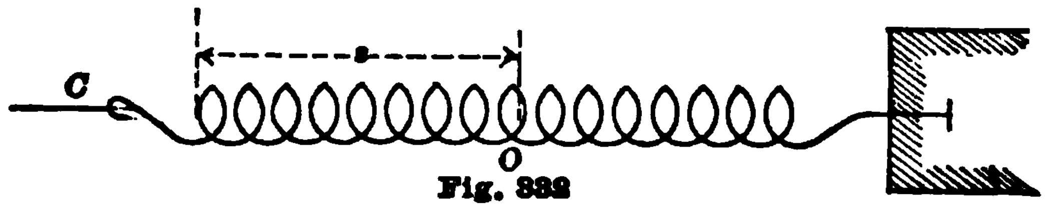


Fig. 332

$$U=\tfrac{1}{2}ps^2.$$

If, after having drawn out the spring a distance s_1, a second epoch begins and the person (or body) pulls the spring out to the full stretch s_2, the work done *during the second epoch* is still the distance (s_2-s_1) multiplied by the *average* pull; viz., $\dfrac{s_1p+s_2p}{2}$, so that the work done is

$$U=\frac{p}{2}(s_2+s_1)(s_2-s_1)=\frac{p}{2}(s_2^2-s_1^2)$$

$$=\tfrac{1}{2}ps_2^2-\tfrac{1}{2}ps_1^2, \text{ which is easily interpreted.}$$

3. When the resistance is neither constant nor uniformly varying, the average may be difficult to find. If R represents the magnitude of the resistance at any general point of the distance, then Rds is an element of the work, and the entire work is represented by the sum of all such elements, so that

$$U=\int_{s_1}^{s_2}Rds, \text{ or } U=\int_0^s Rds.$$

according as the distance is measured from s_1 or from zero. If R varies according to a simple mathematical law, the integration can be made.

320. The potential. Suppose, for the sake of an example, that a body which weighs W at the surface of the earth is drawn by some ideal foreign body, straight away from the earth (which we assume to be at rest) against the resistance offered by the diminishing attraction of the earth, to a distance h from the surface? How much "work" does the foreign body do?

We know from the law of gravitation that the earth's attraction varies inversely as the square of the body's distance from the earth's

center, in mathematical language (see **241**), $F=\frac{k}{s^2}$, in which k is to be determined by the well known fact that $W=\frac{k}{r^2}$, in which r is the radius of the earth; so that our differential equation becomes

$$dU=\frac{Wr^2}{s^2}ds$$

and the required Work is

$$U=Wr^2\int_r^{h+r}\frac{ds}{s^2}=-Wr^2\frac{1}{s}\Bigg]_r^{h+r}$$

$$U=Wr^2\left(\frac{1}{r}-\frac{1}{h+r}\right)$$

If we extend our ideal problem so as to make $h=\infty$, we have $\frac{1}{h+r}=\frac{1}{\infty}=0$ and

$$U=Wr^2\int_r^{\infty}\frac{ds}{s^2}=Wr.$$

The problem might have been: Find the work done by a motor (or mover) which draws an attracted body away from an attracting magnet, assuming that the attraction varies inversely as the square of the distance between the two bodies. The constant k, in such a case, would be determined by a known attraction P_o, at a known distance s_o.

$$P_o=\frac{k}{s_o^2}$$

$$k=P_os_o^2$$

and a general value of F would be $F=\frac{k}{s^2}=\frac{P_os_o^2}{s^2}$. Hence, if the attracted body is moved away from the magnet, supposed to be at rest at a distance s_o, to a distance infinitely great, the "work" done in overcoming the resisting attraction is

$$U=\int_{s_o}^{\infty}Fds=P_os_o^2\int_{s_o}^{\infty}\frac{ds}{s^2}=-P_os_o^2.\frac{1}{s}\Bigg]_{s_o}^{\infty}=P_os_o$$

This quantity of work is known as the "Potential" of the attracting

body with reference to a mass of P units at the distance s. It is represented by some writers by the letter V; hence in general

$$V = Ps$$

so that
$$\frac{dV}{ds} = P$$

which gives the mutual attraction at the distance s.

Since the direction of the line of resultant mutual attraction is not involved, it is evident that the *locus* of points where the attraction on an equal mass is constant is the surface of a sphere whose radius is s, and that the *strength* of that *locus* is measured by the *rate* at *which* **work** *is done* in *increasing the radius* or distance.

The reader should take note of the absence of the *time* element in all this discussion. We are not *now* concerned with time, velocity or acceleration.

321. The work done in a steam cylinder. When the force which moves a body overcomes a resistance, which is constant during a part of its course, and then becomes variable, the total work may sometimes be found. Suppose, by means of an *Indicator* (which see in an Encyclopedia), the intensity of the gas or steam pressure on the piston of an engine at every point in its passage from end to end of a cylinder is known. How much "work" is done by the gas or steam in pushing the piston along?

During a part of the stroke, the intensity of the pressure is constant and equal to p_1. Then, if the area of the piston face is A, the working force is pA; and if the distance moved under full pressure is s_1, the work during the first epoch is $s_1p_1A = U_1$.

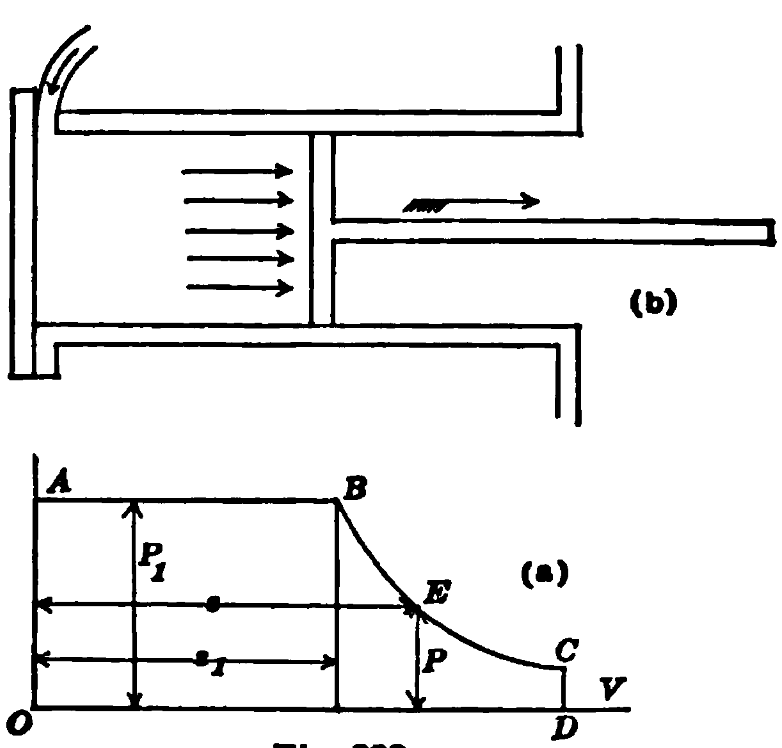


Fig. 338

Next suppose the supply is cut off, and the gas already in the cylinder expands *isothermally, i. e.*, the temperature of the expanding gas is kept constant by means of a hot jacket (without which the gas would cool as it expands). See Chapter XXVII. If the gas is a "perfect gas," like air under ordinary conditions, the pressure will fall according to the law given in physics:—

$$pV = \text{constant} = K$$

in which K is a constant depending upon the temperature, and V is the *volume* of a certain weight, in this case the weight of the steam or

gas used in one "stroke." In a cylinder the volume V increases as s increases, and the volume of the gas at any time is $V=sA$, hence, for an element of motion, the element of work done by the expanding steam is

$$dU_2=pAds=AK\frac{ds}{V}=K\frac{ds}{s}$$

and

$$U_2=K\int_{s_1}^{s_2}\frac{ds}{s}=K\log_e\frac{s_2}{s_1}$$

Hence the entire work done in the two epochs of a full stroke is

$$U=U_1+U_2=p_1As_1+K\log_e\frac{s_2}{s_1}.$$

When p was p_1, V was s_1A; hence $K=p_1s_1A$. So that

$$U \text{ (per stroke)} = p_1s_1A\left(1+\log_e\frac{s_2}{s_1}\right)$$

It is evident that if, in the diagram connected with Fig. 333, the ordinates represent the pressure per sq. unit of piston area, **the area of the figure $OABCD$ represents the work done by the entering and the expanding gas.** The curve BEC is a portion of a hyperbola.

The work done by a team of horses in drawing a grain-reaper a known distance, or the work done by a locomotive in drawing a train of cars a certain distance may be shown by a "dynamometer," if it automatically constructs a diagram similar to Fig. 333(a).

322. In all that has been said about Mechanical *Work*, only *motion* and *force* have been considered. Without motion, no mechanical work is done. A brick in the wall does no work tho it supports a heavy load. When a man *holds* a heavy weight without moving it, he does no *mechanical* work; but when he lifts it against the pull of the earth, he does mechanical work: if he allows it to sink down in his hands, the earth does mechanical work, upon the heavy body and upon the man.

When work is done, at least *one* of the bodies concerned moves; more often both move; when a horse pulls a cart both move; when the earth actually pulls a body to a lower level, only one sensibly moves. Generally, at least three bodies are concerned whenever "Work" is done. Suppose a horse drags a stone up a rough inclined plane; the horse does the work; the rough plane resists the motion by what we call "friction"; the earth pulls *back* by one component of its attraction. The friction and the backward pull constitute the resistance, *if the motion is uniform*; so we have

$$R=fW\cos\theta+W\sin\theta$$

The pull P of the horse must be *more than* R in order to *start* the stone, and if the pull becomes less than R, the stone soon stops moving, and the "work" ceases. When $P > R$, there is an *unbalanced force*,

$$F = P - R$$

which causes the stone to move faster and faster, and the work done per second becomes greater and greater. In this case a *part* of the work is expended in increasing the *momentum* of the moving body. This brings us to a new word:—

POWER.

323. The word "power" is used in mechanics to express the *rate* at which work is done; or the rate at which a motor of some kind, or some moving body, *is able to do work.*

The thing, animal, machine or material which can do 33,000 foot-lbs. of work in a minute is said to be of one "Horse-Power." That is, if it can *actually pull* 2,000 lbs. thru a distance of 16.5 feet in a minute, or 16½ lbs. thru a distance of 2,000 feet in a minute, its capacity is one "Horse-Power." A Horse-Power may be thought of as 33,000 foot-lbs. per min. or 550 foot-lbs. per second, or 990 foot-tons per hour. A horse-power is often written H-P.

Untechnical people often misunderstand the meaning of "horse-power." If a "30 H-P" machine is mentioned, one must not think of a slow moving team of thirty horses. The *distance* factor is as important as the *force* factor. A machine which can pull 220 lbs. a mile in a minute is a 35 H-P machine. This does not mean that it must drag a body which weighs 220 lbs., or that it must carry 220 lbs. weight; it means that it must *actually pull* 220 lbs., as shown by a spring balance, on something *large* or *small*, which moves a mile a minute.

324. Reversible work. When the body upon which work is done is of such a character, or the motion is such that the body can give the work all back again, or do an equal amount of work on another body, the process is said to be *reversible.* For example: If in the case of stretching a spiral spring a distance s as shown in **319**. The "work" done by the agent who (or which) pulled out the spring was $\frac{1}{2}ps^2$. Now conceive of the spring, after a time, acting upon the agent and *pulling it back* to the point of starting, thus doing $\frac{1}{2}ps^2$ units of work upon the agent in the reverse process. Clearly the process of stretching the spring was *reversible.*

During the interval between the two motions, where was the evidence of Work done? Where was the ability or capacity for doing $\frac{1}{2}ps^2$ units of work? Clearly it was "*stored up*" *in the spring.*

Energy.

325. The work thus stored up is called *Energy*. If the spring is perfect, and if the draw hook is secured, the energy may stay in the spring for a long time. Suppose, now, we wish the spring to give up its energy to a rigid mass of metal. Suppose the spring and mass rests upon a smooth horizontal plane, with a thread C between the hook and m. Fig. 334.

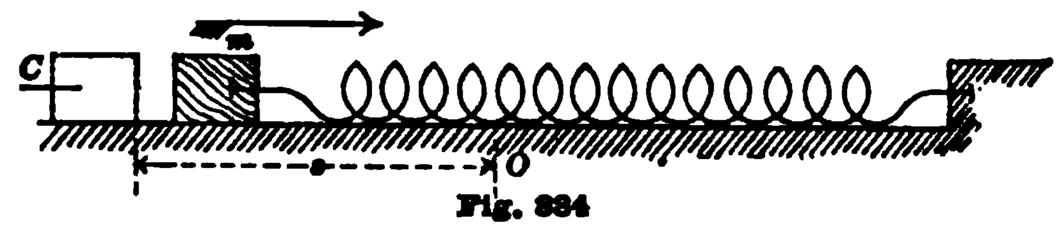


Fig. 334

If now the restraint is removed, the spring will pull the mass, thereby doing work upon m. It will continue to pull, tho with diminishing magnitude, till the spring reaches its neutral unstretched condition. The average pull was $\frac{1}{2}ps$ and the work done on m is $\frac{1}{2}ps^2$, which is just the energy that was stored up in it. Now, suppose the spring is cut off and instantly removed.—Where is the energy now? Of course it is in m. But m is not a spring, its fibers are not distorted in any way; but it is in *motion*; it has velocity, momentum, and in some way the energy it has is a function of its *mass* and its *velocity*. The velocity has already been found in XIV to be

$$v = \sqrt{\frac{ps^2}{m}}$$

Now the energy in the mass m is just *what the spring put into it*, namely, $E = \frac{1}{2}ps^2$. Substituting the value of ps^2 from this last equation, we have

$$E = \frac{mv^2}{2} = \frac{Wv^2}{2g}$$

326. Notation and definition. The letter E, in this Chapter, will numerically represent Energy when it is a quantity; the unit of energy is similar to the unit of work.

Mechanical work will be represented by the letter U as before.

Definitions. A unit of work, U, means an action of one unit of force *actually exerted* upon a moving body in the direction of its motion thru a unit of distance.

A unit of Energy, E, means the *capacity* in a body *to exert* an action of one unit of force thru one unit of distance.

The numerical measure of Energy need not contain either the number of units of force or the number of units of distance. For example, The Energy of a moving mass is numerically expressed by $\frac{mv^2}{2}$,

in which m is the *number* of the units of mass in the moving body, and v the *rate* of its motion in feet per second. There is nothing in E that involves direction. Thus we see that the energy may be due to the velocity of a moving mass, or to a *distortion* or *change of condition* or position of a body which is at rest.

327. Kinetic and potential energy. This distinction is recognized by the use of different names:

Energy due solely to mass and *motion* is called **Kinetic Energy.**

Energy due solely to mass and *position* or *condition* is called **Potential Energy.**

The word *kinetic* is from the Greek kinēo, which means "I move." *Potential* is from the Latin "potens," which means "capable"; so potential energy means energy that in some way is *stored*, but which, when let loose, is *capable* of doing *work*. This is illustrated by the entire motion of the oscillating straight pendulum discussed under harmonic motion in XIV. At first, the energy was (from some source) put into the spring; then it was transferred to the mass of a moving body; then back into the spring. When the oscillating mass was between an extreme position and the neutral center, the energy was partly kinetic and partly potential, *i. e.*, part was in the moving mass and a part was still in the spring.

Energy is stored up in countless ways: in a compressed gas, in explosives, in fuels, in organic and inorganic compounds, in magnetized bars, in storage batteries, etc.

328. Energy due to position. A special form of stored energy is said to exist in bodies which have been raised to storage tanks or platforms above planes where they may, by falling slowly, develop or restore energy for doing useful work. Such bodies are said to hold "*potential energy due to position.*" It would be more accurate to say that in lifting, by any method, a mass m (of solid or liquid) to a height h, the invisible (but quite real) spring or draw-bar which we call gravitation, and which for small heights has a constant tension W, has been *stretched* h feet, so that the work done, Wh, has been stored in that *tension bar of gravitation.* That stored energy is at all times "*capable*" of doing an equal amount of work by pulling the body down again. If its pull is unbalanced, the body will have after a free fall of h feet a velocity $v = \sqrt{2gh}$, so that

$$\frac{mv^2}{2} = \frac{m}{2}\,2gh = Wh.$$

which is just the *work* done in lifting it.

In most cases a work process is not reversible; the work done absorbs the energy in such a manner that it cannot be wholly got back

as mechanical work. The energy spent in plowing the fields, in overcoming friction, in cutting wood and metal, in crushing stones and ores, is **non-reversible.** Where the energy is stored in such processes is not always clear; some of it takes the form of heat, as we know well in the case of friction, in hammering rods of metal, and in turning and boring steel.

329. Illustrations. Many of the problems already used to illustrate the effect of unbalanced forces and couples in producing motion, will serve again to illustrate the transfer of energy, and the spending of energy in the doing of work.

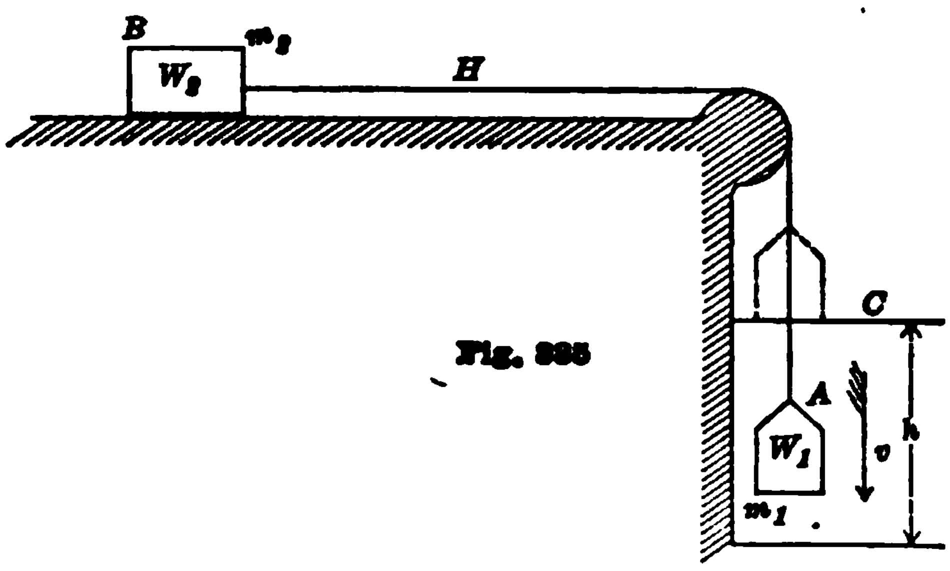


Fig. 335

The potential energy W_1h initially residing in A, when on the plane C, is spent in producing kinetic energy in both A and B and in overcoming friction on the rough horizontal plane H. Fig. 335.

As before, $F = W_1 - fW_2$ is the Unbalanced Force.

$$a = \frac{W_1 - fW_2}{W_1 + W_2} g \text{ is the Acceleration,}$$

$$vdv = \frac{W_1 - fW_2}{W_1 + W_2} gds.$$

$$v_1^2 = \frac{W_1 - fW_2}{W_1 + W_2} 2gh = \frac{W_1 - fW_2}{m_1 + m_2} . 2h$$

The kinetic energy in the two bodies is $\frac{(m_1 + m_2)v_1^2}{2}$ which is by the last equation equal to $\frac{(m_1 + m_2)v_1^2}{2} = W_1h - fW_2h.$

But fW_2h is the *work done* in overcoming friction fW_2 thru a distance h; this is the *non-reversible part* of the energy spent. The rest $\frac{(m_1 + m_2)v_1^2}{2}$ is ready to do work by compressing springs, cutting groves or crushing ores, or other work useful or useless.

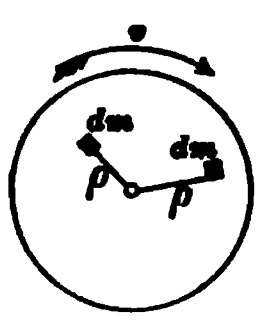


Fig. 336

330. The kinetic energy of a rotating body, turning about an axis thru its center of gravity. Fig. 336.

The Energy of the body is the *sum* of the energy of its *parts*. The energy of a part dm is equal to $\frac{(dm)v^2}{2}$ with-

out regard to the direction in which it is moving. Hence we have

$$(\text{Energy of } m) = \tfrac{1}{2}\int v^2 dm = \frac{\omega^2}{2}\int \rho^2 dm = \frac{\omega^2 I_0}{2}$$

since $v = \rho\omega$, and ω is the same for all the elements in the body, and in the integration is constant. Hence the important formula:

$$K.\ E. = \frac{I_0\omega^2}{2}$$

This is strikingly analogous to the formula for a mass having a velocity of translation, v, viz.:—

$$K.\ E. = \frac{mv^2}{2}$$

as given in the Table of Analagous Formulas (**256**).

When a body has both translation and rotation, both of these formulas are used to find its total energy. A case of this combined motion will now be considered.

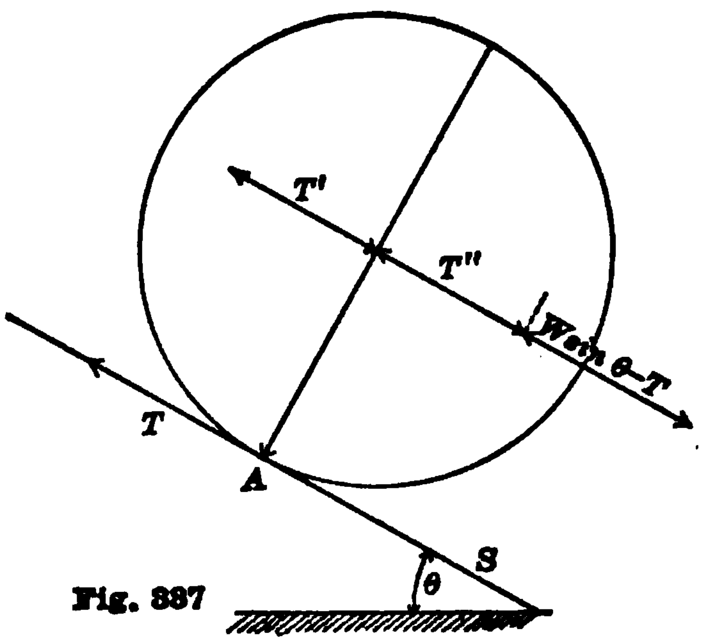


Fig. 337

331. The conservation of energy. Take the case of body rolling down an inclined plane. Let the rolling radius be r, the mass m, the moment of inertia I_0, the length of the plane s, and its inclination θ. Then as in XV, we proceed as follows:—

Canceling $W\cos\theta$ with the normal action of the plane we have left two parallel forces, $W\sin\theta$ and the unknown force T due to friction. Combining these forces, we get an *unbalanced couple*, Tr, and an *unbalanced force*, $W\sin\theta - T$, acting at the center. Fig. 337.

The couple gives us $\qquad Tr = I_0 a, \qquad \therefore\ a = \dfrac{Tr}{I_0}$

The single force gives

$$W\sin\theta - T = ma \qquad \therefore\ a = \frac{W\sin\theta - T}{m}$$

But A is the *instantaneous axis*; hence

$$ra = a \text{ and } r\omega = v.$$

$$\frac{Tr^2}{I_0} = \frac{W\sin\theta - T}{m},$$

whence
$$T = \frac{k_o^2}{r^2+k_o^2} \cdot W\sin\theta$$

$$a = \frac{W\sin\theta}{m+\dfrac{I_o}{r^2}}$$

$$vdv = ads$$

$$v_1^2 = \frac{W\sin\theta}{m+\dfrac{I_o}{r^2}} 2s = \frac{2ghr^2}{r^2+k_o^2}, \text{ since } s\sin\theta = h.$$

$$\omega_1^2 = \frac{v_1^2}{r^2} = \frac{2gh}{r^2+k_o^2}$$

Hence the total energy expended or exerted during the entire descent due to both translation and rotation is

$$\frac{Wv_1^2}{2g} + \frac{I_o\omega_1^2}{2} = \frac{Whr^2}{r^2+k_o^2} + \frac{Whk_o^2}{r^2+k_o^2} = Wh = Ws\sin\theta$$

It thus appears that the Potential Energy due to the elevated position of the body before the rolling began is preserved (or conserved) in the Kinetic Energy of the body when it has reached the base of the plane. This is an instance of the **Conservation of Energy.**

Of course the problem was ideal. It was assumed that no resistance was overcome; that the axis for I_o was thru the axis of the rolling cylinder; tho the body might be solid or hollow, simple or compounded of cylinders, shafts, pulleys, etc.; and finally, that no work was done by the body as it rolled, *i. e.*, it did not crush anything, or rub anything, or slide. There was friction, for that made the body roll, but none of that friction was overcome.

It was shown in **270** that rolling bodies would have different accelerations, velocities, and times, in covering the same distance s; yet everyone satisfies our last equation. The one for which $\frac{k_o^2}{r^2}$ was greatest had the least final velocity; but for the same mass, the Total Energy was the same, tho the division into translation-energy and rotation-energy was very different. It is seen that the last equation is independent of θ, and hence, the path. The equation shows that the potential energy due to the position of the body at a height h above the datum plane thru the foot of the incline, had been transformed into the two forms of kinetic energy.

332. Rolling friction. It is impossible to make wheels and tracks or planes so rigid and smooth that some energy will not be lost as the wheels roll along. This loss arises from the doing of *work* upon the track, or the wheels, or both, which to a measurable extent retards the motion. In the examples in the last two sections, no allowance was made for this friction which is called "Rolling Friction," in distinction from sliding friction, which is much greater. In an example given later in this Chapter, in the case of a trolley car, the co-efficient of friction is given as 0,003, but this includes the friction of bearings. For a single rolling wheel on a level track f would (for metals) not exceed 0,002.

If rolling friction be included in the conditions of the rolling body of **331** the energy $fW\cos\theta.s$ will be lost, and the resulting Kinetic Energy will be less than Wh, the Potential Energy used. The equation will be

$$\frac{mv^2}{2}+\frac{I_0\omega^2}{2}+fsW\cos\theta=Wh.$$

Rolling friction, like sliding friction, is very nearly proportional to the normal pressure, and independent generally of temperature and the area of contact, the breadth of a tire, or the length of a cylinder.

Assuming the Doctrine of the Conservation of Energy, we write, for a freely rolling body, the above equations, and since $v=r\omega$, the values of v, ω, a, α, and t for the plane s are readily found without finding T.

333. The problem shown in Fig. 338, involves points of interest. The Potential Energy stored originally in the mass m_1, that is, W_1h, has been spent in four ways:

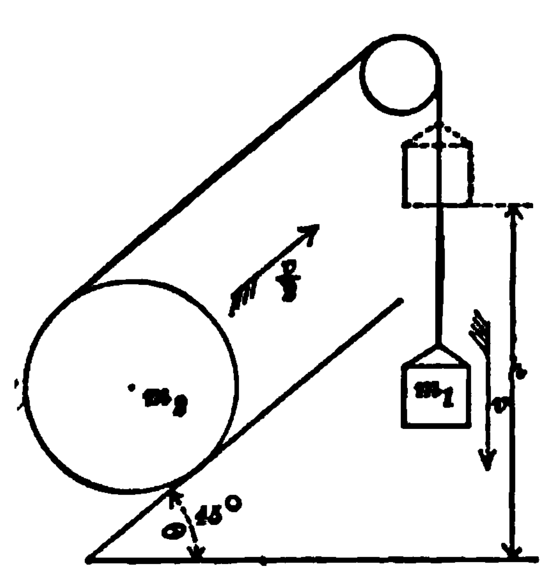


Fig. 338

1. In giving to m_1 a $K.E.=\dfrac{m_1v_1^2}{2}$,

2. In giving to m_2 a $K.E.=\dfrac{m_2\left(\dfrac{v_1}{2}\right)^2}{2}=\dfrac{m_2v_1^2}{8}$

3. In giving to m_2 a $K.E.=\dfrac{I_0\omega^2}{2}=\dfrac{Iv_1^2}{8r^2}$

4. In giving to m_2 a $P.E.=W_2\dfrac{h}{2}\sin\theta$.

The friction is so small that it is neglected.

Let W_1 240 lbs., W_2 400 lbs., and $h=64$ feet.

The student is left to divide W_1h into the four numerical parts correctly. He may also find the values of a, α and t.

334. Rolling and sliding. In the case of the body which both rolls and slides, some of the energy was spent in actually overcoming the friction, so we shall find that the sum of the *K.E.*'s due to translation and rotation will be less than *Wh.*

As T is constantly $fW\cos\theta$, we have $F=W(\sin\theta-f\cos\theta)$.

Hence $$a=\frac{W(\sin\theta-f\cos\theta)}{m}=g\,(\sin\theta-f\cos\theta)$$

$$v_1^2=\frac{W(\sin\theta-f\cos\theta)}{m}\cdot 2s=(g\sin\theta-fg\cos\theta)2s$$

$$\frac{mv_1^2}{2}=Wh-fW\cos\theta.s$$

The second term in the second member of the last equation measures what the work done in overcoming friction *would have been,* had it not rolled at all, but had slid the entire distance. But it did roll and slide too, continuously, all the way down. The turning moment was

$$L=frW\cos\theta=I_0\alpha$$

and its *final* angular velocity was $t_1\alpha=\omega_1$, in which t_1 is the time it took for the body to reach the bottom of the incline. That time is found from the equation $s=\frac{1}{2}at_1^2$ and is $t_1=\sqrt{\frac{2s}{a}}=\sqrt{\frac{2s}{g(\sin\theta-f\cos\theta)}}$.

Now since $$\alpha=\frac{fW\cos\theta.r}{I_0}$$

$$\omega_1^2=t_1^2\alpha^2=\frac{2s}{a}\cdot\left(\frac{fWr\cos\theta}{I_0}\right)^2$$

$$\frac{I_0\omega_1^2}{2}=\frac{s(fWr\cos\theta)^2}{aI_0}=K.E.\quad \textit{due to rotation.}$$

To find the amount of energy lost by sliding.

As was found in **272**, the *acceleration* of sliding is $a-r\alpha$. Hence the distance slid in the time t_1 is

$$\frac{1}{2}(a-r\alpha)t_1^2=s-\frac{sr\alpha}{a}$$

and the energy lost thru sliding is

$$s\left(1-\frac{r\alpha}{a}\right)fW\cos\theta=sfW\cos\theta-\frac{s(frW\cos\theta)^2}{aI_0}$$

Adding Results we have

Energy due to Translation $=Wh-sfW\cos\theta$

Energy due to Rotation $=\dfrac{s(frW\cos\theta)^2}{aI_0}$

Energy lost by Sliding $=sfW\cos\theta-\dfrac{s(frW\cos\theta)^2}{aI_0}$

Total Energy accounted for $=Wh.$

So we see how the energy held by the mass m, due to its elevated position, has been distributed when it reaches the level plane at the foot of the incline.

335. An interesting case of the transfer of energy is found in a steam power plant.

1. The energy stored by cosmic forces long ago in coal is spent in the form of heat upon a mass of water in two ways; (a) *First*, in accelerating the linear or angular motion, or both, of the ultimate parts of the molecules of water to a high degree, according to the pressure to which the water is subjected. Water whose molecules are thus highly accelerated internally, are commonly called *hot*; but their heat is only a form of stored energy.

2. When the expanding instinct of the molecule becomes so intense that it forces away the walls of its restraint, it flashes into steam, as a grain of powder would flash into a large volume of hot gas. The "work" thus done by a grain of powder, or a molecule of water, is relatively enormous.

When a cubic foot of water is converted by heat into steam, some 1700 foot-tons of external work is done, the amount depending upon the temperature (and pressure) at which the conversion takes place. Some of this work may be done upon pistons and so do useful work.

3. In a distant cylinder the steam may do work without expanding and then expand to a greatly increased volume against a diminishing resistance, thereby doing still more mechanical "work."

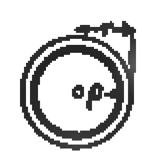
The heat which accelerated the motion of the molecules of water added kinetic energy which may be given back again in the form of heat when condensation takes place. The energy stored in com-

pressed air, as in the case of a stretched spring, is usually called *potential*; but in molecular mechanics the energy in steam or any other gas is called "kinetic," but the difference is one of degree only.

By far the larger part of the energy in fuel is carried off by convection or lost by radiation.

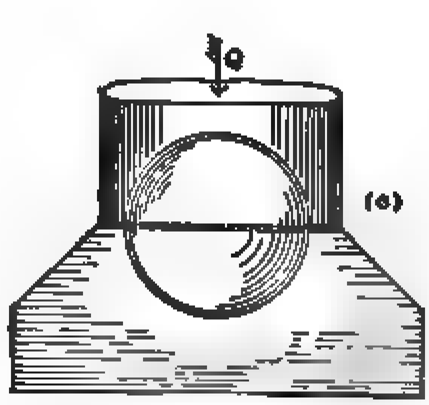
336. In the case of a heavy hammer striking a mass of metal so hot that its elastic limit is reduced nearly to zero, the kinetic energy of the hammer, which may be due to the combined influence of gravitation and expanding steam, is spent in doing work, thereby changing the shape of the metal struck, overcoming internal resistance, and increasing its temperature.

When a hammer strikes an elastic block, the surfaces of both block and hammer are depressed, and the two bodies act like springs, and the action, or force, between the two increases in magnitude till the deformation or depression becomes a maximum, at which point the hammer stops and instantly reverses its motion, flying back into the air again. A blow with a machinist's hammer upon an anvil will illustrate this.



A blow with a club upon an elastic ball, and the rebound of an elastic ball from the hard surface of a wall, are illustrations of motion caused by expanding springs which derive *their* energy from the kinetic energy of a moving body.

The kinetic energy of a falling weight drives a pile into layers of earth; it does work in overcoming friction on the sides and end of the pile, and in crushing it at the end. In all such cases some energy is transformed into heat.



(c)

Fig. 339

337. The friction of pivots. A pivot is a special modification of the end of a shaft, either vertical or horizontal, but subject to thrust, for the double purpose of reducing the **moment of friction** and of preserving the alinement. Pivots are of three kinds: *flat* pivots or rings (a) Fig. 339; *conical* pivots (b); and *cup-and-ball* pivots (c).

(a) **In a flat pivot,** the frictional surface is πr^2, and if the

thrust in the shaft is Q (which may be W, the weight of a vertical shaft and what it carries), the intensity of the pressure is $p=\frac{Q}{\pi r^2}$. The friction upon an elementary ring, whose radius is ρ, is $f.2\pi\rho d\rho.p$ and its moment is $2\pi f p\rho^2 d\rho$, and the moment of all the friction is

$$\text{Mom. of friction } = 2\pi f.\frac{Q}{\pi r^2}\int_o^r \rho^2 d\rho = \frac{2}{3} fQr$$

The *work done* during one revolution in overcoming friction is

$$2\pi M = \frac{4\pi}{3} fQr.$$

The student will do well to note the fact, that the work done by overcoming the friction during one revolution may be obtained by integration, or by multiplying the "moment of friction" by 2π.

Had the bearing surface been a ring, the integration would have been from r to R.

$$\text{Moment of friction} = 2\pi f \frac{Q}{\pi(R^2-r^2)}\int_r^R \rho^2 d\rho = \frac{2}{3} fQ\,\frac{R^3-r^3}{R^2-r^2}$$

The frictional work done in one revolution may be found as follows: The elementary ring $2\pi\rho d\rho$ with a pressure p has a friction $fp2\pi\rho d\rho$. In one revolution the movement on this surface is $2\pi\rho$, so that the work done on it is

$$dU(\text{in one rev.}) = 4\pi^2 fp\rho^2 d\rho$$

$$U\ (\text{lost per rev.}) = \frac{4}{3}\pi^2 fp(R^3-r^3)$$

$$U = \frac{4}{3}\pi fQ\frac{R^3-r^3}{R^2-r^2}$$

If $$r=o,\ U\left(\text{lost}\ \tfrac{\text{per}}{\text{rev.}}\right) = \frac{4}{3}\pi fQR$$

(b) **In the case of a conical pivot** (Fig. 339 *b*), a small hole may be left in the bottom of the step block for the ready removal of foreign matter and for the injection of lubricants. If R and r are the radii of the bases of the bearing surface, the total surface is $\pi s(R-r)$ in which s is the slant height. If p be the normal intensity of pressure at every point of the surface, $p\sin\theta$ is its component parallel to the axis of the shaft, and the resultant of all such components, is

$$\pi s(R-r)p\sin\theta = Q$$

Hence we have

$$p = \frac{Q}{\pi s(R-r)\sin\theta}$$

An elementary zone of the surface is $2\pi\rho ds$; but $ds \sin\theta = d\rho$, so that the elementary surface is $\frac{2\pi\rho d\rho}{\sin\theta}$. The pressure is $p\,\frac{2\pi}{\sin\theta}\rho d\rho$, and the friction is f times the pressure, and the moment of friction is ρ times that product:

Hence
$$dM = fp\frac{2\pi}{\sin\theta}\rho^2 d\rho$$

$$M = \frac{2fQ(R^3-r^3)}{3\sin^2\theta . s(R-r)}$$

But
$$s = \frac{R-r}{\sin\theta},$$

hence
$$M = \frac{2}{3}f\,\frac{Q(R^3-r^3)}{\sin\theta(R-r)^2}.$$

If
$$r = o, \text{ (Mom. F.)} = \frac{2}{3}f\,\frac{fQR}{\sin\theta}$$

The work done in overcoming friction during one revolution is by (a)

$$U = E\left(\frac{\text{lost}}{\text{per rev.}}\right) = \frac{4}{3}\frac{\pi f Q}{\sin\theta}\cdot\frac{R^3-r^3}{(R-r)^2}$$

If
$$r = o, U = E\left(\frac{\text{lost}}{\text{per rev.}}\right) = \frac{4}{3}\frac{\pi f Q R}{\sin\theta}$$

(c) **In the "cup-and-ball" pivot,** or a hemisphere supported in a hemispherical cup, the friction is nearly eliminated if the parts are of hard material and well made. As, however, the surface of contact increases with wear, the moment of friction will increase with the increase of the radii of zones of contact. See Fig. 339 (c).

338. The "Thrust Block" of the U. S. Battleship Delaware. The thrust of a propeller shaft is equally distributed to several ring-shaped bearings by means of several projecting rings on the shaft. Fig. 340.

If the depth of the grooves in the step block (or the height of the rings

The "Thrust-Block" of the U. S. Battleship "Delaware"
Fig. 340

on the shaft) is $R-r$, and the number of rings is n, the bearing surface is $n\pi(R^2-r^2)$, and $p=\dfrac{Q}{n\pi(R^2-r^2)}$.

The *moment of friction* on an elementary zone of *one* ring is $2\pi f p\rho^2 d\rho$ as above, and the moment of friction on *all* the *rings* is

$$(\text{Mom. of Fric.}) = \frac{2}{3} fQ \frac{R^3-r^3}{R^2-r^2}$$

The *work lost* per revolution is $\dfrac{4\pi fQ}{3}\cdot\dfrac{R^3-r^3}{R^2-r^2}$.

Each of the twin shafts of the U. S. Battleship Delaware has *thirteen rings* thrusting against the same number of interior projections of the block. Each projection is *about* $\frac{11}{12}$ of a complete circumference. The radii of the bearing surface are $r=9\frac{1}{8}$ inches, $R=13\frac{1}{2}$ inches.

If Q be the thrust in the shaft, and the pressure on all the bearing surfaces of the block be *uniform*, the intensity of the pressure is *about*

$$p=\frac{Q}{\frac{11}{12}\cdot 13\pi(R^2-r^2)}$$

The moment of friction reduces to

$$M=\frac{2}{3} fQ \frac{R^3-r^3}{R^2-r^2}$$

just as before. The Energy lost in overcoming friction is also as before

$$E\,(\text{lost per rev.}) = \frac{4\pi fQ}{3}\cdot\frac{R^3-r^3}{R^2-r^2}$$

The magnitude of Q cannot be calculated from the data given in Chap. XXI; but if f is known, the *percentage* of Q lost is easily found. The value of f should not differ greatly from $f=0.018$.

Good workmanship is required to so regulate rings and grooves as to make the pressure uniform. However, a slight inequality is removed by wear.

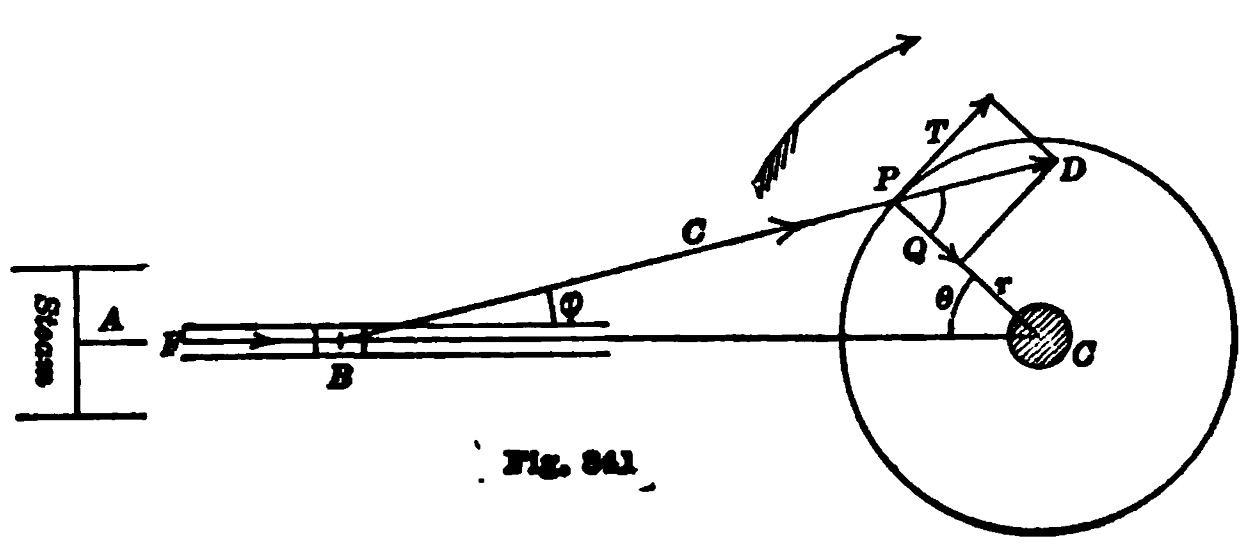


Fig. 341

339. The work done in an engine stroke. It is quite

worth while for the student in his first course in Applied Mechanics to see clearly that no energy, except that due to friction, is lost on account of the pressure against the shaft in the case of a reciprocating crank-engine. Countless so-called inventors have here gone astray. (Fig. 341.)

It has been shown that if the steam pressure against the piston of an engine be $pA = F$, the thrust up the connecting rod (neglecting friction and steam expansion as they do not affect the matter under discussion) is $F \sec\phi$; and the component at right angles with the crank is

$$T = F \sec\phi \sin(\phi+\theta)$$

while against the shaft the thrust is

$$Q = F \sec\phi \cos(\phi+\theta).$$

The thrust Q does no work since it does not move the shaft in the direction of the thrust's action. The component T does move the crank pin; hence it does work. In the time dt its work is

$$dU = Tds = Trd\theta$$

if U stands for the work done, and ds is $rd\theta$. The work done during a stroke is

$$U = \int_0^\pi Trd\theta = pAr\int_0^\pi \sec\phi \sin(\phi+\theta)d\theta$$

Developing, since $c \sin\phi = r \sin\theta$, and $c \cos\phi d\phi = r \cos\theta d\theta$, we have

$$U = pAr\left[\int_0^\pi \sin\theta d\theta + \int_0^0 \frac{c}{r}\sin\phi d\phi\right]$$

The first integral is 2, and the second is zero.

Hence U (the Work done) $= pA.2r$ as it should be; hence:—

The work done in turning the crank equals the work done by the steam, and none is lost on account of the thrust or tension against the shaft.

340. The energy in a moving car. A railway car with 4, 8 or 12 wheels has a velocity v_0 up along a straight inclined track. How far will it move before stopping?

Let the weight of car *body* and *wheels* be W_1. Let there be n wheels, each weighing W_2, and having a radius of r feet, and a moment of inertia $I_0 = m_2k_0^2$, the radius of gyration, k_0, being known. Let f be the co-efficient of rolling friction (0.003), and the grade be θ.

The total kinetic energy in car and wheels must equal the work

done in overcoming the rolling friction, and in overcoming gravitation to a height $s \sin \theta$. Hence

$$K.E. = \frac{W_1 v_0^2}{2g} + n\frac{I_0 \omega_0^2}{2} = fW_1 s \cos\theta + W_1 s \sin\theta$$

Whence
$$s = \frac{v_0^2\left(W_1 + nW_2\frac{k_0^2}{r^2}\right)}{2gW_1(f\cos\theta + \sin\theta)}.$$

If s be known by experiment, the equation will serve to determine a better value of f for local conditions.

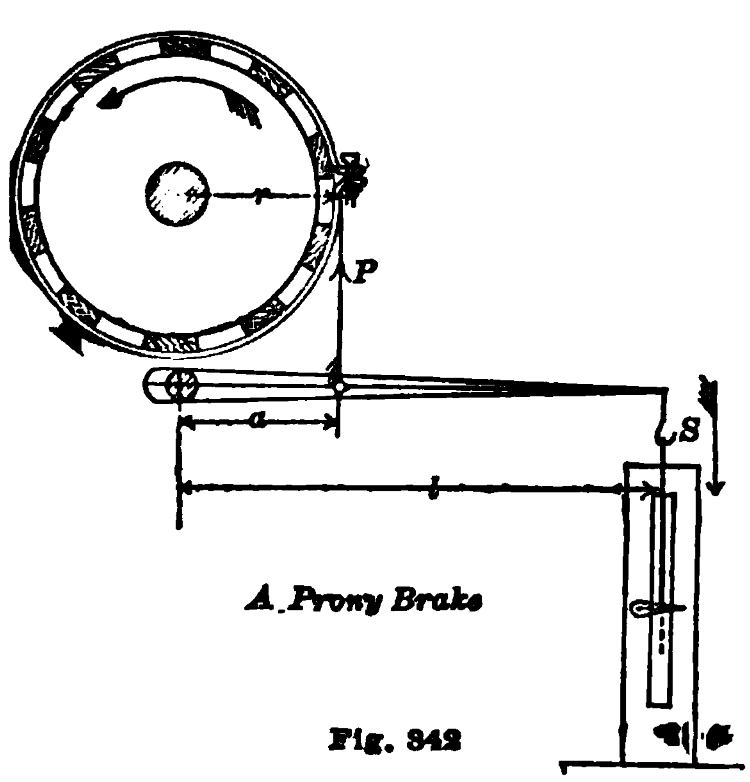
If $\theta = 0$, the distance becomes

$$s = \frac{v_0^2\left(W_1 + nW_2\frac{k_0^2}{r^2}\right)}{2gfW_1}$$

Both formulas show that, v_0 being the same, the car will go further with heavy wheels than with light ones, the initial energy being greater.

341. Measuring the work done by an engine. The work done by the steam in a cylinder is more than the work done by the engine, inasmuch as work is done in overcoming friction, which work is practically lost.

A great variety of methods are in use for measuring the work done by the *engine*, as the reader may see by consulting Professor Wm. Kent's Engineers' Pocket-Book, p. 1280. A common type of dynamometer is the Prony Brake, a form of which is shown in Fig. 342.



A Prony Brake

Fig. 342

The inner surface of the strap carries blocks of wood which are made to press against the face of the driven pulley or fly-wheel of the engine until the moment of friction is all that the engine will overcome and maintain its proper speed.

The unknown moment transmitted to the wheel equals the moment of friction which equals Pr, so that $Pr = M$.

Now, since $Pa = Sl$, we have

$$M = \frac{rlS}{a}$$

Known dimensions and the reading S (the tension on an inverted spring-balance) give the moment transmitted to the wheel. The work done in overcoming the friction during *one* revolution is $2\pi M$; and if N revolutions are made per minute, the work done is

$$U = 2\pi NM = \frac{2\pi rlNS}{a}$$

and the horse-power shown is

$$\text{H-P} = \frac{2\pi rlNS}{33000.a}$$

It is customary to keep a jet of water flowing against the blocks to keep them cool.

This is called an "*Absorption*" *Dynamometer*, since the work done by the engine is all absorbed in overcoming friction, and in generating heat.

342. A transmission dynamometer is one which absorbs practically no work, but measures the power *transmitted by a belt.* A very elegant form is shown in Fig. 343. It does not in any way interfere with the transmission, and it at all times measures the power transmitted provided the ($R. P. M. = N$) of the large driver are known. It can be applied at any point in a train of machinery to show how much power is transmitted by a particular belt. The two upper wheels are "idle" wheels, absorbing no energy since the tension in the belt is constant over the *entire arc of contact on each.* No allowance is here made for the stretching and slipping of the belt.

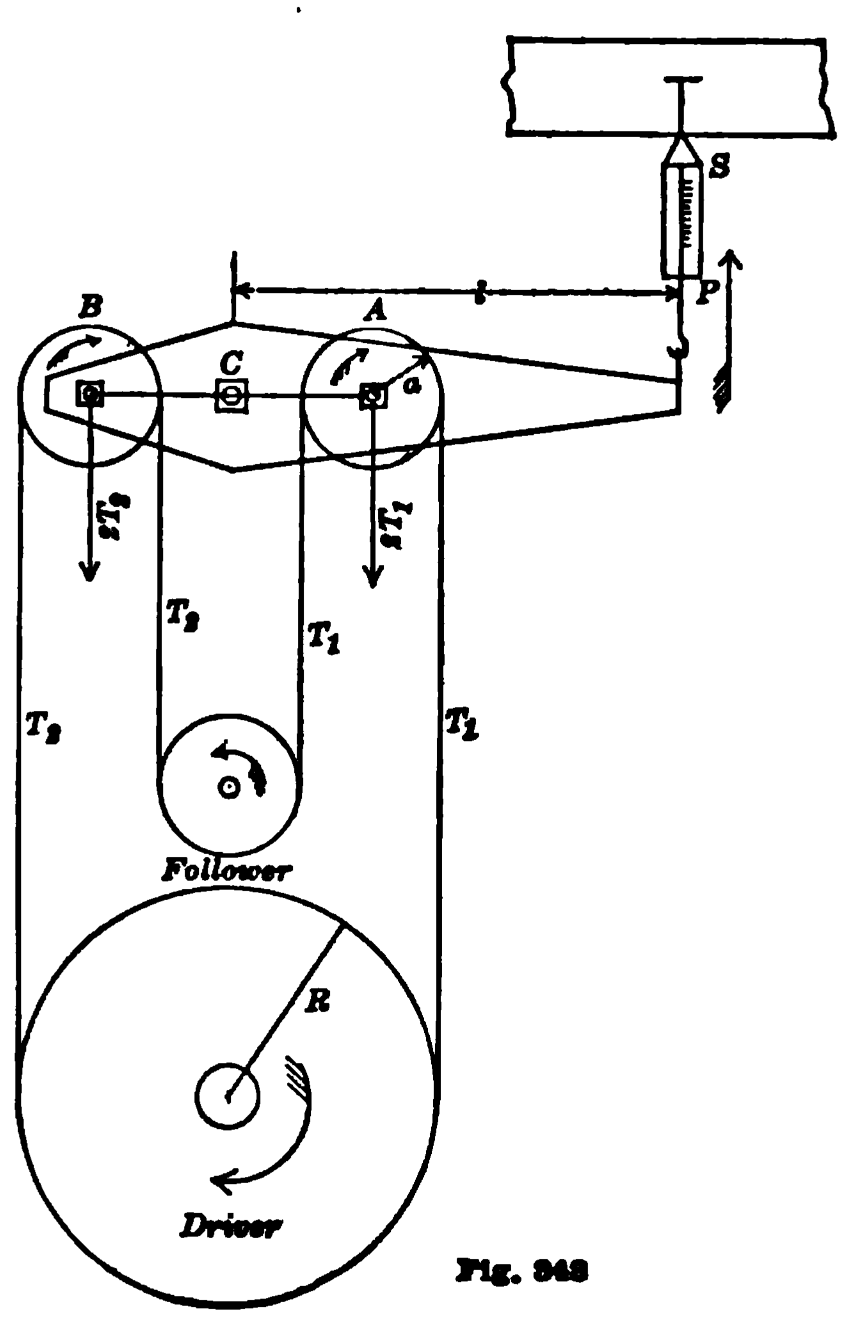


Fig. 343

Let T_1 be the unknown tension in the *driving* ply, and T_2 the tension in the *following* ply. For simplicity let all the plies be parallel, and let the radius of the idle wheels be a. The lever beam is free to turn slightly about the journal box at C. The balance of moments about C gives the equation:

$$2(R-a)(T_1-T_2) = Pl$$

so that $$T_1 - T_2 = \frac{Pl}{2(R-a)}$$

The value of P is read from the spring balance at S.

The energy transmitted per min. in terms of the difference $(T_1 - T_2)$ is $(T_1 - T_2)2\pi RN$.

Hence $$E.\text{ per min.} = \frac{NPl.R\pi}{R-a}.$$

So the horse-power transmitted from Driver to Follower is

$$\text{H-P} = \frac{R}{R-a} \cdot \frac{NPl\pi}{33{,}000}.$$

343. How a brake applied to the wheel of a moving car absorbs its energy *and retards its motion.* Assume that the wheel, Fig. 344, is rolling to the left, on a horizontal track, with just enough tension on the axle towards the left, to overcome the "rolling friction." Now suppose that the brake shown is pressed firmly against the wheel, but is held by the suspension rod from turning with it. The action of the brake, shown by the arrow F, resists the turning of the wheel, and *tends* to turn it *right-handed,* with a moment $Fr = M$. It also lifts upon the wheel at the center by the force F. But the wheel does not turn backward even if it is free to turn without friction on its axle; there must be a second moment which *very nearly* balances Fr. It is found in the action of the track which acts *backward* as shown by the arrow lettered T. This action T is resolved into an equal force T acting backward against the axle, and a *left-handed* couple Tr, which is *very nearly* equal to Fr. It is the force T, acting against the axle, which retards and stops the car.*

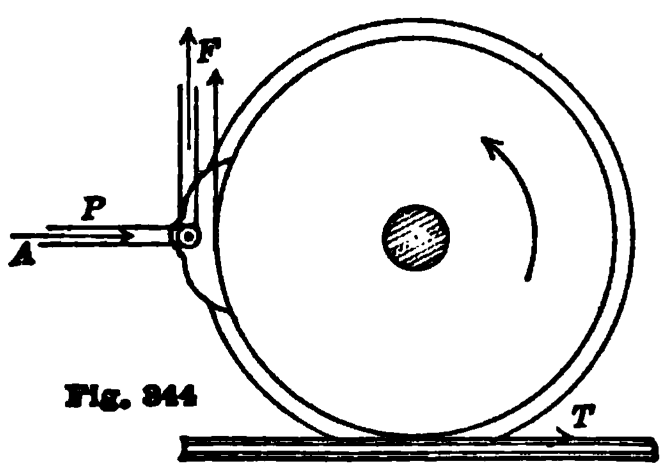


Fig. 344

The action of the brake-rod P is *backward* against the axle, and equally *forward* at the end A, by means of a lever, against the frame of the car; so its resultant action on the car is *nil.* The brake *lifts* on the wheel thereby *tending* to diminish the load or pressure at the bottom of the wheel; but the wheel drags, or *tends* to drag *down* the brake and its support on the frame, thereby leaving the *load* on the wheels *unchanged.*

* The force F must be slightly greater than T in order that there may be a small *unbalanced* moment to retard the rotation of the wheel to correspond with the retardation of the car. The greater I_0, the greater the difference $(F-T)$. It is, however, T which retards the car.

If the brake were at the top of the wheel (Fig. 345) the result would be just the same. The resultant of the brake action and the track action on the wheel would be $F+T$, causing the wheel to exert a retarding force of $F+T$ on the axle; but the brake-rod in this case would act upon the car as an accelerating force F, so that the resultant action upon the car is still T a retarding force.

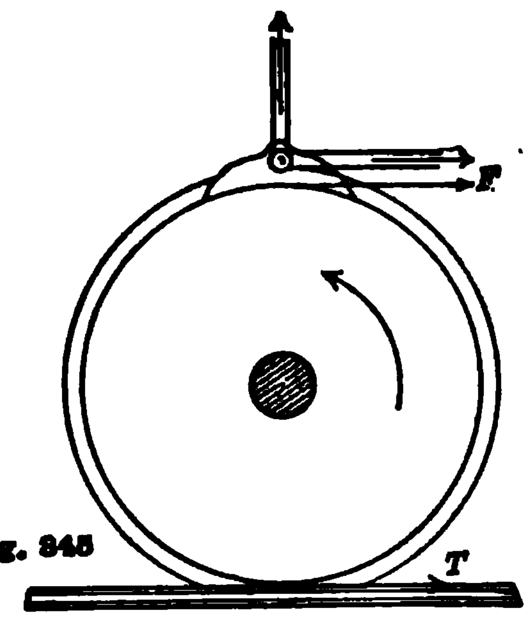


Fig. 345

Problem. The student may prove that the resultant action upon the car is the same when the brake is applied upon the center of the lower quadrant, or on the rear side of the wheel.

344. The kinetic energy in a moving trolley car. Suppose a trolley car having 8 wheels is moving at the rate of 22 ft. per sec. (15 miles per hour). What is its total Kinetic Energy?

The total weight of car, trucks, etc., is W. The wheels are of the same size, with a rolling radius of r feet, a weight of W_1 each, and a radius of gyration of k_1. At the given rate of motion, the angular velocity of the wheels is $\frac{22}{r}=\omega_1$.

There are four motors. The weight of an armature and its shaft is W_2, its radius of gyration is k_2, and its angular velocity is $\omega_2=c\omega_1$.

Accordingly, the $K.E.$ of the car is

$$K.E.=\frac{W}{g}\cdot\frac{v^2}{2}+8\frac{W_1}{g}k_1^2\frac{\omega_1^2}{2}+4\frac{W_2}{g}k_2^2\frac{\omega_2^2}{2}.$$

Mr. Richard McCulloch, Asst. Manager, United Railways, St. Louis, has kindly furnished the following data of a standard electric car:

Total weight of car and trucks, without load, 52,000 lbs. $=W$.

Diameter of each of 8 wheels, $2r=33''=\frac{11}{4}$ feet.

Weight of each wheel, 480 lbs.

Weight of each of 4 armatures, 642 lbs.

Outside diameter of armature, $2r_2=13\frac{1}{4}''=\frac{13.25}{12}$ feet.

Ratio $\frac{\omega_2}{\omega_1}=\frac{69}{17}=4\frac{1}{17}=c$

Approximate value of $k_1^2=\frac{r_1^2}{2}=\frac{121}{128}$

Approximate value of $k_2^2=\frac{r_2^2}{2}=\frac{(13.25)^2}{1152}$

It is seen that if $v=22$ ft. per sec., $\omega_1 = \frac{v}{r_1} = \frac{22}{\frac{11}{8}} = 16$,

and $\omega_2 = 4\frac{1}{17} \times \omega_1 = 65$ nearly.

Substituting these values in the formula for Kinetic Energy, gives

$$K.\ E. = 433{,}845 \text{ foot-lbs.}$$

If the above car is on an *up grade* of one foot in 100 feet, with a rolling-friction co-efficient of $0.003 = f$, how far will it go before stopping, after the electric current is turned off?

Let the required distance be x, then the car will have a vertical *rise* of $h = \frac{x}{100}$ feet, and it will overcome a constant frictional resistance of fW, so that the work done by the Kinetic Energy in overcoming friction is fWx.

Hence
$$K.E. = W\frac{x}{100} + fWx.$$

$$433{,}845 = 52000\left(\frac{x}{100} + 0.003x\right) = 520(1.3)x$$

$$x = \frac{433{,}845}{676} = 642 \text{ ft. nearly.}$$

Had the energy due to the rotation of wheels and armature been omitted, the calculated value of x would have been only 585 feet.

These results should be checked or corrected. The value of g was taken as 32.

It is possible that the value of k_2 has been taken too large on account of the weight of the shaft and gears. It may also be that the coefficient of rolling friction is greater than that assumed, on account of the friction of motor shafts and gears.

345. The difference between momentum and energy. In ordinary discussions these words are often misused and confused. It is important to distinguish them clearly. The student knows that momentum is mv, and that the kinetic energy of a moving mass is $\frac{mv^2}{2}$, but these formulas appeal to the eye only, and the reader asks how their meanings differ. An illustration will help us to answer.

***Momentum* enables us to get the *measure* of the force which caused the motion.** Energy is the measure of the work already done upon a body, thereby creating an internal state of stress or restraint, or

producing motion, whereby, in either case, the body does, or is capable of doing, an equivalent amount of work upon another body.

Suppose we have a smooth bore gun, Fig. 346, containing a charge of "prepared" powder and a heavy shot fitting the bore of the gun, but capable of moving without friction (an ideal gun).

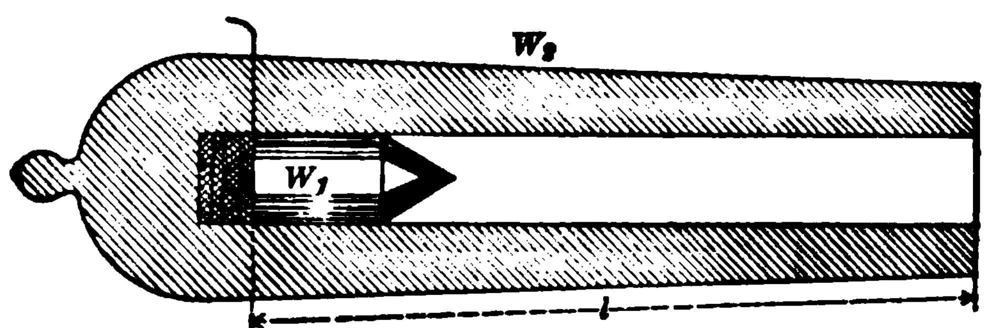


Fig. 346

We will assume (what in gunnery is nearly true) that the powder next the shot burns first and the resulting gas starts the shot, and that as the gas chamber increases in capacity, in consequence of the shot's motion, the powder continues to burn so as to maintain the pressure till the shot leaves the gun, at which time the "prepared" (or gradually burning) powder is all consumed.

Let the area of the rear end of the shot be A, and the *constant* gas pressure be p, so that $F=pA$, is the *unbalanced* pressure on the shot.

The unbalanced pressure in the opposite direction upon the breach of the gun is also $F=pA$. The *forward acceleration* of the shot is

$a_1=\dfrac{pA}{m_1}$, and the (*backward*) *acceleration* of the gun is $a_2=\dfrac{pA}{m_2}$.

In the time t_1, the shot leaves the gun with a velocity

$t_1a_1=\dfrac{t_1pA}{m_1}=v_1$, and the gun recoils with a final velocity $t_1a_2=\dfrac{t_1PA}{m_2}=v_2$.

Comparing these two equations, since t_1 is the same, we notice that $v_1m_1=v_2m_2$ or their *momentums* are equal; that is, the *momentum of the gun is equal to the momentum of the shot.*

Now let us compare their *kinetic energies*. The kinetic energy of the shot measures both the work which the gases did upon it, and *the work which the shot is capable of doing* by overcoming resistances to its motion. A similar statement can be made for the gun.

The energy of the shot is $E_1=\dfrac{m_1v_1^2}{2}=\dfrac{m_1}{2}\left(\dfrac{t_1pA}{m_1}\right)^2$

$$E_1=\frac{t_1^2p^2A^2}{2m_1}. \qquad \text{Similarly } E_2=\frac{t_1^2p^2A^2}{2m_2};$$

so that

$$\frac{E_1}{E_2}=\frac{m_2}{m_1}$$

Hence their *kinetic energies* are *inversely* as their *masses* or weights.

If the gun weighs 80 tons, and the shot 500 lbs., the *energy* of the *shot is 320 times as great as the energy of the gun.*

The energy of the gun is readily employed in compressing a large volume of air in a steel cylinder, the stored energy of which may later assist in handling the gun.

The energy of the shot is employed in demolishing structures, tearing down earth works, or in breaking thru steel armor.

The great quantity of work the shot can do is explained by the great quantity of work done upon it by the gas while inside the gun. In the time t_1 with an acceleration a_1, the distance *thru which the gas continued to push,* is $s_1 = \frac{1}{2}a_1t_1^2$; and the *work done by the gas on the shot* is

$$pAs_1 = \frac{pA}{2}a_1t_1^2 = \frac{p^2A^2gt_1^2}{2W} = \frac{p^2A^2t_1^2}{2m_1} = U_1,$$

just what was found for the $K.E.$ of the shot.

In the same time t_1, with an acceleration a_2, the gun moves backwards $s_2 = \frac{1}{2}a_2t_1^2$, and *the work of the gas in pushing it back is*

$$\frac{pA}{2}a_2t_1^2 = \frac{p^2A^2t_1^2}{2m_2} = U_2$$

The ratio $\frac{U_1}{U_2} = \frac{m_2}{m_1}$ is the same as for their energies.

[Incidentally, it may be stated that $l = s_1 + s_2$.]

In Applied Mechanics, we rarely have occasion to use the word "momentum", or the quantity itself, except in measuring or comparing unbalanced forces. In the proportion $\frac{F}{W} = \frac{a}{g}$ we have the ratio of two (unbalanced) forces put equal to the ratio of the accelerations they produce (on the same mass). When, however, the form of the equation is changed so that it reads

$$F = \frac{W}{g}a,$$

and when we remember that $\frac{W}{g}$ is the *number* of units of mass in the body acted upon by F, we say $F = ma$, which may be read: "An unbalanced force is (numerically) equal to the momentum it can produce in a unit of time."*

* The confusion in the mind of a person, who has read a book on Mechanics without understanding it, is well illustrated by an account given by President R. S. Woodward of the reply made by a young candidate for honors, who had "*been thru*" *Mechanics*, to the question: "What makes a trolley-car continue moving after the current is shut off?" Said the candidate: "It is the *force* of the *power* of the *momentum* of the *energy* of the car." Evidently he had no exact knowledge of anyone of the *four important words* he used; *so he put them all in.*

346. The fundamental equation:— Energy Exerted = Useful Work Done + Kinetic Energy created + Waste in generating heat, overcoming friction, and the *Wear* and *Tear* of materials, — applies with special fitness to devices which, by means of levers and pulleys, modify both force and velocity.

Take for example a **"Block and Tackle,"** with several "plies," as shown in Figs. 347 (a) and (b). Several disks or pulleys (called *sheaves*) are mounted on a shaft which is supported by a sort of stirrup or called a "block." Two such blocks and a rope engaging the sheaves as shown form the combination named above. The upper is called the "*fall-block,*" and the lower the "*running-block.*" The free end of the rope is called the "*tackle-fall.*" If the radii of the sheaves are made proportional to the velocities of the ropes engaging them (as roughly

Fig. 347

shown in the figures above) the sheaves may all be *fixed* upon the shaft; but they may be *loose,* and have equal radii.

Omitting, for the present, the friction and stiffness of the rope on small sheaves, we may say that the tension is uniform from end to end of the rope. A stretch of rope between blocks is called a *ply.*

Suppose by some agency the *tackle fall* is drawn down or out with a velocity v, and that thereby the hook T_2 is drawn up with a velocity u, what is the ratio $\frac{v}{u}$?

The "Energy Exerted" in one unit of time is T_1v; the "Useful Work Done" is Wu; in uniform motion no "Kinetic Energy is Created"; and there is no "Waste" or "Wear and Tear". Hence $T_1v = Wu$.

If T_2 rises at the rate of u feet per second, every ply is shortened at the rate of u feet per second, and the shortening of the rope between blocks will be nu feet per second, if there are n plies. Since the rope is kept taut, the motion of T_1 *must absorb the whole shortening of the plies*; hence $v = nu$ and we have from both methods of reasoning $\frac{W}{T_1} = \frac{v}{u} = n =$ the number of plies.

If the block end of the rope is made fast to the running block, the number of plies is odd. If it is fast to the fall-block, the number is even.

347. In Fig. (a) the ply 1 has no velocity; ply 2 has a velocity $2u$ up, and ply 3 has a velocity of $2u$ *down*. To find the velocity of h on the sheave which receives the plies 3 and 4 (Fig. 348), we may find the instantaneous axis, I, of the sheave and *construct* the velocity of h which is clearly $4u$, for the ply 4. Ply 5 has a velocity $4u$ *down*, and ply 6 a velocity $6u$ up which is v.

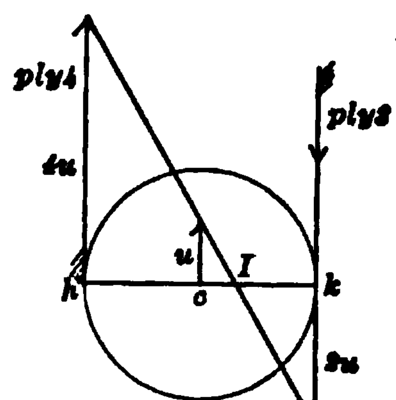


Fig. 348

Let the student *construct* the velocity of the 6th ply.

Another way of reasoning to find the velocity of a ply moving upward is to say that the velocity u must be the *mean* of the two velocities at h and k. Call x the velocity at h, and v_k the velocity at k. Then $u = \frac{x - v_k}{2}$, or $x = 2u + v_k$, which shows that the plies have velocities which form an arithmetical progression.

But a real block and tackle is far from ideal; the internal and external friction of the rope due to bending and straightening is a large factor, and when that is properly taken into account, as well as the friction of bearings, the ratio of useful work is only about 70 per cent of the Energy exerted. Hence if but $\frac{7}{10} T_1$ does useful work we must have, referring again to Fig. 347,

$$\frac{W}{\frac{7}{10} T_1} = n, \text{ or } T_1 = \frac{10W}{7n},$$

in which n is the number of plies.

The friction is distributed around to all the sheaves; consequently the tension is not uniform in the rope, so that it diminishes at every turn, being least at ply No. 1 when W is being raised, and greatest at No. 1 when W is being lowered.

The useless work done and the energy lost in bending and unbending a rope depends upon: The tension, the ratio of diameter of pulley to diameter of rope, and the method of twisting, braiding and binding the strands of the rope. If the ratio of diameters is too small, the internal rubbing is very destructive to the life of a rope. For long service in important positions that ratio for wire ropes should not be less than 40 to 1; that is, **a half-inch wire rope should have a twenty-inch pulley.**

A compound tackle has two or more running blocks and two or

more ropes. Fig. 349. The student will have no difficulty in explaining their operation.

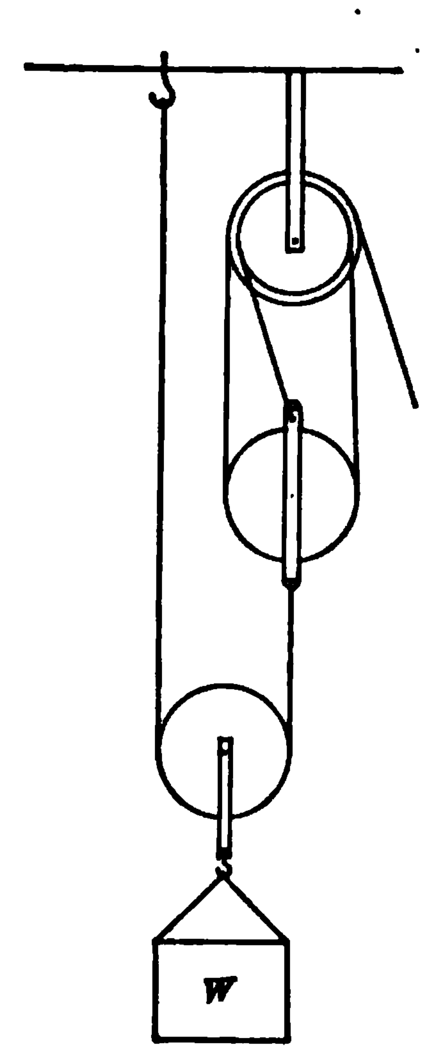


Fig. 349

348. The differential windlass is a tackle with a chain engaging sheaves made like sprocket wheels, the two in the fall-block being, of slightly different radii, and *fixed securely to the same shaft.* The two plies to the running block having practically the same tension exert a turning moment on the upper disk of $T(r_2-r_1)$; but the factor r_2-r_1 is made so small that moment of friction is sufficient to balance the turning moment; hence

$$T(r_2-r_1)=k$$

The free loop C, by means of its weight hanging on sheaves of unequal radii, has a slight tendency to prevent W from "running" down. However, a small tension on the ply L lowers the weight.

If W is to be raised, a tension on the ply C must first balance the moment $T(r_2-r_1)$ and then it must overcome the friction moment k. Hence W can be raised if the moment applied by hand be *greater* than $2T(r_2-r_1)$ or

$$T_1r_2>2T(r_2-r_1)$$

$$T_1>W\left(\frac{r_2-r_1}{r_2}\right)$$

If $r_2=4''$ and $r_1=3.75$

$$T_1>\frac{W}{16}$$

If $r_2=4''$ and $r_2=3''.5$

$$T_1>\frac{W}{8}$$

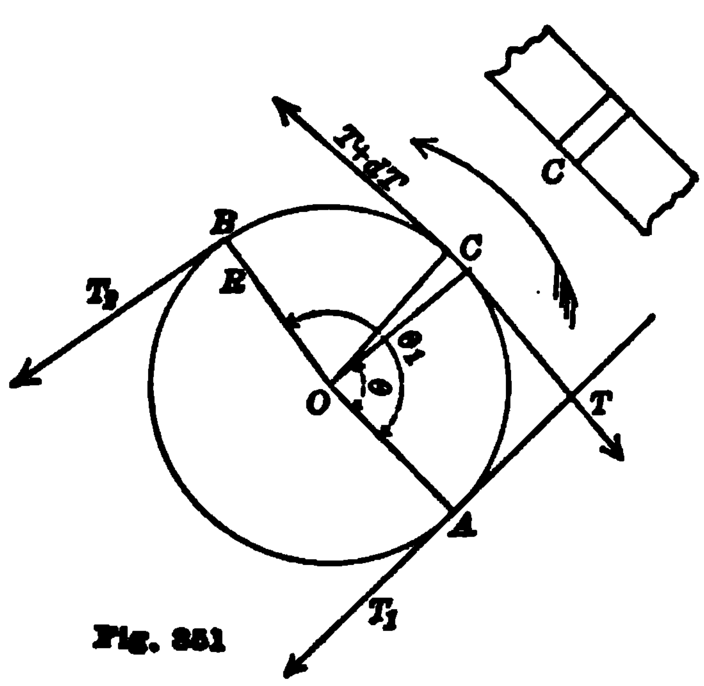


Fig. 351

The device is exceedingly useful for lifting and sustaining weights too heavy to be lifted by hand alone.

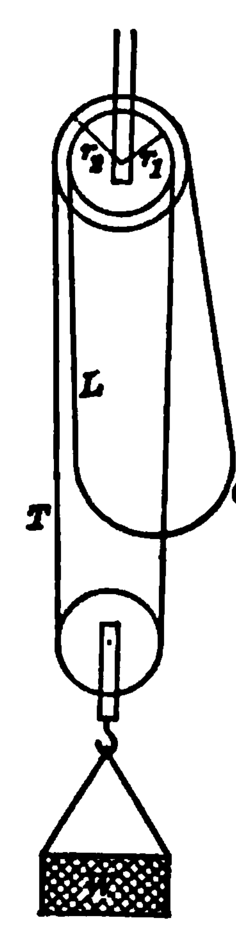


Fig. 350

349. The transmission of energy by belts. Suppose a pulley and its connections is being driven by a belt. Fig. 351. As the belt moves, the friction between it and the pulley causes the pulley to turn. The tension of the belt at A is T_1, at B it is T_2; the turning moment is $(T_2-T_1)R$.

The tension increases along the arc of contact, and the difference T_2-T_1 is made up by infinitesimal increments, due to the holding friction, or grip, between the belt and the face of the pulley.

At the point C, the tension is T, and on the element of the pulley face, $cRd\theta$, the *resultant* of the two adjacent T's acts; c is the width of the belt. These T's are perpendicular, respectively, to the radii at θ and $\theta+d\theta$, and their angle is $(\pi-d\theta)$; hence their resultant is $Td\theta$, *normal to the face of the wheel.*

If the student will graphically get the resultant of (T) and $(T+dT)$, the difference between whose lengths is invisible, he will readily see that the resultant is $Td\theta$, which is equally invisible. It is best to make both dT and $d\theta$ *just visible,* and then *reason* on the result.

Now the pressure $Td\theta$ upon the area $cRd\theta$ produces a certain frictional grip, which is (within a definite limit) proportional to the total pressure and independent of the area of action. Let the grip be $aTd\theta$, which must be just the infinitesimal difference between the T's on the two sides of the element; hence

$$aTd\theta=dT$$

$$ad\theta=\frac{dT}{T}$$

The co-efficient a is *not* the "co-efficient of friction"; it is less than f.

Integrating this equation from A, where $\theta=0$ and $T=T_1$, to B where $\theta=\theta_1$ and T_1 is T_2, we have

$$\left.\begin{aligned} a\int_0^{\theta_1} d\theta = \int_{T_1}^{T_2}\frac{dT}{T} = a\theta_1 = \log_e\frac{T_2}{T_1} \\ T_2 = T_1\epsilon^{a\theta_1} \end{aligned}\right\} \qquad (1)$$

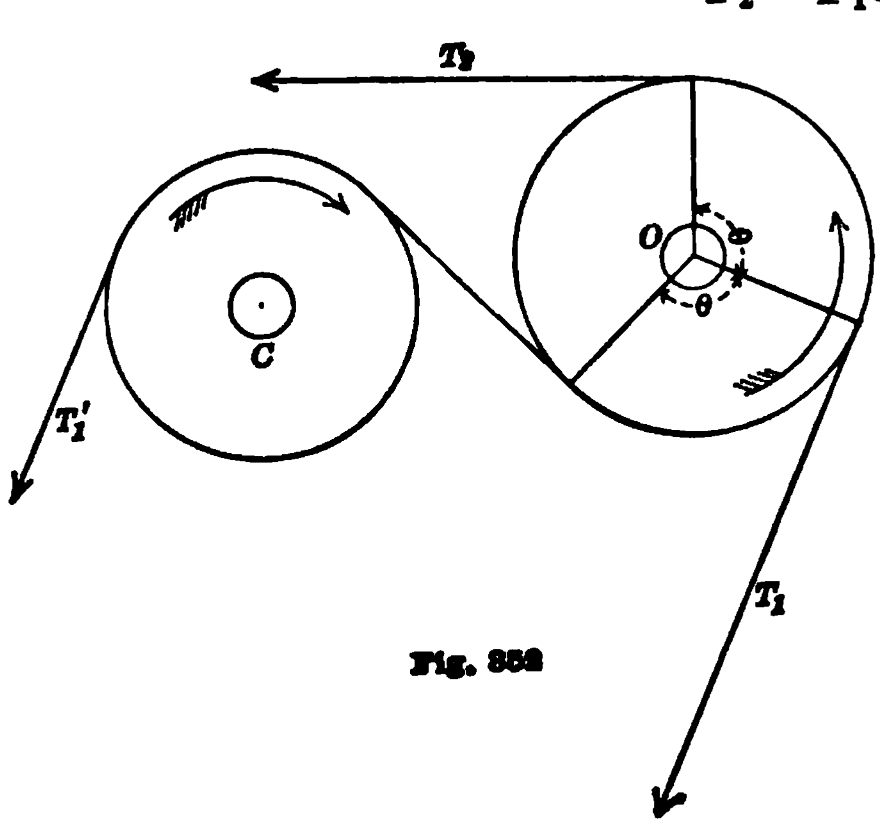


Fig. 352

Since $\epsilon=2{\cdot}7+$ it is plain that the ratio of $\frac{T_2}{T_1}$ is greatly increased by increasing θ_1. If θ_1 be doubled by a deflecting idle pulley C, as seen in Fig. 352, the ratio between T_2 and T_1 is squared in value.

It should be noted that the radius of the wheel has no effect upon the ratio $\frac{T_2}{T_1}$.

It is further evident that the factor a should be less than the coefficient of friction between belt and wheel. If it reaches f, the belt is in danger of slipping, and slipping causes heat, loss of energy and a complete change of conditions. The "tightener" does two things: increases the arc of contact, and tightens the belt.

The values of T_1 and T_2 may be calculated if the power to be transmitted, and the $R.P.M.=N$ (the Revolutions per Minute), of the pulley be known.

There is always a **certain amount of shipping** in a belt or rope engaging a "working" pulley. As the tension increases on the "follower", the belt slips *forward*; and as it diminishes on the driver, it slips *backward*. Hence the "follower" lags behind an amount depending upon the elasticity of the belt.

1. **Problems.** Given H-P, N, θ_1, and a, to find T_1 and T_2.

By formula already used (See **341**).

$$\text{H-P}\times 33000=(2\pi RN)(T_2-T_1) \qquad T_2-T_1=\frac{16500\ \text{H-P}}{\pi RN} \tag{2}$$

which combined with (1) gives

$$T_1=\frac{16500\ \text{H-P}}{\pi RN(\epsilon^{a\theta_1}-1)} \qquad T_2=\frac{16500\ \text{H-P}}{\pi RN(1-\epsilon^{-a\theta_1})} \tag{3}$$

2. Given $\text{H-P}=240$, $R=3'$, $N=120$, $a=\frac{1}{4}$, $\theta_1=2$ (radians). Find numerical values for T_1 and T_2.

350. The work done in turning a screw. The right-and-left screw described on p. 268, was turned by a wrench which had an effective lever length of 7 feet. Six men by means of a double-geared "crab," and a "block-and-tackle" of six plies, exerted a tension of approximately 20,000 pounds upon the end of the wrench. Hence the "Energy exerted" in one turn of the screw was

$$\text{Energy exerted}=2\pi.\ 7.12.\ 20,000 \text{ inch lbs.}$$

$$=10,560,000 \text{ inch lbs.} \left(\text{using } \pi=\frac{22}{7}\right).$$

Assuming that one-fourth of this Energy was spent in overcoming friction between threads, and calling the thrust produced between the segments of the adjustable tube (the two "nuts" operated upon by the screw) P, we have the equation of "Energy usefully exerted" and "Useful Work done"

$$\tfrac{3}{4}.\ 10,560,000=P.\tfrac{3}{4}$$

The pitch of each screw was $\frac{3}{8}$ of an inch; hence the *distance* thru

which *resistance was overcome* was $\frac{3}{4}$ of an inch each revolution. Accordingly the thrust was

$$P = 10{,}560{,}000 \text{ lbs. or } 5{,}280 \text{ tons.}$$

There were 24 adjustable tubes. Three wrenches were broken during the work of adjustment.

CHAPTER XIX.

Elasticity, Deformations and Deflection of Beams.

351. Elasticity is that property of a body which permits a certain amount of deformation or change of shape, as the effect of surface actions from other bodies, retaining, however, the power of actively recovering its original form gradually as the deforming actions are gradually diminished to zero. Like a rubber band, but much less in degree, a steel rod is stretched by tension, but it recovers its length when relieved, if the tension is not too great. All bodies are elastic to some extent, tho that extent is very small in such bodies as lead, tin, zinc, gold and loose granular or earthy matters. Pure gases are elastic without limit under ordinary conditions. Steel, platinum, iron, brass and other alloys are perfectly elastic between certain limits of stress. By "perfect" we mean that the deformation is proportional to the stress. If a rod is stretched a quarter of an inch by a certain stress, and is stretched a half inch by twice that stress, and a whole inch by four times that stress, and recovers itself every time when relieved, we say its elasticity is "perfect" for such stress.

352. The elastic limit. Every element in a uniform bar which is under tension shares in the deformation which consists of an increased length in the direction of the stress, and a slight lateral shrinkage; the shrinkage is, however, small compared with the longitudinal extension. When the pulling bodies cease to pull, the bar resumes all its original dimensions provided the stress was not too great.

When, however, the stress passes a certain magnitude, called the "Elastic Limit," the bar, when relieved of the tension, recovers itself only partly: it is then said to have taken a "permanent set." Nevertheless the deformeed bar may still be perfectly elastic within the original limit; in fact, in some cases the limit may have been slightly raised.

To repeat what has been already said, the amount of deformation is called the **strain** as distinguished from the **stress** which produced

it. The amount of strain produced by a certain stress varies greatly in different bodies.

Thus far in this book we have generally treated solids and liquids as incompressible, often, as rigid; but that condition was only *ideal*, for there are no rigid bodies; but thus far compressibility was too small to be worth consideration. Now, however, elasticity must be taken into account. We have calculated bending moments; we must now find how much the beam actually *bends*, and how much it is *deflected*.

We can find deflections only when we know the *stiffness* or degree of rigidity of the material. This can be found only by careful experiments, such as are found in every engineering laboratory. The practical details of experiments are left to "Laboratory Practice"; a typical case will suffice here.

353. The modulus of elasticity. Suppose a uniform bar between large ends prepared for clutches is carefully measured for length and diameter, and subjected to a known moderate tension. The added length which

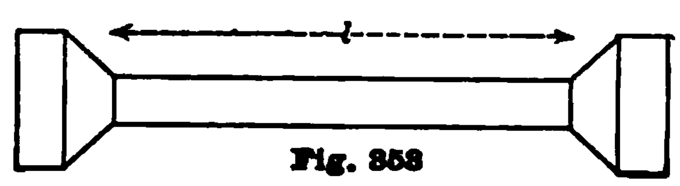


Fig. 353

we have called λ, is measured with the greatest care (see p. 35), and both the *Modulus* of *Elasticity* and the *Elastic Limit*, as fully defined and explained in (**138**), are determined for the specimen under consideration.

It must be now remembered that in this Chapter E means, *not* energy, but the *Modulus* of *Elasticity*, and that it means *stress per unit of surface*, that is, lbs. per square inch, for example. Its formula value is

$$E = \frac{\textit{unit stress}}{\textit{unit strain}} = \frac{Tl}{A\lambda} = \frac{pl}{\lambda}$$

so that

$$\frac{p}{E} = \frac{\lambda}{l}$$

which can be read, *in words*, either forward or backward, with a clear recognition of the meaning.

In the APPENDIX a table is given containing the *Ultimate Strength*, the *Elastic Limit*, and the *Modulus* of *Elasticity* under *Tension* of *woods* and *metals*, according to the most trustworthy authorities.

A smaller table is given for materials much used for Direct-Thrust or compression.

A third table gives similar information for building materials and metals under shearing stress, concerning which see Chapter (XXI).

354. Flexure. Radius of Curvature. The effect of a Bending Moment is called *Flexure.* The methods of finding M at any point have already been given; they had nothing to do with internal stress or with the resulting flexure. The deformations were assumed to be too small to effect the values of M appreciably.

The flexure at a point is determined by its radius of curvature. Fig. 354 represents a portion of a bent beam which was originally straight. The lamina between the cross-sections AC and BD was of uniform thickness before the load was applied. Under the load the fiber at the top has been lengthened; at the bottom it has been shortened. It is assumed (and the assumption is correct for ordinary safe bendings) that the cross-sections are still plane surfaces, or normal planes, which intersect in a line at O, distant from the neutra laxis of the sections by the *Radius* of *Curvature,* $\rho = ON$. Now the triangle ONN'

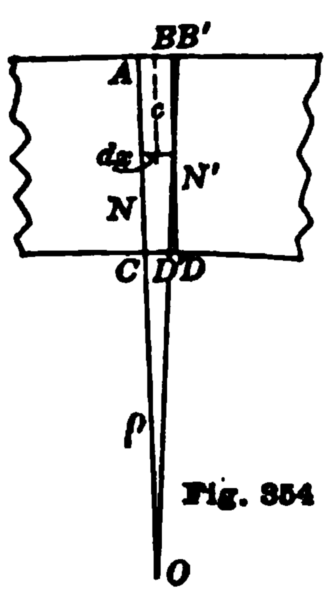


Fig. 354

and $BN'B'$, are similar, hence $\dfrac{BB'}{dx} = \dfrac{c}{\rho}$ in which c is the distance from the neutral axis of the beam to the extreme fiber. The short arc BB' is the elongation of a fiber whose length was dx; hence $\dfrac{BB'}{dx} = \dfrac{p}{E}$ in which p is the intensity of stress at B, and E is the modulus of elasticity of the material of the beam. Let, as in **142**, the letter a represent the intensity of the normal stress at a unit's distance from the neutral axis.

Hence as $$p = ac$$

$$\frac{c}{\rho} = \frac{p}{E}, \text{ or } \rho = \frac{cE}{p} = \frac{E}{a}$$

But by **145** $$a = \frac{M}{I},$$

so that $$\rho = \frac{EI}{M}$$

This very important equation shows the relation between four quantities:—The *rigidity* of the *material* which is represented by E; the *cross-section* of the beam, shown by I; the way it is *loaded* and *supported,* shown by M; and the *flexure produced,* shown by ρ.

The formula of ρ, as found in the Calculus, is:

$$\rho = \frac{(dx^2 + dz^2)^{\frac{3}{2}}}{dxd^2z}$$

Let the beam be a cantilever, Fig. 355 (load not shown). The neutral plane of the beam was originally horizontal and parallel to OX. OZ is vertical, positive downward, so that ρ is positive (that is, d^2z is positive with dx positive). [If a bent beam has a point of inflexion the sign of $\frac{d^2z}{dx}$ changes sign. This must be kept in mind.]

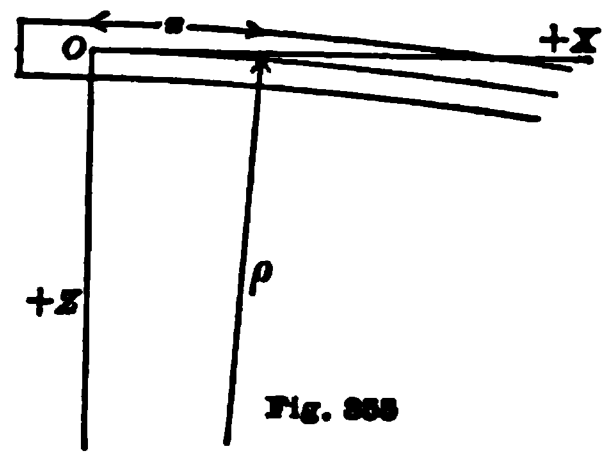


Fig. 355

Dividing both terms of the fraction, giving the value of ρ, by $(dx)^3$ we have

$$\rho=\frac{\left(1+\left(\frac{dz}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2z}{dx^2}}.$$

But the beam, tho bent, is so nearly parallel to OX that *the square of* $\frac{dz}{dx}$ *may be neglected* when compared with unity; hence the closely approximate value of ρ in common use is

$$\rho=\frac{1}{\frac{d^2z}{dx^2}}, \text{ or } \frac{d^2z}{dx^2}=\frac{1}{\rho}=\frac{M}{EI};$$

hence the useful equation $$EI\frac{d^2z}{dx^2}=M.$$

which, **if the very slight distortion due to shear be neglected,** is the differential equation of the curve on an elastic beam when bent. The co-ordinates x and z locate a point on the bent neutral line of the beam; and if the axis OX is tangent to the curve at O, z measures the *deflection from it.*

355. The function of a "Fixed End." It must be made clear that the fixed end of a cantilever beam furnishes both a vertical support and a bending moment.

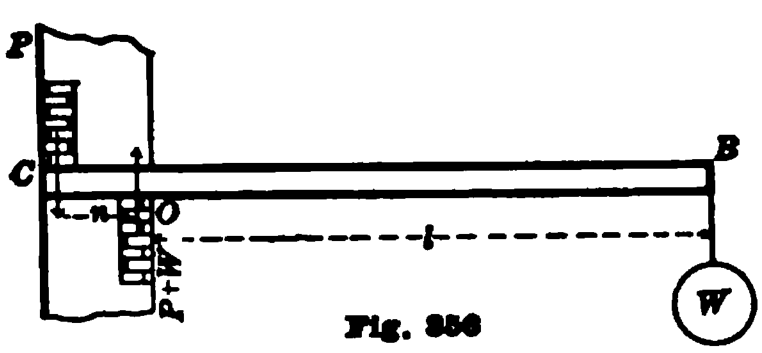


Fig. 356

Fig. 356 represents a cantilever beam with one end fixed in a wall. It may be supposed to rest *on* a cross-beam at O, and to rest *under* a second cross-beam at C, in such a way that a tangent to the axis of the beam at O is horizontal. In place of the two cross-beams, a broad

block of masonry above and another below may serve to "fix" the end of the beam in the same way.

The upper cross-beam or block acts *down* with a force P, and the lower beam or block acts *up* with a force $P+W$. The two forces P and P form a left-handed couple; and the two W's (one on the end of the beam) form a right-handed couple, and as the couples balance, the moments are numerically equal, *i. e.* $Wl=Pn=M_0$.

If we are thinking of the action of the part OB upon the part OC, or upon the wall of masonry, we see that it is pulling *down* W, and applying a turning moment Wl, *right-handed*. If we are thinking of the action of the wall or part OC upon the part OB, we see that it is acting *up*, $V=W$, and applying a moment $Pn=M_0$, which is *left-handed*. The support V and the moment M_0, *together*, balance the force W acting at the end of the beam.

The *wall's action* on the section of the beam at O, consists of a shearing stress equal to W, and a uniformly varying normal stress whose resultant action is equivalent to a left-handed couple whose movement exactly balances the couple lW. Hence $M_0=lW$. Fig. 357.

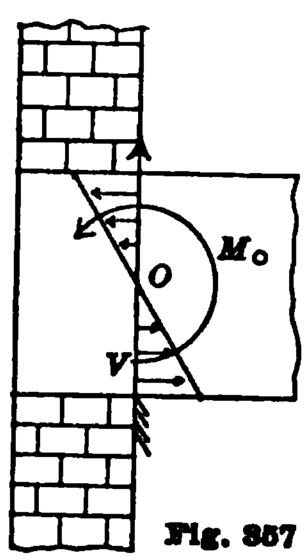


Fig. 357

For methods of "fixing" the ends of steel beams and columns, the reader is referred to books issued by makers of structural steel.

356. In all the problems which now follow, E and I are constant. The reader may have in mind any constant cross-section; a rectangle, a circle, a tube, an I-beam, or a built beam or post like one of those shown in **156.** The entire side view of a beam, bent or straight, will generally be represented by a heavy line.

The reader must not fail to remember that every straight beam bends when loaded, but as a rule the bending is so small that it is detected only by careful measurement. In the drawings which follow, the distortions are *greatly exaggerated* in order that they may be visible. It does not follow, however, that, because deflections and slopes are small, they are of no account. Precise levels are often necessary, and some materials of construction crumble or crack at very small flexures produced by temporary loads. For the most part the loads here considered will be loads which come and go; *they do not usually include the weight* of the beam itself. The effects produced are to be added to those produced by the beam's own weight, when these last effects are worth consideration.

357. The *stiffness* of a beam is often of more importance than its *strength*.

"In some instances, *deflection* rather than *absolute strength* may become the governing consideration in determining the proportions of the beam to be used. For beams carrying plastered ceilings, for example, it has been found by practical test that, if the deflection exceeds $\frac{1}{360}$ of the distance between supports, there is danger of the ceiling cracking."

We are now ready to find

MOMENTS, SLOPES, DEFLECTIONS, AND SAFE LOADS.

for loaded beams which are straight and horizontal when unloaded, or for posts which are vertical when not acted upon by horizontal forces.

358. CASE I. **Cantilever beams with load at the end.** In accord with the notation shown in Fig. 358, we find an expression for the bending moment at P due to the load W at the end, distant $l-x$ from the neutral axis at P. Hence we have from **354**

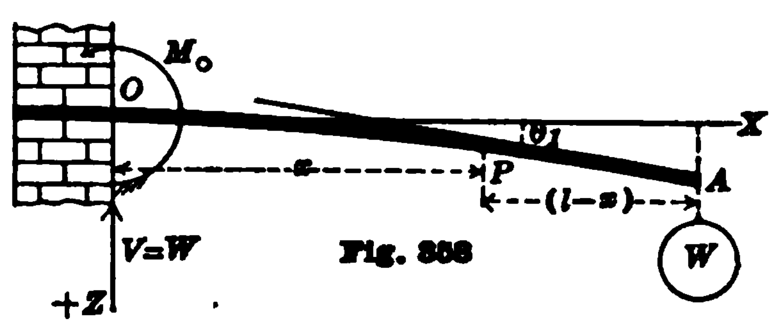


Fig. 358

$$(1)\qquad EI\frac{d^2z}{dx^2}=M=W(l-x)$$

Multiplying by dx, and integrating, we have

$$(2)\qquad EI\frac{dz}{dx}=W\left(lx-\frac{x^2}{2}\right)+(H=0)$$

As the beam is horizontal at O, when $x=0$, $\frac{dz}{dx}$ is also equal to zero, and hence $H=0$. Integrating again

$$(3)\qquad EIz=W\left(\frac{lx^2}{2}-\frac{x^3}{6}\right)+(K=0)$$

The quantity z measures the deflection of the bent beam at the point x; when $x=0$ there is no deflection, hence K must be zero.

Equation (1) shows that M will be maximum when $x=0$, hence we have

$$\text{Max. }(M)=Wl \qquad \text{(I)}$$

The **slope equation** (2) shows that the slope is the greatest when $x=l$, hence

$$\text{Max. slope}\left(\frac{dz}{dx}\right)=\frac{Wl^2}{2EI}=\tan\theta_1 \qquad \text{(II)}$$

The equation (3) is the algebraic equation of the curve of which the bent beam is a part; z is the deflection of P; evidently the greatest deflection is at the end when $x=l$. Hence

$$\text{Max. } (z) = \Delta = \frac{Wl^3}{3EI} \tag{III}$$

If for Max. (M) we put the working *Max.* Moment of Resistance $\frac{fI}{c} = Wl$, we have

$$\text{Max. working Load } (W) = \frac{fI}{cl} \tag{IV}$$

The letter f denotes the *working strength* of the material, *not* its breaking strength, nor even the stress at the elastic limit. If the elastic limit is 40,000 lbs. per sq. inch, f may be 10,000 lbs. per sq. inch.

The number c measures the distance from the neutral axis of the cross-section to the extreme fiber, either above or below.

N. B.—In the integrations which follow, we shall introduce "Constants of Integration" instead of integrating between limits; and when it is readily seen that for a simple value of x the constant is zero, the fact will be indicated thus: $+(H=0)$, or $+(C=0)$, without explanation.

359. Case II. **Cantilever beams with uniformly distributed load** of intensity w lbs. per linear foot. With the notation of Fig. 359, we find the moment about an axis at P, of the distributed load $w(l-x)$ between P and A. Its moment is seen to be: Its weight $w(l-x)$ times its mean arm, $\frac{l-x}{2}$; hence we have

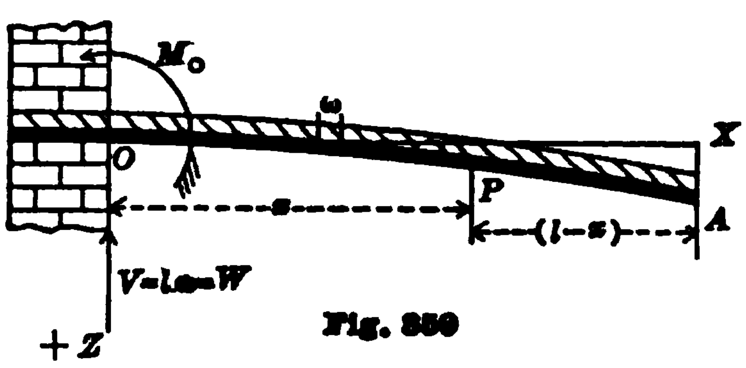


Fig. 359

$$\text{(1)} \qquad EI\frac{d^2z}{dx^2} = M = \frac{w}{2}\cdot(l-x)^2$$

Integrating

$$EI\frac{dz}{dx} = -\frac{w}{6}(l-x)^3 + H$$

When $x=0$, the slope is zero, hence $H = \frac{wl^3}{6}$, and

$$\text{(2)} \qquad EI\frac{dz}{dx} = \frac{w}{6}\left(l^3-(l-x)^3\right) = \frac{w}{6}(3l^2x - 3lx^2 + x^3)$$

[Had we developed the binomial in (1) before integrating, the constant to be added would have been zero, but the result would have been the same. Sometimes we integrate a binomial as such, and sometimes we develop it.]

Integrating (2) we get

$$(3)\qquad EIz = \frac{w}{6}\left(\frac{3l^2x^2}{2} - lx^3 + \frac{x^4}{4}\right) + (K = 0.)$$

We note that all the equations of Moment, Slope, and Deflection are one degree higher than in Case I. As before we evidently have for $x = 0$.

$$\text{Max. }(M) = \frac{wl^2}{2} = \frac{Wl}{2} \qquad \text{(I)}$$

in which we write $wl = W$.

Both slope and deflection are Maximums when $x = l$,

$$\text{Max. }\left(\frac{dz}{dx}\right) = \frac{wl^3}{6} = \frac{Wl^2}{6EI} \qquad \text{(II)}$$

$$\text{Max. } z = \Delta = \frac{wl^4}{8} = \frac{Wl^3}{8EI} \qquad \text{(III)}$$

$$\text{Max. Load (distributed)} = W = \frac{2fI}{cl} \qquad \text{(IV)}$$

360. Case III. **A simple beam carries a load in the center.** Fig. 360. This is like Case I inverted. The halves "fix" each other horizontally at O, and the supports act *up* at the ends. However, the student should solve this case independently and then get the same results by inserting $\frac{l}{2}$ for l, and $\frac{W}{2}$ for W in formulas I-IV, Case I.

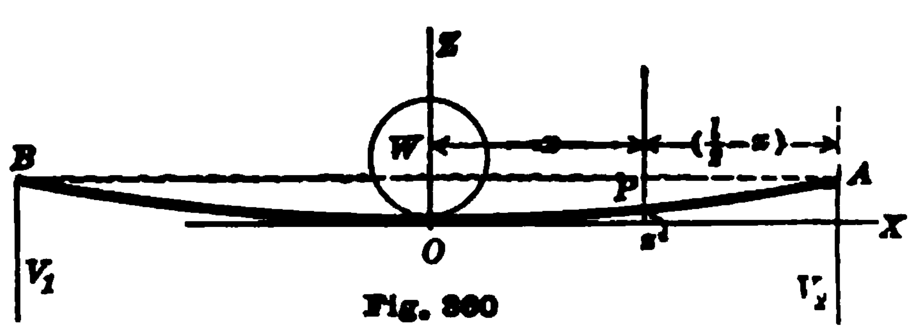


Fig. 360

$$\text{Max. }(M.) = \frac{Wl}{4} \qquad \text{(I)}$$

$$\text{Max. }\frac{dz}{dx} = \frac{Wl^2}{16EI} \qquad \text{(II)}$$

$$\text{Max. }(z) = \Delta = \frac{Wl^3}{48EI} \qquad \text{(III)}$$

$$\text{Max. load} = \frac{4fI}{cl} \qquad \text{(IV)}$$

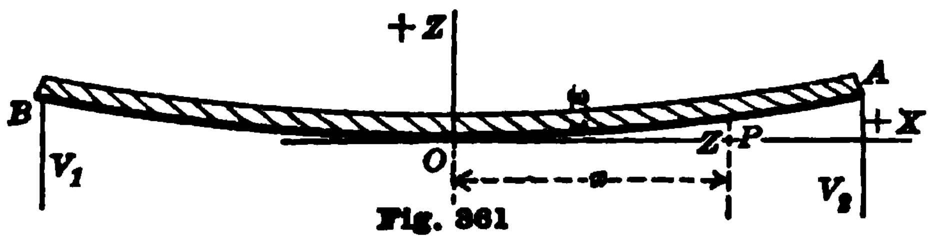


Fig. 361

361. CASE IV. **A simple beam carries a uniform load.** Fig. 361. If $W = lw$, the equations are readily written.

(1) $$M_x = V_2\left(\frac{l}{2} - x\right) - \frac{w}{2}\left(\frac{l}{2} - x\right)^2$$

(2) $$EI\frac{dz}{dx} = -\frac{V_2}{2}\left(\frac{l}{2} - x\right)^2 + \frac{w}{6}\left(\frac{l}{2} - x\right)^3 + H$$

(3) $$EIz = \frac{V_2}{6}\left(\frac{l}{2} - x\right)^3 - \frac{w}{24}\left(\frac{l}{2} - x\right)^4 + Hx + K$$

Making $x = 0$ in (2), we have

(4) $$H = \frac{V_2 l^2}{8} - \frac{Wl^2}{48} = \frac{Wl^2}{24}$$

Similarly using (3), making $x = 0$,

(5) $$K = -\frac{V_2 l^3}{48} + \frac{Wl^3}{384} = -\frac{Wl^3}{128}$$

The greatest moment is evidently at the center where the slope is zero.

$$\text{Max. } (M) = \frac{Wl}{8} \qquad \text{(I)}$$

The greatest slope is at the support when $x = \frac{l}{2}$.

$$\text{Max. } \left(\frac{dz}{dx}\right) = \frac{H}{EI} = \frac{Wl^2}{24EI} \qquad \text{(II)}$$

The ordinate z does not directly measure the deflection of the beam. The *value* of the deflection at the center is found by making $x = \frac{l}{2}$.

$$\text{Max. Def.} = \Delta = +\frac{H\frac{l}{2} + K}{EI} = \frac{5Wl^3}{384EI} \qquad \text{(III)}$$

The maximum allowable distributed load is found by making the Max. $(M) = \frac{Wl}{8}$ equal to the Moment of Resistance $\frac{fI}{c}$, hence

$$\text{Max. } (W) = \frac{8fI}{cl} \qquad \text{(IV)}$$

362. Case V. **A beam is fixed at both ends and carries a uniform load.** Fig. 362. The curve of the bent beam is evidently symmetrical with respect to a vertical thru its center, and the vertical supports are equal. Since the ends are "fixed" at the ends, as was shown in **355**, it is evident that there must be a moment M_o at each end as shown.

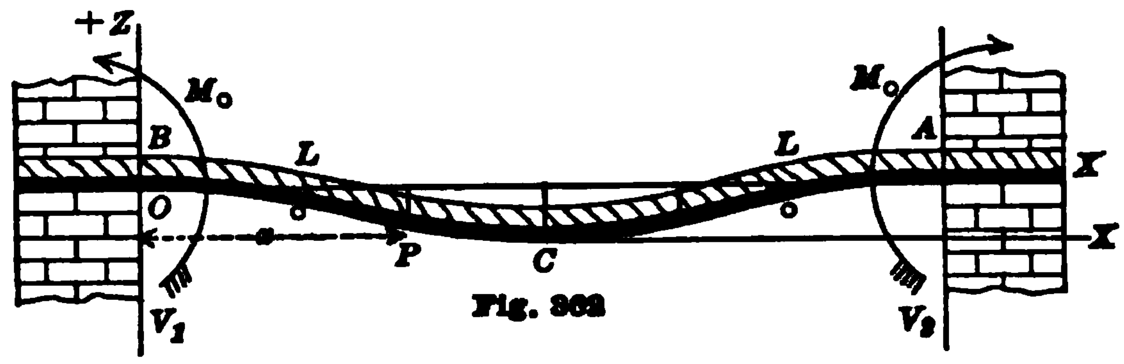


Fig. 362

$$V_1 = V_2 = \frac{wl}{2}.$$

The moments M_o have equal magnitudes.

The moment at P is, considering forces acting on the segment OP,

(1) $$M_x = V_1 x - \frac{wx^2}{2} - M_o$$

(2) $$EI\frac{dz}{dx} = V_1\frac{x^2}{2} - \frac{wx^3}{6} - M_o x + (H=0)$$

(3) $$EIz = \frac{V_1 x^3}{6} - \frac{wx^4}{24} - \frac{M_o x^2}{2} + (K=0)$$

There are three ways of finding the value of M_o, viz.:

Let $x = l$ in (2)

Let $x = \frac{l}{2}$ in (2)

Let $x = l$ in (3)

In every case we get

(4) $$M_o = \frac{wl^2}{12} \qquad \text{(I)}$$

The moment at the center is from (1) when $x = \frac{l}{2}$

(5) $$M_c = \frac{wl^2}{4} - \frac{wl^2}{8} - \frac{wl^2}{12} = \frac{wl^2}{24}.$$

The greatest deflection is evidently at the center

$$EI\Delta = \frac{wl^4}{96} - \frac{wl^4}{384} - \frac{wl^4}{96}$$

$$(6) \qquad \Delta = -\frac{wl^4}{384EI} = -\frac{Wl^3}{384EI} \qquad \text{(II)}$$

If the values of V_1 and M_0 are substituted in (1) and (2), we have the moment and slope curves:

$$(7) \qquad M = \frac{w}{2}\left(lx - x^2 - \frac{l^2}{6}\right)$$

$$(8) \qquad \frac{dz}{dx} = \frac{w}{2EI}\left(\frac{lx^2}{2} - \frac{x^3}{3} - \frac{l^2x}{6}\right)$$

From (7) we see that $M = 0$ when $x = 0.21l$, and $x = 0.79l$. This shows that the beam is really a cantilever. The cantilever arms BL and AL being each $\frac{21}{100}l$, and the suspended portion LL' being $\frac{58}{100}l$.

If we make $\frac{dM}{dx} = 0$, we find $x = \frac{l}{2}$, and $\frac{wl^2}{24}$ for a *maximum* value of M; and yet we see that $M_0 = \frac{wl^2}{12}$, is twice as large. The explanation of the fact that the *maximum* moment is not the *greatest* will be found in the moment curve **364.**

If, in order to get the points of maximum slope, we place the *differential of* $\frac{dz}{dx}$ equal to zero, we shall get the equation we just used for finding where $M = 0$. We actually find Max. $\frac{dz}{dx}$, by substituting in (2) the value $x = 0.21l$, or $0.79l$. We thus find the slopes.

$$\text{Max.}\ \frac{dz}{dx} = \tan\theta = \pm - \frac{Wl^2}{125EI} \qquad \text{(III)}$$

We may remark in passing that if cantilever beams were really made with terminal pins at L and L, and a simple beam were hung thereon with a uniform load thruout, E and I being constant, the three parts would assume the same continuous curve, and have the same moments as does the continuous beam.

The maximum working load is found as before from the equation

$$M_0 = \frac{Wl}{12} = \frac{fI}{c}$$

$$\text{Max.}\ W = \frac{12fI}{cl} \qquad \text{(IV)}$$

363. The economy of an inexpensive "fixing" of the ends of a beam is seen if the load this beam will carry is compared with the same

beam with the ends merely supported as in the last case. The practical ability to carry a distributed load has been increased 50 per cent by "fixing" the ends.

In *stiffness*, the superiority of the beam with fixed ends is still more remarkable. Comparing the deflections, we see that it is *five times as stiff*.

The reader will see in the above simple illustration, the suggestions which led to the Cantilever Bridge. By the use of two cantilevers and a suspended span, the gain may be, not in the ability to carry an increased load, but in the ability to carry the same load over a longer span between piers.

When instead of constant cross-sections, the sections are so proportioned that the extreme fibre stress, p_1 is constant, the deflection of a double cantilever beam is a little more complicated. See Chap. XXII on "Beams of Uniform Strength."

364. The graphical analysis of a bent beam. It may now be stated, once for all, that inasmuch as every slope equation can be obtained from the deflection equation (which is the equation of the elastic curve) by differentiating it; and as the moment equation can be obtained from the slope equation by differentiation; and the shear equation in like manner from the moment equation, we have

Deflection is a maximum when *Slope* is zero.

Slope is a maximum when *Moment* is zero.

Moment is a maximum when *Shear* is zero.

It thus appears that a complete analytical diagram of a loaded beam should contain five curves:

No. 1. The elastic curve of the beam itself whose ordinates give the Deflection.

No. 2. The curve whose ordinates give the Slope, $\dfrac{dz}{dx}$.

No. 3. The curve whose ordinates give the Moment, $EI\dfrac{d^2z}{dx^2}$.

No. 4. The curve whose ordinates give the Shear, $\dfrac{dM}{dx}$.

No. 5. The straight line whose ordinate gives the distribution of the load, w.

These curves are drawn for the last problem in Fig. 363. The dotted line shows the unloaded beam, the axis of X.

No. 0, shows the unloaded beam and its supports.

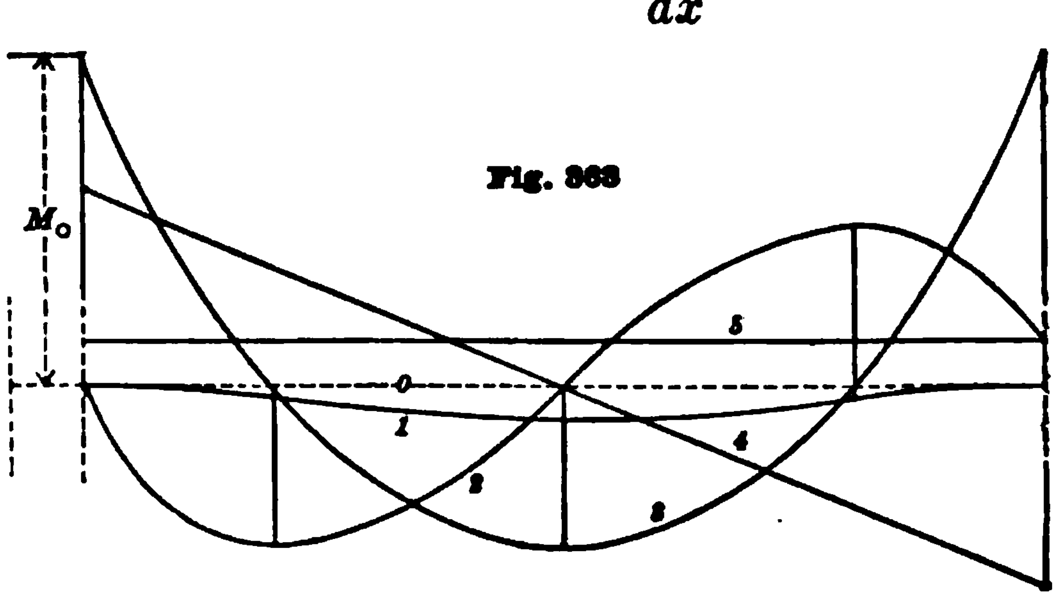


Fig. 363

No. 1, shows the bent beam, a curve of the 4th degree.
No. 2, shows the slopes in No. 1, a cubic equation.
No. 3, shows the moments in No. 1, a parabola.
No. 4, shows the shear in No. 1, an oblique straight line.
No. 5, shows the uniform load on No. 1, parallel to OX.

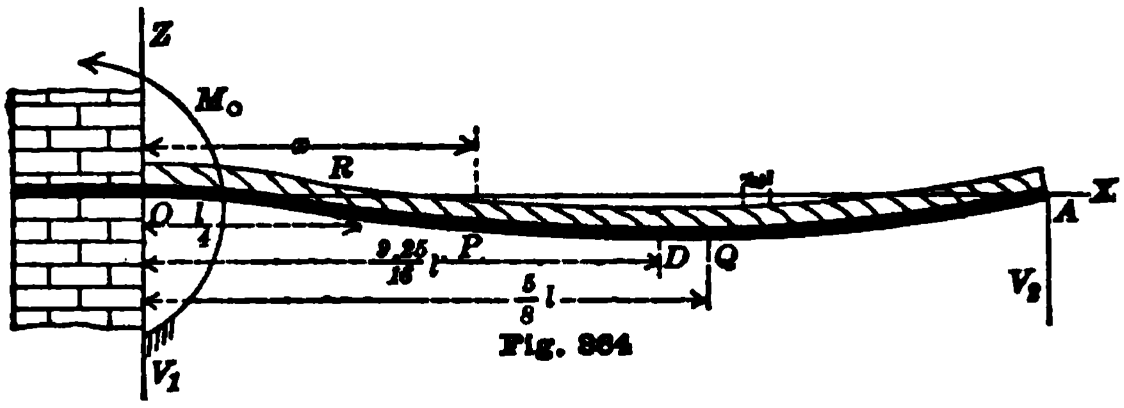


Fig. 364

365. CASE VI. **A beam has one end fixed and carries a uniform load.**

With the notation of Fig. 364, we get at once

(1) $$M = V_1 x - \frac{wx^2}{2} - M_o$$

(2) $$EI\frac{dz}{dx} = \frac{V_1 x^2}{2} - \frac{wx^3}{6} - M_o x + (H=0)$$

(3) $$EIz = \frac{V_1 x^3}{6} - \frac{wx^4}{24} - \frac{M_o x^2}{2} + (K=0)$$

From (1) when $x=l$ we know that $M=0$; hence

(4) $$M_o = V_1 l - \frac{wl^2}{2}$$

From (3) we have when $x=l$, $z=0$; hence

$$0 = \frac{V_1 l}{6} - \frac{wl^2}{24} - \frac{M_o}{2}$$

(5) $$M_o = \frac{V_1 l}{3} - \frac{wl^2}{12}$$

From (4) and (5) we get

(6) $$V_1 = \frac{5}{8}wl = \frac{5}{8}W$$

(7) $$M_o = +\frac{Wl}{8}$$

This gives the numerical value, and the positive sign shows that *we were correct* when we made it left-handed. Differentiating (1) to get the equation of shear for the purpose of finding where M is a maximum we get

$$\frac{dM}{dx} = 0 = V_1 - wx,$$

(8) hence $$x = \frac{V_1}{w} = \frac{5}{8}l,$$

This value in (1) gives

(9) $$M = \frac{5}{8}V_1 l - \frac{w}{2}\cdot\frac{25l^2}{64} - \frac{Wl}{8} = \frac{9}{128}Wl.$$

This is the maximum for *that part of the beam,* but is *much less* than M_O, the moment at O.

To find the length of the cantilever end, we make $M=0$, and solve (1) for x. We get

$$x = l, \text{ and } \frac{l}{4}\cdot \quad \text{Hence}$$

(10) $$\text{Cantilever } OR = \frac{l}{4}\cdot$$

The slopes at R and A are readily found to be

$$\left(\frac{dz}{dx}\right)_R = -\frac{11Wl^2}{768EI}\cdot$$

$$\left(\frac{dz}{dx}\right)_A = +\frac{Wl^2}{48EI}$$

Making $\frac{dz}{dx} = 0$ in (2) we find for the point of greatest deflection $x = \frac{9.25l}{16}$, with which z becomes in (3)

$$(z) = -\frac{Wl^3}{185EI}$$

We can now write our several maximum values for a uniform beam, with one fixed end and an evenly distributed load:

$$\text{Max. } M. = \frac{Wl}{8} = M_0 \qquad \text{(I)}$$

$$\text{Max. Slope} = \frac{Wl^2}{48EI} \qquad \text{(II)}$$

$$\text{Max. Deflection } \Delta = \frac{Wl^3}{185EI} \qquad \text{(III)}$$

$$\text{Max. allowed Load} = \frac{8fI}{cl}\cdot \qquad \text{(IV)}$$

It is interesting to note that the maximum moment at the fixed end is the same as it *would have been at the center,* had the end not been fixed.

Comparing these results with those of the beam in Case 4, we see that while only *just as strong*, it is *two-and-a-half times as stiff, i. e.*, its deflection is only about $\frac{2}{5}$ as great.

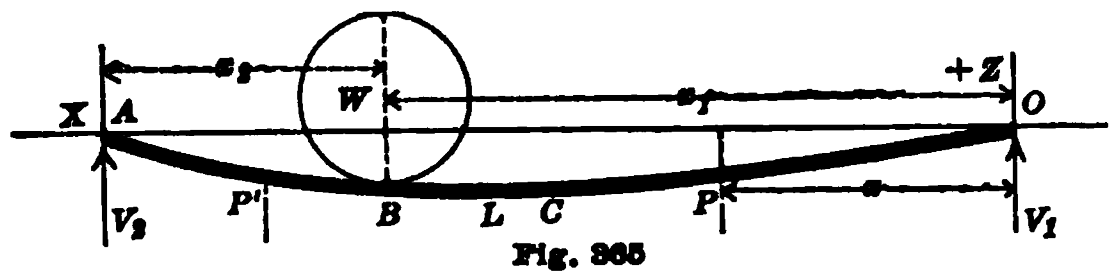


Fig. 365

366. Case VII. A simple beam carries a single eccentric load. Fig. 365.

It will be convenient to take axes as shown and *left-handed moments positive.**

There are two elastic curves, one containing OB, and another containing BA; hence we shall have *two sets of equations.*

Eq's for BA.	*Eq's for OB.*
(4) $M_{P'} = V_1x - W(x - x_1)$	(1) $M_P = V_1x$
(5) $EI\frac{dz}{dx} = \frac{V_1x^2}{2} - W\left(\frac{x^2}{2} - x_1x\right) + C$	(2) $EI\frac{dz}{dx} = V_1\frac{x^2}{2} + H$
(6) $EIz = \frac{V_1x^3}{6} - W\left(\frac{x^3}{6} - \frac{x_1x^2}{2}\right) + Cx + D$	(3) $EIz = V_1\frac{x^3}{6} + Hx + (K=0)$
In these equations, x must not *be less than* x_1.	In these equations, x must not *exceed* x_1. V_1 and V_2 are known.

We have four conditions for finding the values of the four constants of integration:

1st. The deflection, *i. e.*, z is zero at G, hence from (3) $K=0$.

2nd. The deflection at A is zero, hence (6) gives, when $x=l$,

$$0 = \frac{V_1l^3}{6} - W\left(\frac{l^3}{6} - \frac{x_1l^2}{2}\right) + Cl + D \tag{7}$$

3rd. The slope of the two curves at B are the same; hence if $x = x_1$ in (2) and (5) we get

$$\frac{V_1x_1^2}{2} + W\frac{x_1^2}{2} + C = \frac{V_1x_1^2}{2} + H.$$

$$W\frac{x_1^2}{2} + C = H \tag{8}$$

* One should always be free to take axes and moments as the situation requires, but when they have once been chosen, no change should be made in *the same problem.* For example one must *not* get the equations for the segment OB with O as the origin, and then take A as the origin for the equations for the segment AB.

4th. The deflections of the two curves at B are equal. Hence if $x=x_1$ in (3) and (6) we have $W\frac{x_1^3}{3}+Cx_1+D=Hx_1$. (9)

From (8) and (9) we get $D=\frac{Wx_1^3}{6}$. (10)

Putting the value of D in (7) we get $C=-\frac{W}{6l}(x_1^3+2x_1l^2)$; (11)

and finally from (8) we get $H=\frac{Wx_1}{6l}(3x_1l-x_1^2-2l^2)$;

or $$H=-\frac{Wx_1}{6l}(2l-x_1)(l-x_1)=-\frac{Wx_2}{6l}(l^2-x_2^2) \quad (12)$$

if we put $l-x_1=x_2$.

Having found the values of the constants, we will now find the four important matters which every engineer and architect is at times required to know, either about an existing beam, or in designing a new beam.

The *greatest moment* is always at the point where the load is, since the shear changes sign at that point; hence letting $x=x_1$ in (1) we have, since $V_1=\frac{Wx_2}{l}$, $$\text{Max. }(M)=\frac{x_1x_2}{l}W \quad (I)$$

If $x_1=x_2=\frac{l}{2}$, the moment becomes $\frac{Wl}{4}$ as in Case 3.

The *greatest slopes* are found at the ends where $M=0$. At $O, x=0$, and the slope is, from (2),

$$\left(\frac{dz}{dx}\right)_{x=0}=\frac{H}{EI}=-\frac{Wx_2}{6lEI}(l^2-x_2^2) \quad (II)$$

At A, $x=l$, and the slope is, from (5),

$$\left(\frac{dz}{dx}\right)_{x=l}=\frac{Wx_1}{6lEI}(l^2-x_1^2) \quad (II')$$

These slopes are different unless $x_1=\frac{l}{2}=x_2$, when they are

$$\left(\frac{dz}{dx}\right)_{x=\frac{l}{2}}=\pm\frac{Wl^2}{16EI}.$$

The *deflection* is *greatest* at the point where the tangent to the elastic curve is horizontal, and consequently $\frac{dz}{dx}=0$. To find that we point we put $\frac{dz}{dx}=0$, in (2) and solve for x. It thus appears that

$$x^2=-\frac{2H}{V_1}=\frac{1}{3}x_1(l+x_2)$$

whence
$$x=\pm\frac{1}{3}\sqrt{3x_1(l+x_2)}$$

The positive value gives the point required; the negative value gives a maximum point on the algebraic curve, *not* on the beam.

If we substitute $\frac{1}{3}\sqrt{3x_1(l+x_2)}$ for x in (3) and solve for z, the maximum deflection is found for the load W placed x_1 distant from the right hand support.

367. Case VIII. **Moving or live loads.** Beams carrying a concentrated load are generally designed on the assumption that the load may be moved from point to point. It is therefore necessary to find *where the load must be* to produce the greatest possible stress or deformation for that load. In the last problem, the greatest value of M for an eccentric position of W was $\frac{Wx_1x_2}{l}$, and this is a real maximum only when $x_1=x_2=\frac{l}{2}$; hence the greatest possible moment caused by a concentrated load moving across a simple supported beam is at the point where the load is when it is at the center:

$$\text{Max. } M=\frac{Wl}{4}. \tag{I}$$

and
$$\text{Max. } W=\frac{4fI}{cl} \tag{II}$$

But the position which makes M a maximum is evidently not the position for the greatest slopes at the ends. The questions to be now answered are:

Where must W be on the beam, to make the *slope* at O a maximum? and what will that maximum be?

The slope at O is $\frac{H}{EI}$, which will be a maximum when $\frac{dH}{dx_1}=0$.

Hence we have, by differentiating (11) in **365**, the equation for finding x_1,

$$6x_1l-3x_1^2-2l^2=0$$

$$x_1=\frac{3-\sqrt{3}}{3}\cdot l=0.423l.$$

Thus we get the greatest possible slope at O by placing W at a point $x_1=0.423l$ from O. Substituting $x_1=0.423l$ in the expression for the slope at O in the last section, we get

$$\text{Max. Slope}\left(\frac{dx}{dz}\right)=-\frac{Wl^2}{15.6EI},\ \text{for a } \textit{rolling load} \qquad \text{(III)}$$

which is numerically a little greater than was the end slope when the load was at the center in Case III.

The maximum deflection will be at the center, when the load is at the center, viz.:

$$\Delta=-\frac{Wl^3}{48EI}$$

368. The two elastic curves of a beam. A few words must be said about the two elastic curves (3) and (6), in **365**, which together include the beam. Fig. 366 shows the two cubics with deflections greatly exaggerated. The curves have a common tangent at B and yet they intersect each other there; just as do an ellipse and its osculatory circle at a general point on the circumference.

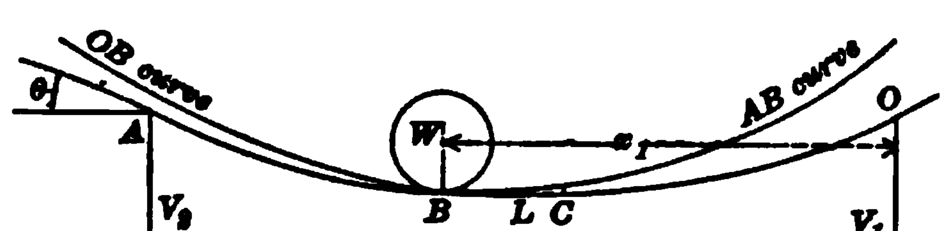


Fig. 366

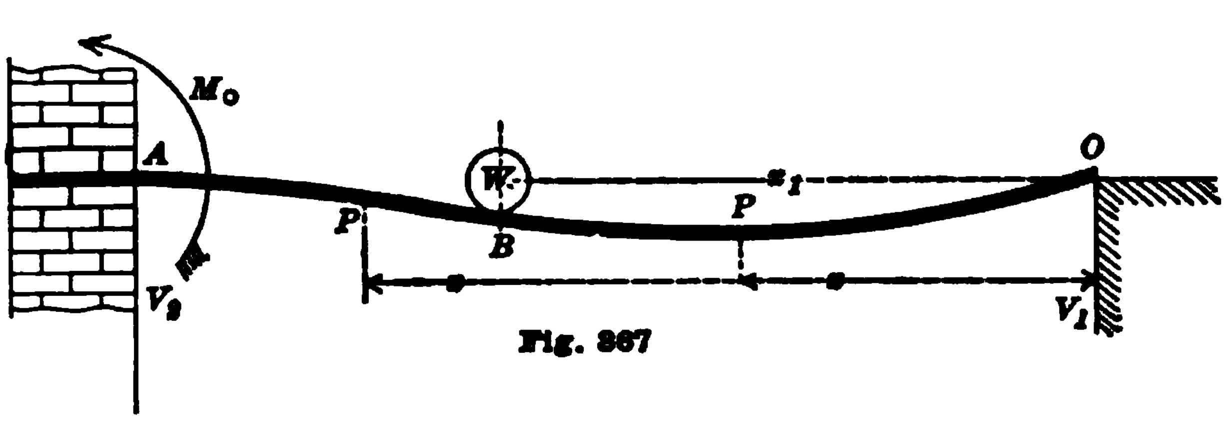


Fig. 367

369. Case IX. **A beam with a single concentrated load, has one end fixed.** To find Maximum Moment, Slope and Deflection, as the load passes over the beam. With the notation of Fig. 367, the equations 1-6 *appear* to be quite the same as in the last problem for the two curves.

Equations for BA.

(4) $M = V_1x - W(x-x_1)$

(5) $EI\dfrac{dz}{dx} = \dfrac{V_1x^2}{2} - \dfrac{W}{2}(x-x_1)^2 + C$

(6) $EIz = \dfrac{V_1x^3}{6} - \dfrac{W}{6}(x-x_1)^3 + Cx + D$

When $x=l$, $\dfrac{dz}{dx}=0$, hence

(7) $0 = \dfrac{V_1l^2}{2} - \dfrac{W(l-x_1)^2}{2} + C;$

When $x=l$, $z=0$, hence

(8) $0 = \dfrac{V_1l^3}{6} - \dfrac{W(l-x_1)^3}{6} + Cl + D;$

Equations for OB.

(1) $M = V_1x$

(2) $EI\dfrac{dz}{dx} = \dfrac{V_1x^2}{2} + H$

(3) $EIz = \dfrac{V_1x^3}{6} + Hx + (K=0)$

When $x=x_1$ the slopes are equal in the two equations (2) and (5); hence

(9) $H=C;$

When $x=x_1$ in (3) and (6), the values of z are the same, hence, since $H=C$,

(10) $D=0.$

Combining (7), (8) and (10), we get

(11) $$V_1 = \frac{W(l-x_1)^2(2l+x_1)}{2l^3}$$

(12) $$C = H = -\frac{Wx_1(l-x_1)^2}{4l}$$

(13) $$V_2 = W - V_1 = \frac{Wx_1(3l^2-x_1^2)}{2l^3}$$

Having found all the constants for any given x_1, which determines the position of the load, the *moment*, *slope*, and *deflection*, of every point are given by equations (1) to (6).

Thus far we have been supposing the load is stationary at x_1.

370. The load now moves across. It is now necessary to find *where the load must be* to produce the maximum moment, the maximum slope, and the maximum deflection; and *what those maximum values are.* Equation (1) shows that the greatest moment in OB, Fig. 367, must always be at the point B where $x=x_1$, so that

(14) $$M_B = V_1x_1 = \frac{W}{2l^3}(l-x_1)^2(2lx_1+x_1^2)$$

For maximum value (for M changes as x_1 changes)

$$\frac{dM}{dx_1} = \frac{W}{2l^3}[(2l+2x_1)(l-x_1)^2 - 2(l-x_1)(2lx_1+x_1^2)] = 0;$$

whence $$x_1 = \frac{l}{2}(\sqrt{3}-1) = 0.366l, \text{ and } l-x = 0.634l.$$

This value of x_1 substituted in V_1x_1, Eq. (14), gives the

$$\text{Max. } (M_B) = \frac{Wl^4}{2l^3} \cdot (0.634)^2(0.366)\,(2.366) = Wl0{\cdot}174 \qquad (\mathrm{I_B})$$

These values of x_1 and $l-x_1$ tell us where the load must be to cause a maximum value of M; substituted in (14) they give us the greatest possible moment at B in the central portion of the beam, which is, when $x_1=0{\cdot}366l$,

$$\text{Max. } (M_B) = W0{\cdot}174l,$$

But this is not the greatest possible in the beam. The moment M_0 at A is the greatest as must now be shown.

The general value of M_0 is found by making $x=l$ in (4); hence

$$M_0 = V_1l - W(l-x_1).$$

This is a *maximum* when $\dfrac{dM_0}{dx_1}=0$.

Substituting the value of V_1, differentiating with respect to x_1 and solving for x_1 we get $x_1=0{\cdot}577l$, which is the value of x_1 when M_0 is a maximum.

Substituting for x_1 in the above value of M_0 we get

$$\text{Max. } (M_0) = -Wl(0{\cdot}192) = -\frac{Wl}{5{\cdot}2} \qquad (\mathrm{I_A})$$

which is clearly greater than the maximum at B in the central part of the beam.

When we look for the *greatest slope* in the bent beam, we readily see that we must investigate two points, the unfixed end at O, and the point of inflection which is always between A and B. At each of these points the Moment is zero. It is almost self-evident that the greater slope is at O, and it will be assumed below that such is the case. It will be left to the student as a problem for practice to investigate the slope at the point of reverse curvature. The steps in the process are easily seen, and the algebraic work may be a bit involved, but one ought never to be afraid of algebra for the simple reason that an equation is long or somewhat beyond ordinary limits.

We now assume that the maximum *slope* is at O, where $x=0$ and the slope is from (2)

$$\left(\frac{dz}{dx}\right)_O = \frac{H}{EI}.$$

This quantity H varies with x_1; hence to find its greatest value as the load moves over the beam, we put $\frac{dH}{dx_1}=0$. From (12) we thus get

$$x_1=\frac{l}{3},$$

which determines the *position of the load* for the greatest possible slope at O. Substituting this value in H we get

$$\text{Max. Slope at } O, \left(\frac{dz}{dx}\right)_{x_1=\frac{l}{3}}=-\frac{Wl^2}{27EI} \qquad \text{(II)}$$

Now we must find the greatest possible *deflection* as the weight W moves across. So long as W is on the beam, every point is deflected; but the point where W is and the point where z is the greatest (numerically) are two different points. While this statement is generally true, there is *one* exception, and that is the very one we want, viz.:— when the two points coincide; that is to say, when the value of x which makes $\frac{dz}{dx}=0$, is the value which x_1 must have when the deflection is a maximum.* The fact that $x=x_1$ is almost self-evident, since

* The reasoning in the text, that x is necessarily x_1 when the deflection is the greatest possible, may not seem satisfactory; accordingly the following mathematical proof is added:—

When z is a maximum, $\frac{dz}{dx}=0$.

Solving the equation $\frac{dz}{dx}=0=\frac{Vx^2}{2}+C$, we get

$$x^2=-\frac{2C}{V}=\frac{x_1l^2}{2l+x_1}$$

(Substituting in (3) we get for the general value of z

$$z=-\frac{W}{6EI}\left(\frac{(l-x_1)^2x_1^{\frac{3}{2}}}{(2l+x_1)^{\frac{1}{2}}}\right)$$

To get the maximum value of z we place $\frac{dz}{dx}=0$, and solve for x_1 getting the same value as in the text, viz:

$$x_1=l(\sqrt{2}-1)$$

Moreover, if this value of x_1 be substituded in the value of x^2 giiven above, we get

$$x^2=\frac{l^3(\sqrt{2}-1)}{l(\sqrt{2}+1)}=l^2(\sqrt{2}-1)^2$$

$$x=l(\sqrt{2}-1)=x_1$$

if free to *roll*, the body would surely roll to the lowest possible point. Hence we find that common value by making $x = x_1$ in (2)

$$\frac{V_1 x_1^2}{2} - H = 0$$

whence $$x_1 = l(\sqrt{2} - 1).$$

This value we substitute for x_1 and for x in (3), and get the maximum deflection, when $x_1 = 0{\cdot}414l$, $x_2 = 0{\cdot}586l$,

$$\text{Max. } \Delta = -\frac{Wl^3}{3EI}(17 - 12\sqrt{2}) = -\frac{Wl^3}{102EI} \tag{III}$$

The *largest allowable load* to move across is evidently found from (I_A); hence

$$\text{Max. safe load} = (W) = \frac{5.2fI}{cl} \tag{IV}$$

If, as often found in works on Applied Mechanics, the concentrated load is placed at the center of the beam when one end is fixed, the value of the supports V_1 and V_2 are easily found from (11) and (13) by making $x_1 = l - x_1 = \frac{l}{2}$; whence

$$(16) \qquad \left(V_1\right)_{\frac{l}{2}} = \frac{5}{16}W \qquad \left(V_2\right)_{\frac{l}{2}} = \frac{11}{16}W$$

371. Characteristic positions of a rolling load. The solution is now complete. We have found the four maximum values we wished to find. It may be well to summarize our results pictorially so that their meaning may be fully appreciated. Fig. 368 is a panoramic

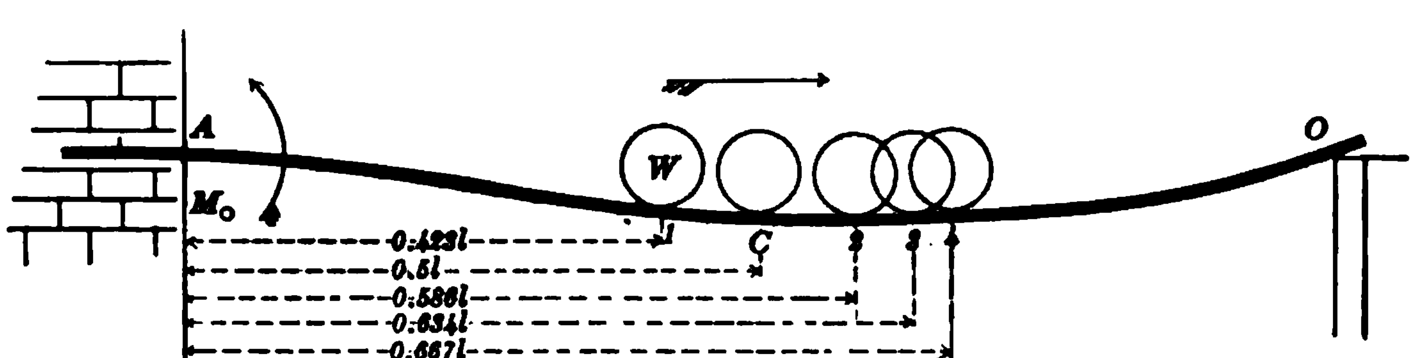


Fig. 368

view of the rolling load, as it passes from A to O, stopping for reflection at *four* important stations. Of course the bending is many times magnified.

The first stop is at No. 1 when $x_2 = 0{\cdot}423l$. During this stop the moment at A, which is M_0, is the *greatest the beam will have to bear.*

$$\text{Max. } M_0 = \frac{Wl}{5.2} = W(0.192l)$$

The load does not stop at the *center* of the beam inasmuch as that point has no present interest, as it involves no maximum.

There is great interest connected with the second stop at No. 2, which is 0.586l distant from A, for that is the *lowest point reached* by the rolling load.

$$\text{Max. } \Delta = \frac{Wl^3}{102EI}$$

The third stop at No. 3, which is 0.634l from A, gives the point where the *moment under the load* is a maximum.

$$\text{Max. } M_B = W(0.174l = \frac{Wl}{5.75}.$$

The fourth stop is at No. 4, which is the place where the load stands when the *slope at O* is the greatest:

$$\text{Max. slope at } O. = \frac{Wl^2}{27EI}.$$

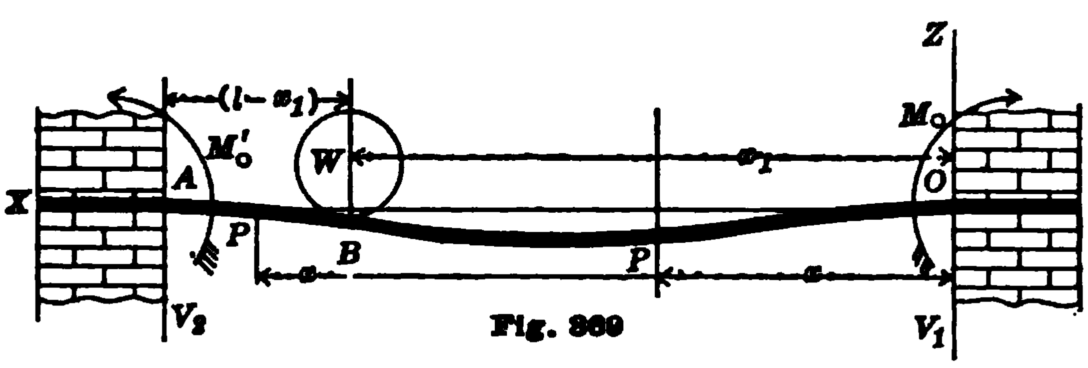


Fig. 369

372. Case IX. **A load W moves across a horizontal beam fixed at both ends.** A general position is shown in Fig. 369.

The equations for the two segments are:

For BA.

(4) $M = V_1x - M_0 - W(x - x_1)$

(5) $EI\dfrac{dz}{dx} = \dfrac{V_1x^2}{2} - M_0x - \dfrac{W}{2}(x-x_1)^2 + C$

(6) $EIz = \dfrac{V_1x^3}{6} - \dfrac{M_0x^2}{2} - \dfrac{W}{6}(x-x_1)^3 + Cx + D$

From (5), when $x = l$

(7) $0 = \dfrac{V_1l^2}{2} - M_0l - \dfrac{W}{2}(l-x_1)^2 + C$

For OB.

(1) $M = V_1x - M_0$

(2) $EI\dfrac{dz}{dx} = \dfrac{V_1x^2}{2} - M_0x + (H=0)$

(3) $EIz = \dfrac{V_1x^3}{6} - \dfrac{M_0x^2}{2} + (K=0)$

From (6) when $x = l$

(8) $0 = \dfrac{V_1l^3}{6} - \dfrac{M_0l^2}{2} - \dfrac{W}{6}(l-x_1)^3 + Cl + D$

The slopes are equal when $x = x_1$; hence

(9) $$C = 0$$

The deflections are also equal when $x = x_1$; hence

(10) $$D = 0$$

Solving (7) and (8) for V_1 and M_o, we have

(11) $$M_o = \frac{W}{l^2} \cdot x_1(l-x_1)^2$$

(12) $$V_1 = \frac{W}{l_3}(l+2x_1)(l-x_1)^2$$

It is evident from symmetry that the maximum moment *in the concave part of the beam*, and the maximum deflection, are at the center, when $x_1 = \frac{l}{2}$ and $V_1 = \frac{W}{2} = V_2$. This value of x_1 makes

$$M_o\left(\text{when } x_1 = \frac{l}{2}\right) = \frac{Wl}{8}$$

and also

(13) $$M_{x=x_1=\frac{l}{2}} = \frac{W}{2} \cdot \frac{l}{2} - \frac{Wl}{8} = \frac{Wl}{8}$$

That is, if W is at the center, the moments, at center and at the ends are the same, $\frac{Wl}{8}$. It must however be carefully noted that $M = \frac{wl}{8}$ is *not the greatest possible moment.*

Eq. (11) gives the general value of M_0 dependent on the position of the load. To find its maximum value we put $\frac{dM_o}{dx_1} = 0$,

$$l^2 - 4lx_1 + 3x_1^2 = 0 = (l-3x)(l-x)$$

Whence, $$x_1 = \frac{1}{3}l.$$

This value in (11) gives

$$\text{Max. Value of } M_o = \frac{4}{27}Wl. \tag{I}$$

Hence the *danger point* (if there is one) is at the *end*, when the load is distant only $\frac{1}{3}l$.

The greatest *slope* is always at the point of inflection where $M=0$. From (1) we find that $M=0$ when $x = \frac{M_o}{V_1}$. Substituting this value of x in (2), we have a general value of the slope depending on x_1:—

$$EI\frac{dz}{dx} = \frac{R}{2}\frac{M_o^2}{R^2} - \frac{M_o^2}{R} = -\frac{M_o^2}{2R} = -\frac{W}{2l} \cdot \frac{x_1^2(l-x_1)^2}{l+2x_1}.$$

This expression is readily found to be a maximum when

$$x_1 = \frac{l}{6}(\sqrt{13}-1) = 0.434l.$$

This value in (16) gives

$$\text{Max. Slope} = -\frac{10Wl^2}{309EI} \qquad \text{(II)}$$

The position of the point of *inflection* when the load is at $x_1 = 0.434l$, is found from $x = \frac{M_o}{V_1}$ when this vlaue of x_1 is substituted.

Hence $$x_s = \frac{lx_1}{l+2x_1} = 0.23l$$

The position of S changes for every change in place of W.

The Deflection at the center is readily found from (3) by substituting $x = x_1 = \frac{l}{2}$, $V_1 = \frac{W}{2}$ and $M_o = \frac{Wl}{8}$ giving

$$\Delta = \frac{Wl^3}{192EI}. \qquad \text{(III)}$$

The maximum Safe load is from (I)

$$\text{Max. safe } W = \frac{27fI}{4cl}. \qquad \text{(IV)}$$

373. Characteristic positions. The results of this discussion may be pictured as before. Fig. 370. Diagram showing characteristic positions of a single load rolling from O to A.

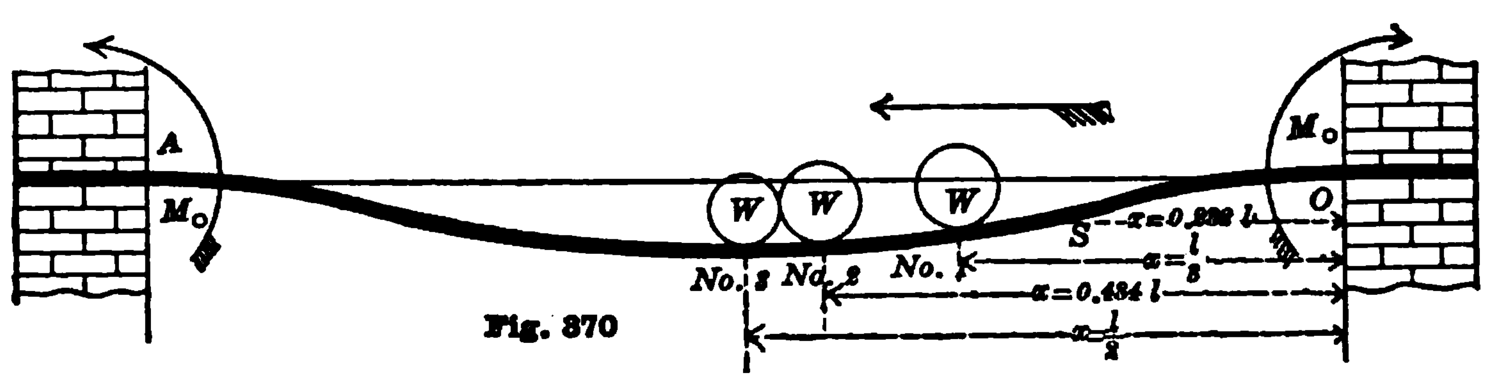


Fig. 370

No. 1 shows load when M_o is Max. $= \frac{4}{27}Wl$.

No. 2 shows load when Slope (at S) is Max. $= -\frac{10Wl^2}{309EI}$.

No. 3 shows load when Δ is Max. $= \frac{Wl^3}{192EI}$.

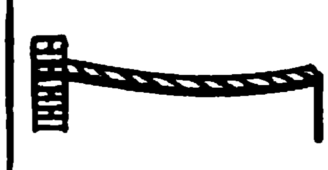
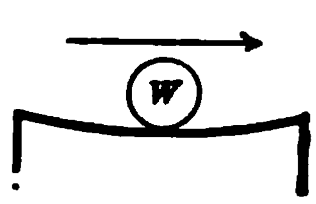
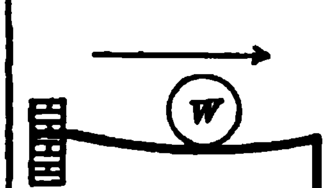
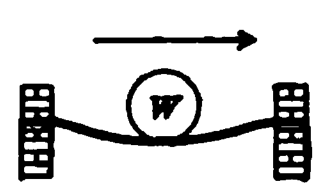
374. The utility of fixing one or both ends of a prismatic beam under a uniform load, is plainly seen if we compare the results already reached.

Every engineer and architect, and student even, asks:—What is gained in either strength or stiffness by fixing one or both ends of a plain beam with constant section? This natural question is answered by placing results in a table so that comparisons are easily made.

375. Table.

MAXIMUM STRENGTH, SLOPE, DEFLECTION AND SAFE LOADS IN PRISMATIC BEAMS.

f=greatest allowed stress. c=distance of extreme fiber from neutral axis.

$W=lw$				
Max. Moment.	$\frac{Wl}{8}$	$\frac{Wl}{8}$ at fixed end	$\frac{Wl}{12}$ at ends	$\frac{Wl}{2}$
Max. Slope	$\frac{Wl^2}{24EI}$	$\frac{Wl^2}{48EI}$	$\frac{Wl^2}{125EI}$	$\frac{Wl^2}{6EI}$
Max. Deflection	$\frac{5Wl^3}{384EI}$	$\frac{Wl^3}{185EI}$	$\frac{Wl^3}{384EI}$	$\frac{Wl^3}{8EI}$
Max. Working Load	$\frac{8fI}{cl}$	$\frac{8fI}{cl}$	$\frac{12fI}{cl}$	$\frac{2fI}{cl}$
Weight of Beam is not included				
Max. Moment.	$\frac{Wl}{4}$	$\frac{Wl}{5.2}$ at fixed end	$\frac{4Wl}{27}$ at the ends	$\frac{Wl}{1}$
Max. Slope	$\frac{Wl^2}{15.6EI}$	$\frac{Wl^2}{27EI}$	$\frac{Wl^2}{31EI}$	$\frac{Wl^2}{2EI}$
Max. Deflection	$\frac{Wl^3}{48EI}$	$\frac{Wl^3}{102EI}$	$\frac{Wl^3}{192EI}$	$\frac{Wl^3}{8EI}$
Max. Working Load	$\frac{4fI}{cl}$	$\frac{5.2fI}{cl}$	$\frac{6.75fI}{cl}$	$\frac{fI}{cl}$

376. Case IX. 32. **A simple beam carries a uniform and a single concentrated load.**

(*a*) If the concentrated load be at the center, add the maximum values of corresponding quantities: Thus:

$$\text{Max. } M = \frac{W_1 l}{8} + \frac{W_2 l}{4}\text{, in which } W_1 = lw.$$

$$\text{Max. Slope} = \frac{l^2}{EI}\left(\frac{W_1}{24} + \frac{W_2}{15.6}\right)$$

$$\text{Max. Def.}^{\text{n}} = \frac{l^3}{EI}\left(\frac{5W_1}{384} + \frac{W_2}{48}\right.$$

(*b*) If the concentrated load is *not* at the center, the values of M, dz/dx and z are to be found for the concentrated load from the *general* equations of Case VII, and the results for different points are to be added to the results for the same points for the uniform load.

(*c*) Or the problem can be solved anew by the method already used. Fig. 371. Thus:

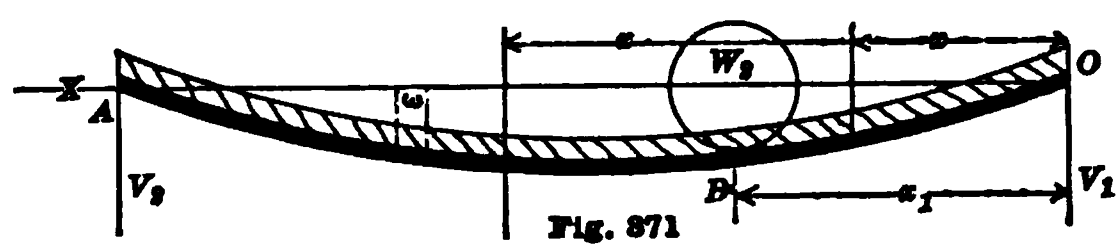


Fig. 371

V_1 and V_2 are known.

Eqs. for AB	*Eqs. for OB*
(4) $M = V_1 x - \frac{wx^2}{2} - W_2(x - x_1)$	(1) $M = V_1 x - \frac{wx^2}{2}$
(5) $EI\frac{dz}{dx} = \frac{V_1 x^2}{2} - \frac{wx^3}{6} - \frac{W_2}{2}(x - x_1)^2 + C$	(2) $EI\frac{dz}{dx} = \frac{V_1 x^2}{2} - \frac{wx^3}{6} + H$
(6) $EIz = \frac{V_1 x^3}{6} - \frac{wx^4}{24} - \frac{W_2}{6}(x - x_1)^3 + Cx + D$	(3) $EIz = \frac{V_1 x^3}{6} - \frac{wx^4}{24} + Hx + (K=0)$

It will be found that $H = C$, and that $D = 0$. The value of C can then be found from (6) by letting $x = l$, and the maximum values can be found as in Cases VII and VIII.

377. Case XI. **If a beam carries two (or more) concentrated loads** there will be three (or more) elastic curves, and there will be three (or more) sets of equations, and six (or more) constants of integration. All such constants can be found by the methods already used but the process may be long and the equations of the three (or more)

elastic curves may be much involved. If one (or more) of the loads is negative becoming thereby a support (or supports), we have the case of a continuous girder over one (or more) supports which will be discussed later.

378. CASE XII. **A simple beam carries two equal concentrated loads equally distant from the center.** Fig. 372 shows both the bent and the unbent beam. We are to find the *moment* and *deflection* at C and at B.

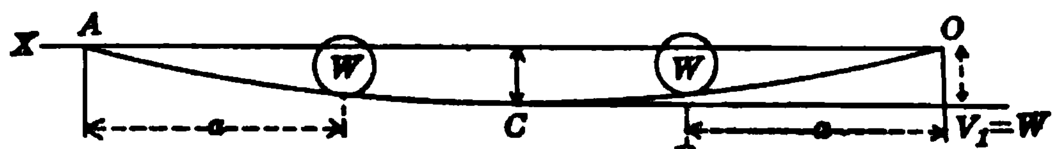


Fig. 372

Knowing that $\frac{dz}{dx} = 0$ at C, the student will have no difficulty of finding moments and deflections if he takes the two curves OB and BC, as in other Cases. He may (or may *not*) take his origin at the center of the beam.

There is however **a second method.**

379. Another solution. Consider the half beam CBO, Fig. 373, as an inverted cantilever, horizontal and fixed at C. $V_1 = W$ at the end causes an *upward* deflection (by Case 1) of

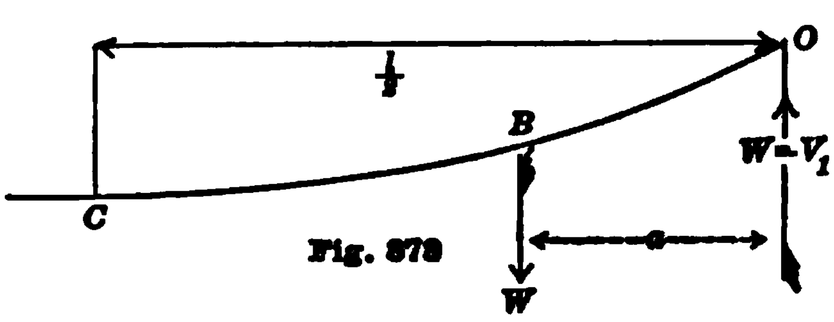


Fig. 373

$$\frac{W(\frac{l}{2})^3}{3EI} = \frac{Wl^3}{24EI}.$$

The load at B, acting at the end of the cantilever BC, whose length is $\frac{l}{2} - a$, produces a *downward* deflection *at* B of $\frac{W(\frac{l}{2}-a)^3}{3EI}$. If the force acting at B, were *acting alone*, either up or down, the free end BO would *remain straight*, and the outer end would have a greater deflection than B. That greater amount is found by multiplying the distance a by the tan θ, which is the *slope at B*. That slope (by Case 1) is $\frac{W(\frac{l}{2}-a)^2}{2EI}$. Multiplying by a and adding to the deflection at B, we have the total deflection at O, *caused by the load at B*, namely:—

$$\frac{W(\frac{l}{2}-a)^3}{3EI} + \frac{Wa(\frac{l}{2}-a)^2}{2EI} = \frac{W(\frac{l}{2}-a)^2}{EI}\left(\frac{l-2a}{6} + \frac{a}{2}\right) = \frac{W(l-2a)^2(l+a)}{24EI}$$

$$= \frac{W(l^3 - 3al^2 + 4a^3)}{24EI}$$

Subtracting the deflection due to the force at B from the deflection due to V_1, we have

$$\Delta = \frac{Wa(3l^2-4a^2)}{24EI} \tag{1}$$

If we put the total load on OCA (Fig. 372) as W_1, then $W=W_1/2$, and the above deflection is

$$\Delta = \frac{W_1 a(3l^2=4a^2)}{48EI}. \tag{2}$$

[See Carnegie Pocket Companion, p. 94.]

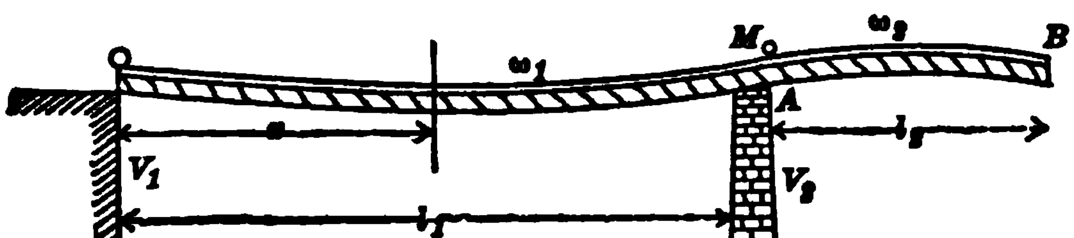


Fig. 374

380. Prismatic beam with a cantilever on one end, under uniform load. (Fig. 374.)

The supporting forces are readily found to be:

$$V_1 = \frac{l_1^2 w_1 - l_2^2 w_2}{2l_1}, \qquad V_2 = \frac{l_1^2 w_1 + l_2^2 w_2 + 2l_1 l_2 w_2}{2l_1}.$$

It is also evident that $M_o = \frac{w_2 l_2^2}{2}$. The shear at any point in l_1 is $V_1 - w_1 x$, which is zero when $x = \frac{V_1}{w_1} = \frac{l_1}{2} - \frac{l_2^2 w_2}{2l_1 w_1}$. The moment at any point in the part OA is (taking forces acting on the *left* of our general point).

(1) $$M = V_1 x - \frac{x^2 w_1}{2}$$

This is a maximum when the shear is zero, that is for $x = \frac{l_1}{2} - \frac{l_2^2}{2l_1} \cdot \frac{w_2}{w_1}$, and maximum M is readily found for any practical case.

The slope equation for the part OA is found from (1) in the usual way (multiplying (1) by dx and integrating) so that

(2) $$EI\frac{dz}{dx} = \frac{V_1 x^2}{2} - \frac{x^3 w_1}{6} + H.$$

Integrating again we have

(3) $$EIz = \frac{V_1 x^3}{6} - \frac{x^4 w_1}{24} + Hx + (K=0)$$

When $x = l_1$ $z = 0$, so that

$$H = \frac{l_1^3 w_1}{24} - \frac{V_1 l_1^2}{6}.$$

This value of H substituted in (2) and (3) will give the slope and deflection at any desired point. Perhaps the most important questions

are:—first, what is the slope at A? and second, what should the relation between l_2 and l_1 be to make the slope at A zero?

The slope at A is found by making $x=l$, and putting the value of H in (2).

$$\frac{dz}{dx}\,(\text{at } A)=\frac{1}{EI}\left[\frac{V_1l_1^2}{2}-\frac{l_1^3w_1}{6}+\frac{l_1^3w_1}{24}-\frac{V_1l_1^2}{6}\right]=\frac{1}{EI}\left[\frac{V_1l_1^2}{3}-\frac{l_1^3w_1}{8}\right]$$

If $\frac{dz}{dx}$ (at A) is zero, $\qquad 0=8V_1-3l_1w_1$

whence $\qquad l_1=2l_2\sqrt{\frac{w_2}{w_1}}$, and $l_2=\frac{l_1}{2}\sqrt{\frac{w_1}{w_2}}$

From which we see how long the cantilever should be to have the beam horizontal on the top of the support. If $w_2=w_1$; $l_2=\frac{1}{2}l_1$, and the portion OA is under Case VIII, of a beam fixed horizontally at one end, and uniformly loaded. For $l_2=\frac{1}{2}l_1$, the moment at A is M (at A) $=-\frac{wl_1^2}{8}$, as in **365**, and V_1 is $\frac{3}{8}l_1w_1$, and the greatest moment in the portion OA for $x=\frac{3}{8}l$, is $M=+\frac{9}{128}wl^2$, which is less than at A as already found in Case VIII.

381. The cantilever part. In general the slope or direction of the cantilever at the point of support is *not* zero (tho the inclination is presumably small) in which case the cantilever is analyzed independently. As the beam, Fig. 375, is *very nearly* horizontal, the formulas for vertical loads may be applied without serious error.

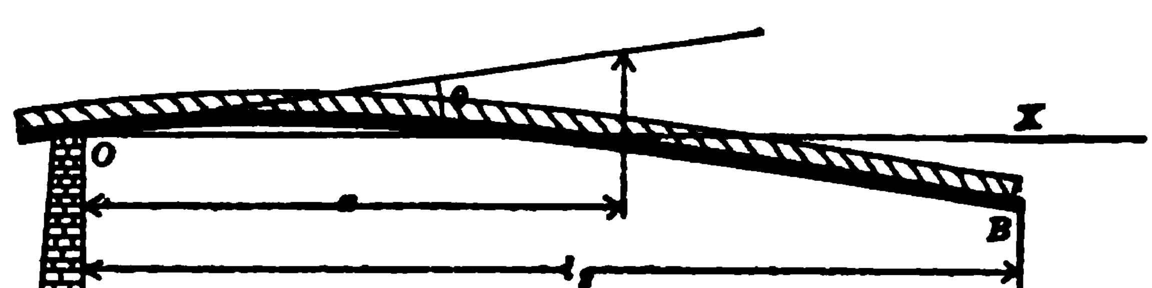


Fig. 375

(1) $$M=\frac{(l_2-x)^2w_2}{2}=-EI\frac{d^2z}{dx^2}$$

(2) $$-EI\frac{dz}{dx}=\frac{w_2}{2}\left(l_2^2x-l_2x^2+\frac{x^3}{3}\right)+H$$

When $x=0$ we have $H=-EI\tan\theta$. Proceeding

$$-EIz=\frac{w_2}{2}\left(\frac{l_2^2x^2}{2}-\frac{l_2x^3}{3}+\frac{x^4}{12}\right)+Hx+(K=0)$$

(3) Hence $$z=-\frac{w_2}{24EI}\left(6l_2^2x^2-4l_2x^3+x^4\right)+x\tan\theta,$$

from which it appears that the deflection of a point, distant x, *below* the axis of OX is *diminished* by $x \tan\theta$, from what the deflection *would have been* had θ been zero.

Equations (2) and (3) give the slope and deflection for every point in the beam OB, as θ is supposed to be known.

382. Case XIII. **A moving group of concentrated loads** at fixed distances apart, rolls on and across a given girder or beam. Fig. 376.

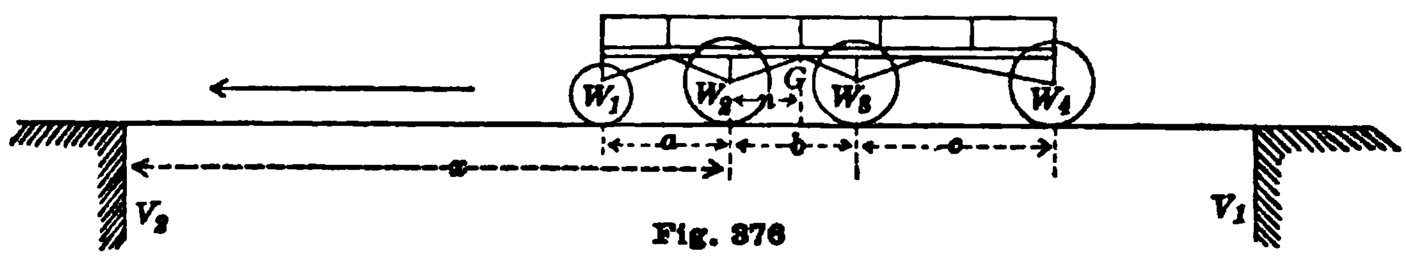


Fig. 376

Under each wheel there is a moment which varies as the group moves across. The problem is to find the maximum moment to which the beam or girder is subjected. The greatest moment under some one wheel will be the maximum sought. Let us find the position of W_2 when the moment *under it* is a maximum, and then find what that maximum is.

Let x be the distance of W_2 from the left-hand support V_2, the moment at W_2 will then be

(1) $$M_x = V_2 x - W_1 a$$

But V_2 depends upon x, and the position of the center of gravity of the group. Accordingly we find the position of that center G. Let the distance from W_2 to G be n. Then

$$n(\Sigma W) = W_3 b + W_4(b+c) - W_1 a$$

$$n = \frac{W^3 b + W^4(b+c) - W_1 a}{W_1 + W_2 + W_3 + W_4}.$$

Hence $$V_2 = \frac{(l-x-n)}{l}(\Sigma W) \text{ by Chap. III.}$$

and

(2) $$M_x = \frac{lx - x^2 - nx}{l}\Sigma(W) - Wa.$$

This is the general value of the moment under W_2 as *that group* moves across the girder: it will be a maximum when $\frac{dM_x}{dx} = 0$. Differentiating, reducing, and solving for x, we get

$$x = \frac{l}{2} - \frac{n}{2}.$$

This shows that the maximum moment under W_2 will occur when the *central point* of the *girder* is *midway* between W_2 and G the center of gravity of the group. This value of x substituted in V_2, and both in M_x will give the maximum moment sought for the point under W_2.

We may next find the greatest moment under W_3 (if there is any doubt as to the result) and compare the two to find the greatest of all. The formulas apply only for an *unchanging* group.

Example. Let $l=100$ feet, $W_1=6$ tons, $W_2=10$ tons, $W_3=12$ tons, $W_4=4$ tons. Let $a=8$ feet, $b=10$ feet, $c=12$ feet. Find the Max. moment, applied to the beam as the group passes over.

383. Special example. Two equal loads W and W_1, whose distance apart is a, roll across a simple beam. Find where the loads are when the bending moment is a maximum, and find that maximum. Fig. 377.

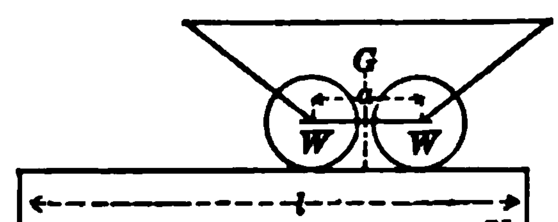


Fig. 377

Prove that there are two positions, one on each side of the center, with G distant $\frac{a}{4}$ from the center, where the moment reaches a maximum value, viz.:—

$$\text{Max. } M = W\left(\frac{l}{2} - \frac{a}{2} + \frac{a^2}{8l}\right)$$

How much greater is this than what the constant moment between the *loads would be*, if the wheels were equally distant from the center of the girder?

384. The exceeding stiffness of thin tubes as compared with *solid rods* of the *same weight* per unit of length, is very suggestive.

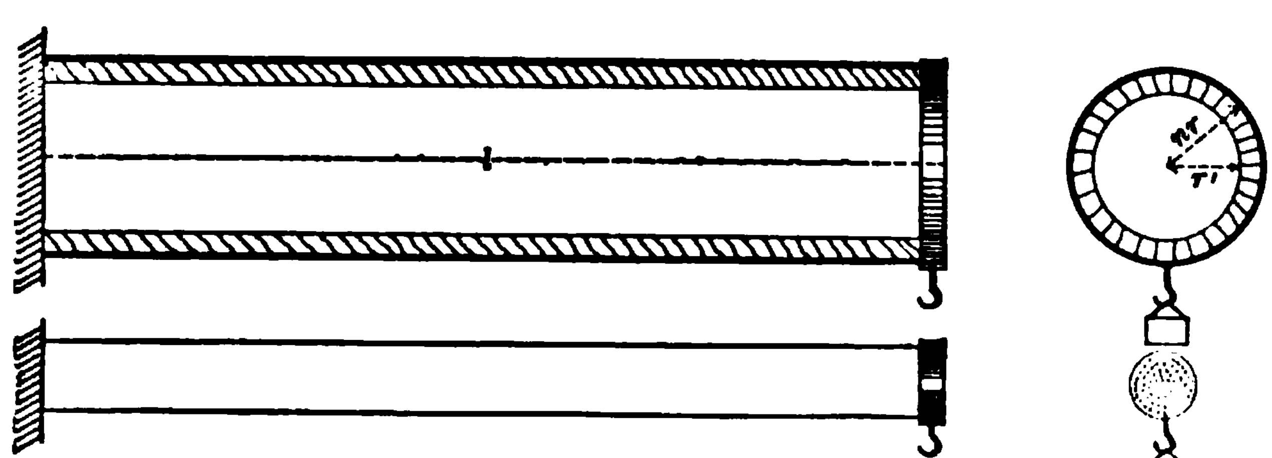


Fig. 378

Problem. Compare the deflection of a solid rod of uniform radius, when used as a cantilever beam carrying a concentrated load at the end with that of a tube of the same length, material and weight. Fig. 378. In each case let the load be W.

Let the radius of the rod be r, and the outer radius of the tube be nr. If r' be the inner radius of the tube, we have

$$\pi r^2 = \pi(n^2r^2 - r'^2)$$

$$r'^2 = r^2(n^2 - 1)$$

The deflection of the solid cylinder is found readily. Measure x from the fixed end.

$$M = W(l-x)$$

$$EI\frac{dz}{dx} = W\left(lx - \frac{x^2}{2}\right) + (H=0)$$

$$EIz = W\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + (K=0)$$

$$\Delta_1 = \frac{Wl^3}{3EI_1} = \frac{4Wl^3}{3\pi Er^4}, \text{ since } I_1 = \frac{\pi r^4}{4} \tag{1}$$

The deflection of the tube is

$$\Delta_2 = \frac{Wl^3}{3EI_2}$$

$$I_2 = \frac{\pi(n^4r^4 - r'^4)}{4} = \frac{\pi r^4}{4}(2n^2-1). \tag{2}$$

Hence

$$\Delta_2 = \frac{4Wl^3}{3E\pi r^4(2n^2-1)} \tag{3}$$

and

$$\frac{\Delta_1}{\Delta_2} = 2n^2 - 1 \tag{4}$$

That is, if

$\left.\begin{matrix} n=2 \\ n=3 \\ n=5 \end{matrix}\right\}$ the deflection of the solid cylinder is $\left.\begin{matrix} 7 \\ 17 \\ 49 \end{matrix}\right\}$ *times as great* as the deflection of the tube.

CHAPTER XX.

Graphical Representations of Bending Moments, Shear, Slope, and Deflections in Beams.

385. When a simple beam OB, Fig. 379, has concentrated loads $W_1W_2W_3$, the bending moments at points along the beam are conveniently represented by the ordinates to a broken line drawn as follows (Fig. 379): Let the supports at the left

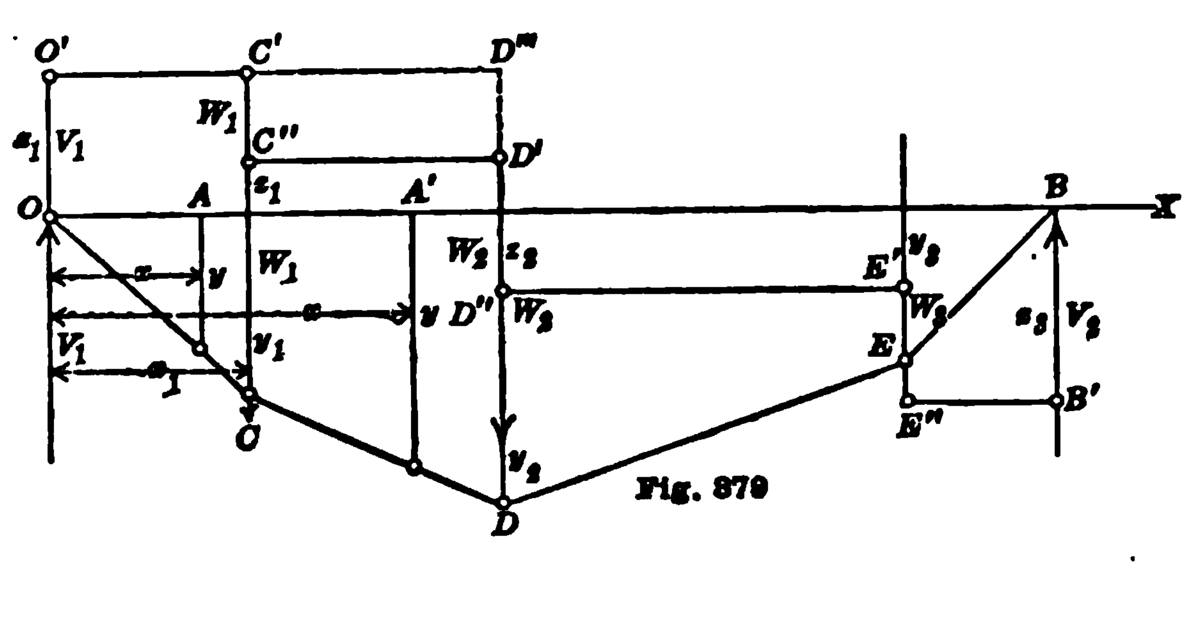


Fig. 379

and right be V_1 and V_2 whose values are supposed to have been found. At the point A, at a distance x from O, the bending moment is V_1x, and if the ordinate y be dropped from the beam, whose length is V_1x, we have the equation

$$y = V_1x$$

which is the equation of a straight line passing thru O and which holds until it crosses the line of action of W_1, at which point its value is V_1x_1, x_1 being the distance from O to W_1.

Now taking a point A' between W_1 and W_2 at a distance x from O, the bending moment is

$$V_1x - W_1(x - x_1) = y$$

which is the equation of another straight line passing between the lines of action of W_1 and W_2, and connecting with the former moment line. When x becomes x_2, which is the distance from O to W_2

$$y_2 = V_1x_2 - W_1(x_2 - x_1)$$

In the same way the moment line is continued to its intersection with W_3, the value of y at that point being

$$V_1x_3 - W_1(x_3 - x_1) - W_2(x_3 - x_2) = y_3$$

From that point the moment line continues in a straight line to the point B where the moment is zero. Thus we have the broken line *OCDEB*, the ordinate to which at any point gives the magnitude of the bending moment.

386. The moment may change sign. If the beam is acted upon by vertical forces, some positive and some negative, the moment curve may cross the axis of X, showing that the bending moment itself has changed its sign. This is best illustrated by a beam with a cantilever portion.

Let *OBC*, Fig. 380, be a beam supported at O and B, and carrying concentrated loads on both sides of B, namely $W_1W_2W_3$, distant respectively from O: x_1, x_2, x_3. Let the distance OB be l_1, and the distance BC be l_2.

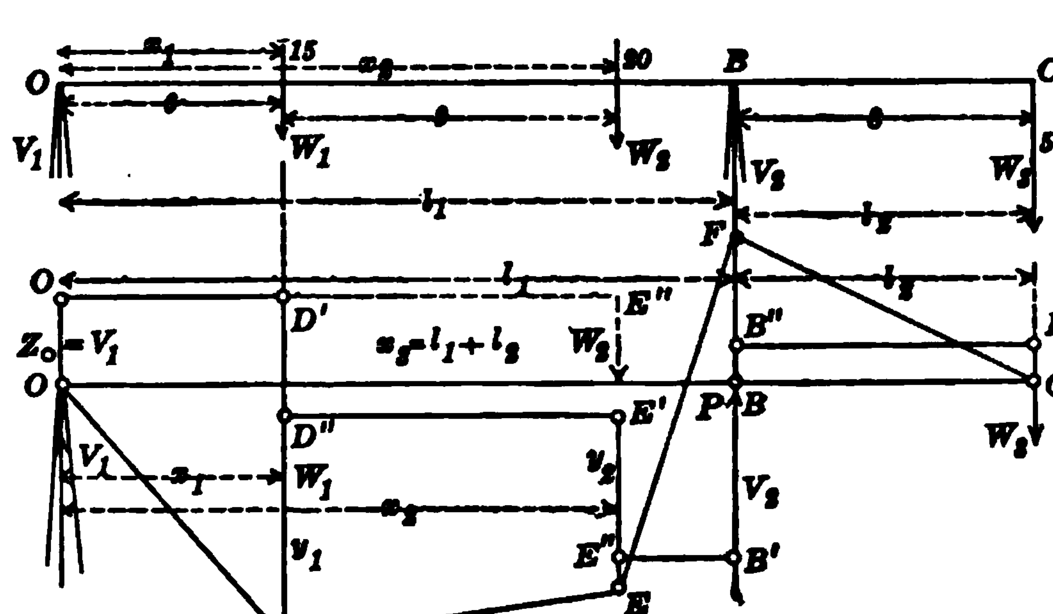


Fig. 380

Having found the supports V_1 and V_2, proceed as before. Assuming that V_1, as well as V_2, acts vertically upwards, we draw the moment curve

as in the previous example, *ODE*. When we proceed to draw the line between W_2 and B, we find that when $x=l_1$, the moment on top of the second support—

$$V_1l_1-W_1(l_1-x_1)-W_2(l_2-x_2)=y_l$$

is negative. Accordingly y_l must be laid off not downward but upward from the point B; thus the moment line, which is still a straight line, crosses the axis of the beam at the ponit P at which there is no moment. The exact position of that point is found by careful drawing or by solving the equation

$$V_1x-W_1(x-x_1)-W_2(x-x_2)=0$$

We thus learn that the bending moment changes its sign at P.

Proceeding now from the point F where the last moment line cuts the line of action of V_2, we proceed to find the value of y, where the line crosses W_3. Its value at that point is

$$y_3=V_1x_3-W_1(x_3-x_1)-W_2(x_3-x_2)+V_2(x_3-l_1)=0$$

giving us the point C.

The fact that y_3 must be zero was almost self-evident. We, therefore, have a broken line *ODEFG* and its distance from the axis of x at any point shows both the *magnitude* and the *kind* of *bending moment* in the beam. If the moment curve is below the axis, the beam is concave upwards, and the upper fibers are in compression. When the moment curve is above the axis, the beam is concave downwards, showing that the upper fibers are in tension. The point P where the moment changes sign is therefore a point of **Inflection in the beam.**

387. Shear diagrams. Closely related to the bending moments in a beam is the *shear* at different points. This, too, may be represented graphically. It will be helpful in thinking of shear to imagine every beam made up of thin plates lying between consecutive cross-section planes, as if one were to glue together all the leaves of a big dictionary six inches thick, and then by means of a die cut out a prism six inches long containing a piece of leaf from every page in the book, and then use that prismatic block as a beam made up of thousands of laminae. Shearing stress, as it will be remembered, is the action between two adjacent laminae tangential to the two surfaces. In the case of two adjacent laminae in the beam, one is dragging down, and the other is dragging up.

Let us take the beam *OB* in Fig. 379, and draw the shear diagram for every point in the beam. Beginning at the point O we will suppose that the support V_1 acts directly under the end lamina of the beam thereby lifting or pushing up with a force V_1. Then V_1 is exactly the

measure of the shearing stress between the first and second laminae; as the sheer is upward we will draw the ordinate OO' upward whose length to scale is V_1. Since we are considering only the shear produced by loads W_1, W_2, W_3, we take no account of the weight of the different laminae.

Accordingly the action between the second and third laminae is the same as between the first and second, that is V_1; and the shear is evidently the same at every point between O and the line of action of W_1. Consequently the shear diagram is bounded by a straight line running parallel to the axis of the beam at a distance $z = V_1$ above it from O' to C'.

Taking the next lamina to the right of W_1 we notice that V_1 acts up and W_1 acts down. Consequently the shear is $V_1 - W_1$. Therefore C' drops to C'', the distance from C' to C'' being W_1. The diagram then continues in a line parallel to the axis of the beam to D' where the shear drops again to a point D'' which is below the axis of the beam. This shows that the shear changes from being up to being down. It remains constant then from D'' to E' where it changes again from E' to E'' and then remains constant until we reach the support V_1, where the last lamina in the beam is acted upon downward by the lamina adjacent. It will be noticed that all the way thru we have been considering the action of a lamina on the *left* of the ideal plane of section upon the lamina on the *right* of the same. The first lamina acted *up* upon the second, and the *last but one* acted *down* upon the last. The change of direction and the change of sign was at the point where W_2 acted. The shear diagram is therefore made up by the broken line $OO'C'C''D'D''E'E''B'B$ as shown in the figure.

388. The moment at a point is represented by an area. In practice it is a good plan to have the shear diagram and the moment diagram drawn on the same sheet of paper with a new axis of x used in common directly under the drawing of the loaded beam so that the vertical lines of action of the supports and the loads will correspond. This is illustrated in Fig. 380.

Taking the same cantilever beam, we find the shear diagram as follows: Beginning at the point O as before, we erect the shear ordinate $z_0 = V_1$, and draw the line $O'D'$ until it intersects the line of action of the first load. Then drop from D' to D'' a distance equal to W_1 drawn to scale and the shear line moves parallel with the axis of the beam to E', where it intersects the line of action of W_2. It then drops a distance W_2 and moves parallel with the axis until it cuts the line of V_2 at B'. V_2 acts upward and therefore the shear diagram rises from B' to B'', a distance equal to V_2 by scale. It then moves parallel with the axis

until it cuts the line of W_2 at H' and drops to C, a distance W_2. The shear diagram is just as simple in this case as it was in the last.

In accord with the general relation between Bending Moment and Shear shown later on, it should now be noted that the *length* of the ordinate y_1, (see Figs. 379 and 380), is numerically equal to the *area* of the shear rectangle OC'; and that the *length* of y_2 is equal numerically equal to the *area* OD'' *less* $C''C'''$. In general it may be stated that, if shear and moment ordinates be drawn for *any* point in the beam, the *length* of the *moment ordinate* equals (numerically) the *shear area* bounded by the ordinate x.

389. Beams under distributed loads. Let the beam OB, Fig. 381, supported at the ends as before by V_1 and V_2, carry a uniformly distributed load of w lbs. per linear foot. Let l be the length of the beam so that each support is $\frac{lw}{2}$. The moment at any point A distant x from O is readily seen to be

$$y = V_1 x - \frac{wx^2}{2} = \frac{w}{2}(lx - x^2)$$

which is the equation of a parabola with a vertical axis, passing thru the points O and B.

Fig. 381

By symmetry we note that the vertex of that parabola is below the center of the beam and its distance below is found by giving to x the value $\frac{l}{2}$ whence

$$y_1 = \frac{wl^2}{8}$$

which is the maximum bending moment in the beam.

390. **Had this entire distributed load** wl **been concentrated at** the center of the beam (see dotted circle), Fig 381, the moment at that point would have been $\frac{wl^2}{4}$, which is just twice as great as it is when the load is uniformly distributed. If the moment figure be drawn for the concentrated load wl_1 at the center, it will consist of two straight lines meeting at a point below the center with y_2 equal to just twice y_1

found above. By a familiar property of the parabola the straight lines OC' and $C'B$ are tangent to the parabola at the points O and B.

The shear diagram for this beam and load starts with an ordinate erected at O with the length $z_0 = V_1$. At a point A distant x from O, the shear ordinate is $z = V_1 - wx$.* But this is the equation of a straight line. As x increases z decreases uniformly, and when we get to the middle of the beam where $x = \frac{l}{2}$ and $wx = V_1$ we have $z = 0$. Accordingly the *shear diagram* runs from the top of z_0 down in a straight line to the axis of the beam at its center. When we pass the center z keeps on diminishing, being negative, and continues to be negative more and more until it reaches the line of V_2 where it becomes $-V_2$, so that the shear diagram in this case is simply one straight line running from a point above O to a point below B.

It is hardly necessary to point out that the numerical values of y and y_1 are respectively equal to the shear areas $OO'A'A$ and $OO'C''$.

391. If on a beam there are both **concentrated** and **distributed loads,** the moment diagram will show a boundary line consisting of parts of different parabolas intersecting on the lines of action of the concentrated loads. A single example will suffice.

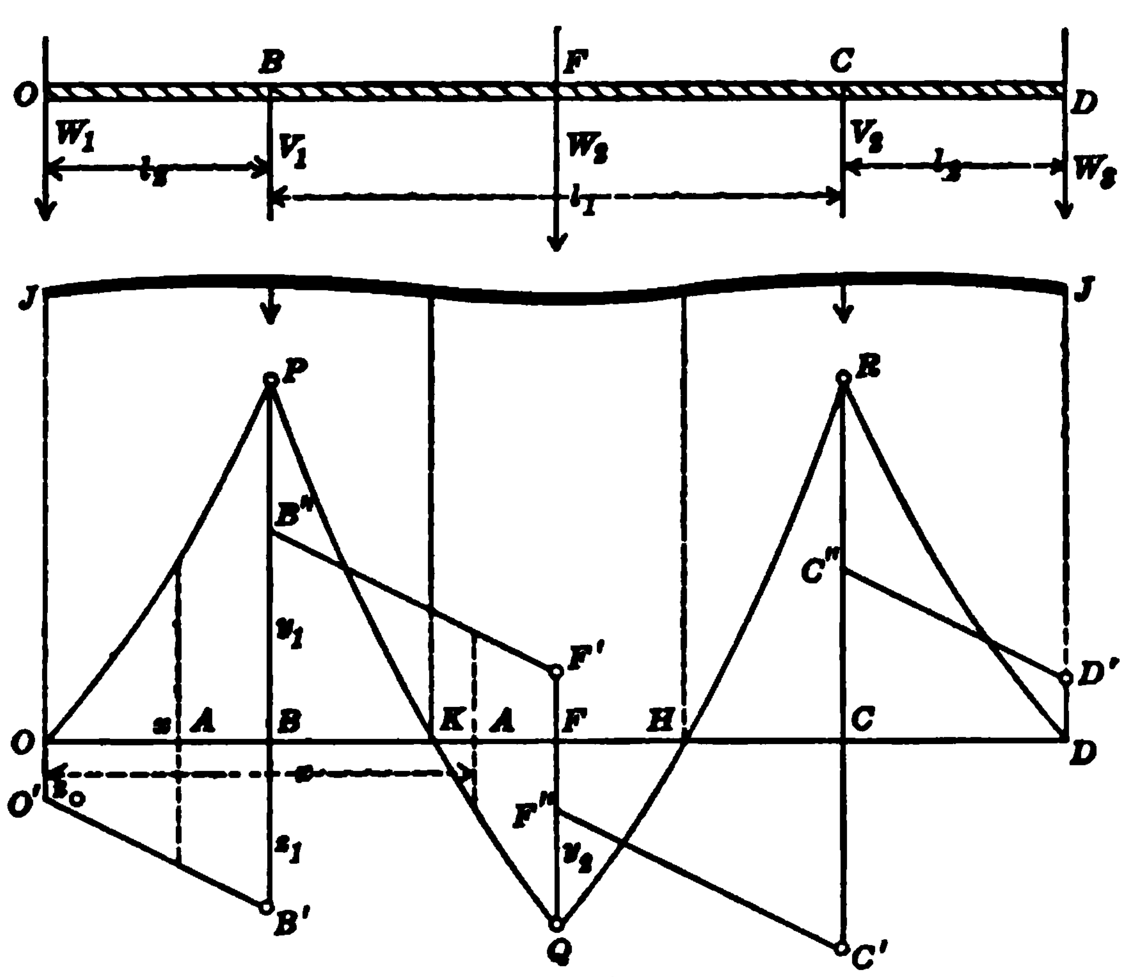


Fig. 382

Take the double cantilever beam $OBCD$ (Fig. 382) with supports at B and C, V_1 and V_2. Let there be a distributed load of magnitude w per foot; and suppose also a load W_1 is suspended at O; W_2 at the center between B and C; and W_3 at the end D. We will suppose that the end loads are equal, and that W_2 is the greatest. Let each cantilever have a length l_2, and the distance between the supports be l_1.

* In order to distinguish shear ordinates from moment ordinates, we will call the former z, z_0, etc., positive upwards. The moment ordinates y, y_0, etc., positive downwards.

Moment Equations.

The equation of OP is $y = -W_1x - \frac{wx^2}{2}$.

The equation of PQ is $y = -W_1x - \frac{wx^2}{2} + V_1(x - l_2)$.

The equation of QR is $y = -W_1x - \frac{wx^2}{2} + V_1(x - l_2) - W_2\left(x - l_2 - \frac{l_1}{2}\right)$.

The equation of RD is $y = -W_1x - \frac{wx^2}{2} + V_1(x - l_2) - W_2\left(x - l_2 - \frac{l_1}{2}\right)$
$+ V_2(x - l_1 - l_2)$.

Shear Equations.

Equation of $O'B'$, $z = -W_1 - wx$.
Equation of $B''F'$, $z = -W_1 - wx + V_1$.
Equation of $F''C'$, $z = -W_1 - wx + V_1 - W_2$.
Equation of $C'''D''$, $z = -W_1 - wx + V_1 - W_2 + V_2$.

The Significance of Shear Areas.

The length PB = Area of trapezoid OB'.
The length FQ = Area of trapezoid OB', *minus* area of trapezoid BF'.
The length CR = Area of trapezoid OB', *minus* area of BF', *plus* area FC'.
It should be further seen that

$$\text{Area } OB' + \text{Area } FC' = \text{Area } BF' + \text{Area } CD'.$$

A few words of explanation may be helpful. Beginning at O, where the moment is necessarily O, we take a point A at a distance x from O. We find the value of y to be

$$y = -W_1l_2 - \frac{wl_2^2}{2}$$

This negative moment shows that y must be drawn upward and that the curve is convex downward. The curve is a parabola and it continues until it meets the line of action of the support V_1 at P. At V_1 we have

$$y_1 = -W_1l_2 - \frac{wl_2^2}{2}$$

Passing to a point A between B and W_2 at a distance x from O we have

$$y = V_1(x - lz) - W_1x - \frac{wx^2}{2}$$

This is the equation of a parabola but a different parabola from the former one. It intersects W_2 at a point Q, whose distance below the beam is

$$y_2 = \frac{V_1 l_1}{2} - W_1\left(l_2 + \frac{l_1}{2}\right) - \frac{w}{2}\left(l_2 + \frac{l_1}{2}\right)^2$$

If this value turns out to be positive, the parabola crosses the beam as shown in the figure. If the loads and dimensions had been such as to make y_2 negative, then the point Q would have been *above* the beam, showing that *the beam* from end to end was concave downwards. As it is drawn, the beam is concave *downwards* from O to the point K where the parabola crosses the axis of the beam, is concave upward from K to H, where the next parabola crosses the axis of the beam, which is then concave *downward* to the end at D. See JJ in Fig. 382.

Since W_3 has the same value as W_1, the moment curves are symmetrical with respect to a vertical line thru the center.

The position of the greatest bending moment will be either in the center where W_2 is applied, or above the supports. The figure shows the greatest moments at the supports.

392. Ex. Let the student make his drawing and illustrate the last section using the following numerical data: Let $W_1 = W_3 = 2$ tons; let $W_2 = 15$ tons, let $w = \frac{1}{2}$ of a ton per foot; let $l_2 = 8$ feet and let $l_1 = 20$ feet.

In drawing the parabolic arcs the student should find in every case at least three intermediate points on each arc by assuming three separate values for x.

393. Having fully grasped all the explanations and instructions thus far, the student should assume numerical values for beams and loads, calculate all values, and make moment and shear diagrams to scale. A little practice in this way will make the matter very clear and it will seem to be very simple and easy. It should be noticed that in these diagrams representing moments and shear it is assumed that the beam is practically straight or but slightly bent, and no attention is paid whatever to the character of the beam, whether it is cylindrical, prismatic, wood or steel. We have been dealing simply with external forces and with scarcely a thought for internal stress. The most that we did in reference to stress was to note whether the upper fibers were in tension or in compression.

394. Varying loads. Considering now both the moment and the shear diagrams, we notice that under concentrated loads alone the moment diagram consists of a broken rectilinear figure and the shear diagram of stepped lines parallel and perpendicular to the axis of the beam.

On the other hand, with a continuous load uniformly distributed, the moment diagram consists of a parabola and the shear diagram of an inclined straight line. When the beam carries uniformly distributed loads and several concentrated loads, the moment diagram consists of a succession of parabolic arcs and the shear diagram is made up by a series of inclined straight lines as in Fig. 382.

Had the load been distributed differently but still according to some mathematical law depending upon x, both the moment diagram and the shear diagram would have been bounded by curves of a higher order. The mathematical order of the shear diagram is always one degree lower than the mathematical order of the moment diagram.

For example, suppose we have a simple beam OB, Fig. 383, with end supports V_1 and V_2. Suppose in addition to a uniform load of intensity w an additional load is put on which at any point has the intensity cx so that the intensity of this additional load will be zero at O and that it increases uniformly from O to B where it is cl. It is evident that

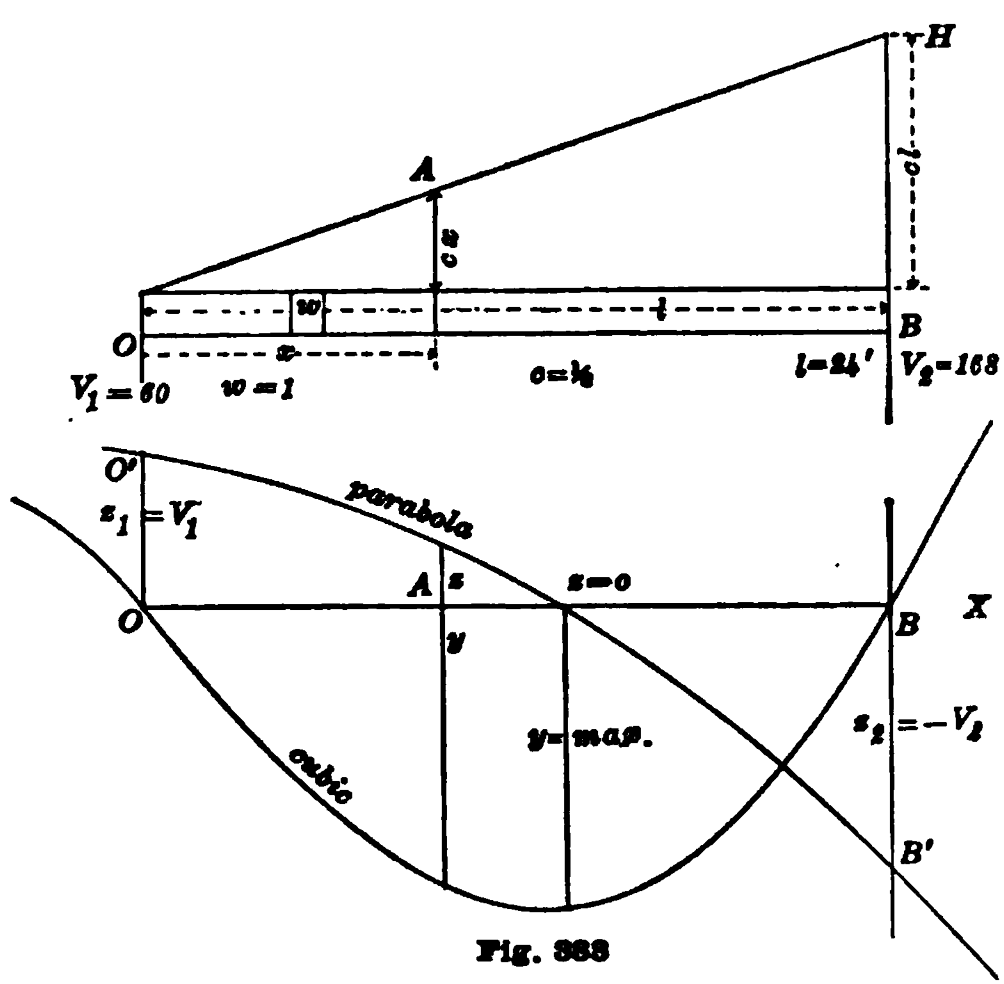


Fig. 383

$$V_1 = \frac{wl}{2} + \frac{1}{3}\frac{cl^2}{2},$$

and that

$$V_2 = \frac{wl}{2} + \frac{2}{3}\frac{cl^2}{2}.$$

Now, drawing the moment diagram with a single line denoting the axis of the beam, we have at the point A, a distance x from O, a moment

$$y = V_1x - \frac{wx^2}{2} - \frac{cx^3}{6}.$$

This is an equation of the third degree. The curve passes through both O and B, and is concave upward between those points. This curve is not symmetrical with respect to a vertical axis thru the center of the beam.

The shear diagram begins at O and is drawn vertically with a value $z_1 = V_1$. At A we have

$$z = V_1 - wx - \frac{cx^2}{2}.$$

This is the equation of a parabola which does not pass thru either O or B, but which does pass thru the points O' and B', has its axis vertical, and is concave downward.

It thus shows that the curve of the shear is one degree lower than the curve of moments, and the mathematical reader will be interested to see that the shear equation is the "first derived equation" from the moment equation. That is to say, if we differentiate the moment equation and find the value of $\frac{dy}{dx}$, we shall find that it is exactly equal to z of the shear equation. But $\frac{dy}{dx}$ measures the *Slope* of the Moment curve at the point x. Hence we see that while the *shear area* measures the *ordinate* to the moment curve, the *Shear* Ordinate measures the Slope of the Moment Curve.

If we carry this investigation one step further down and derive by differentiation a new equation from the sheer equation, in the case of the last problem, we find that we get $\frac{dx}{dz} = -w-cx$, which is exactly the *intensity* of the *load* at the *point* x.

Since the slope of the moment curve is measured by the ordinate to the shear diagram, it follows that when the moment is a *maximum*, the shear diagram must cross the axis of x. Hence *if we find where the shear diagram crosses the axis of x we have the point of maximum moment.*

395. An illustration. In this last example the equation of the shear diagram was

$$z = V_1 - wx - \frac{cx^2}{2}$$

If, now, we let $\quad w=1T,\ c=\frac{1}{2}T,\ l=24,$ we have $V_1=60,$

and

$$z=60 - x - \frac{x^2}{4}.$$

Since z must be 0 where the shear curve crosses the axis of X, we have, solving the quadric, $x=+13.6$ or -17.6.

This positive value of x substituted in the moment equation will give the *maximum moment in the beam.* This moment equation is

$$y=M=60x-\frac{x^2}{2}-\frac{x^3}{12}.$$

The *negative* value of x gives the position (off the beam) whence a negative ordinate to the "*Cubic*" gives an algebraic *minimum.*

It may be useful to the student to know that the point of maximum

moment in the above beam is not the point under the center of gravity of the load.

The use of the shear diagram for finding the points in a beam where the moment is a maximum should never be overlooked.

396. Equilibrium polygons. It has just been shown that both bending moments and shearing stresses can be clearly represented by diagrams, formed by curves or broken straight lines. Another and a much more useful method of showing at a glance the bending moment in a beam, arch or truss, is by means of the Equilibrium Polygon, explained in **89**.

Suppose we have a system of vertical concentrated loads, $F, F_1, F_2 \dots F_4$, which are to be borne by a beam resting on two supports L and X. Fig. 384, (a) and (b), represents the force diagram and the equilibrium polygon complete. The moment at any point in the beam, as Q, is equal to the (tension eo) times QN_1 (its arm), *minus* the thrust ao times QS (its arm). But the triangle QNN_1 is similar to the triangle EOH, in (b), hence tension in $eo = H \sec \theta$ and $eo.QN_1 = H.QN_1 \sec \theta = H.QN$.

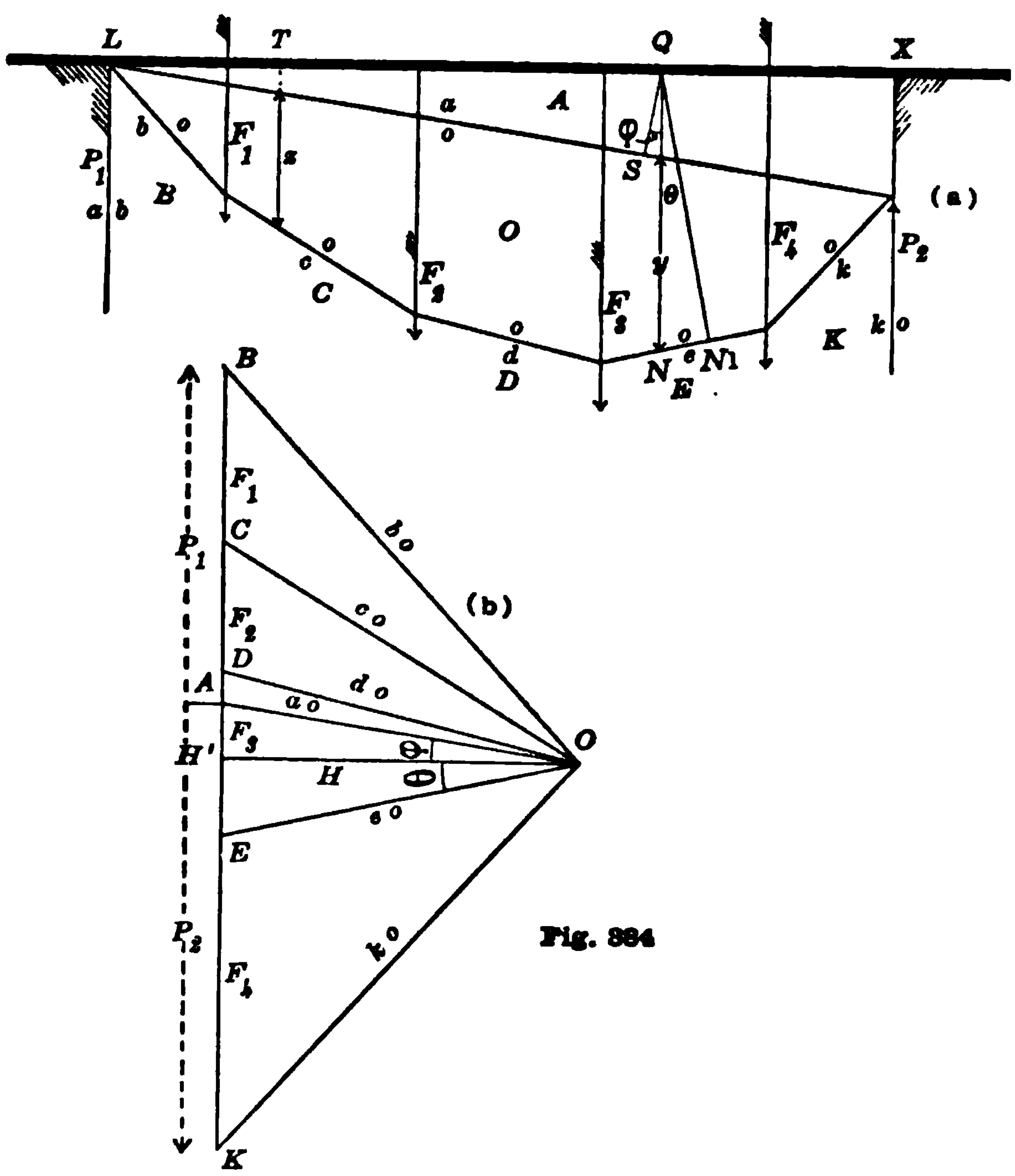


Fig. 384

In like manner the moment $ao.QS_1 = H \sec\phi.QS_1 = H.QS$, hence

$$M = H(QN - QS)$$
$$M = H(SN) = Hy$$

Similarly, the moment at the point T is

$$M_T = Hz$$

The force H is known as the "*Horizontal thrust*" (or "tension") of every link in the chain, and its length in the force diagram is called the "*pole distance.*"

The rational of the above is as follows: For the purpose of ascertaining the stresses in the beam, we, in imagination, substitute a closed chain in its place, having the same loads and the same supports. The chain shown in Fig. 384 has five links in tension and one in compression. At any point O, the bending moment of the external forces acting on the beam, is balanced by the resisting moments of its internal stresses. Now the moment of the loads with reference to a point is the same whether the loads act on the beam, or at joints of the chain, and the balancing moment is easily found by means of the tension in the *lower link*, and the thrusts in the *upper link*, since the magnitudes of their stresses are given in the force diagram.

Since both H and y are found graphically, and since the factor H is constant for a given chain, the *intercepts on vertical lines multiplied by H* give the bending moments at corresponding points on the beam. Thus the chain polygon becomes in effect the bending moment diagram. H is given in units of force (according to the scale used in the force diagram) and y is given in units of length (according to the scale used for the length of the beam).

In the case of a distributed load, the chain becomes a cord which forms a curve. However, trusses are always loaded at joints, and even a distributed load on a beam may, without sensible harm, be assumed as concentrated at frequent points.

397. In subsequent uses of the equilibrium polygon, it will be convenient to transform it to an equivalent figure having the closing link horizontal. Thus, Fig. 385, having the polygon $OZCD$, it is evident that so far as values of y and areas are concerned, the figure $OXC'D'$ is equally representative and more convenient.

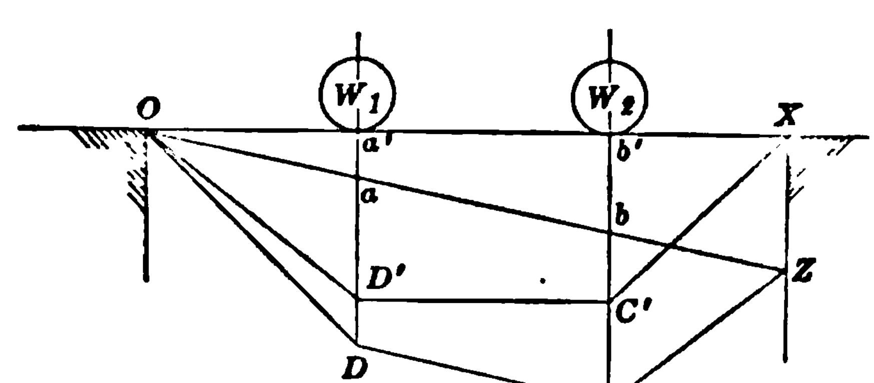


Fig. 385

This transformation must be shown to be legitimate. It was distinctly stated that the "*pole*" O, of the force polygon could be taken at random. Now, having so taken it, we have drawn the force polygon and found the supports V_1 and V_2. Let us now assume a new pole, so that the line $O'A$ is a horizontal line, and draw a new equilibrium polygon, whose compression member, ao, will coincide with the axis of the unbent beam. If $O'A = H$, we have the transformed figure given in Fig. 385.

If now we go one step further and make $O'A = H = Unity$, according to some convenient scale, the bending moment at W_1 will be numerically represented by the ordinate $A'D'$, and our equilibrium polygon becomes the *Curve of Moments*, due to the given loads, as shown in **385.**

Before showing the value of the above discussion, some of the relations between the elastic curve of a bent beam and the curves of its *derived equations* must be pointed out.

398. Derived equations and graphic relations between their curves and areas. It has been seen that the elastic curve of a loaded beam usually consists of one or more curves of the 3d or 4th order, the ordinates to which are known as Deflections, which in this book have generally been represented by z. The equation of a single elastic curve takes the form

$$EIz = f(x) \tag{1}$$

The quantities E (Modulus of Elasticity) and I (Moment of Inertia) are here regarded as constants.

The first-derived equation is

$$EI\frac{dz}{dx} = f'(x)$$

If the second member is placed equal to a new ordinate z_1, we have the equation

$$z_1 = f'(x) \tag{2}$$

and a curve, every ordinate to which represents $EI \tan\theta$, in which $\tan\theta$ is the slope of the elastic curve at the point (x).

The second derived equation is

$$EI\frac{d^2z}{dx^2} = f''(x) = M = z_2$$

in which M is the *Bending Moment* applied to the beam at the point (x). The equation

$$z_2 = M = W''(x) \tag{3}$$

gives the *Moment Curve.*

The third derived equation is

$$EI\frac{d^3z}{dx^3} = f'''(x) = S. = z_3. \tag{4}$$

in which S is the *shear* at the point (x). The shear equation, and its graphical representative gives the line or curve of Shear in the beam.

The above statements are not wholly new to the reader, but the full significance of the areas involved in the drawings may be new.

399. In studying a real or *imaginery* beam, we generally begin with the Shear, or Bending Moment at the point (x), and develop the higher equations by integration. If we start with the shear equation, we multiply by dx, and integrate, and get

$$EI\frac{d^2z}{dx^2} = \int z_3 dx = M = \text{Area } A_1 = z_2,$$

since z_3dx is an elementary area reaching to the line or curve of Shear. So that the ordinate z_2 is measured by an Area.

This relation between the moments and the shear diagram was found in **388**.

In like manner when we integrate again we get

$$EI\frac{dz}{dx}=\int z_2dx=z_1=\text{Area}A_2$$

of the Moment diagram, which area measures the value of an ordinate to the *Curve* of *Slopes*.

Finally we integrate again and get

$$EIz=\int z_1dx=A_3,$$

an area of the *slope diagram*, which measures the value of the Deflection of the elastic curve of the loaded beam.

We thus see that if the Shear diagram is correctly drawn, the *moment* at (x) is seen at a glance from the shear area; that if the moment curve is correctly drawn we see at a glance the *slope* as measured by a moment area; and finally, if the slope curve is correctly drawn, we see at a glance the *deflection* as measured by an Area in the slope diagram. This much will now be illustrated.

400. A simple case will be a cantilever beam carrying a uniform load. With axes as shown, Fig. 386, the shear at P_1 is given by the equation

$$EI\frac{d^3z}{dx^3}=S=-w(l-x)=z_3$$

The *locus* of P_1, the point (x, z_3) is the straight line CB, Fig. (a). The shear acts up and is therefore negative, forming a right-handed couple with the weight $w(l-x)$.

Multiplying by dx and integrating, we have

$$EI\frac{d^2z}{dx^2}=M=z_2=w\int(l-x)dx$$

$$=+\frac{w}{2}(l-x)^2+(K=0)=A_1$$

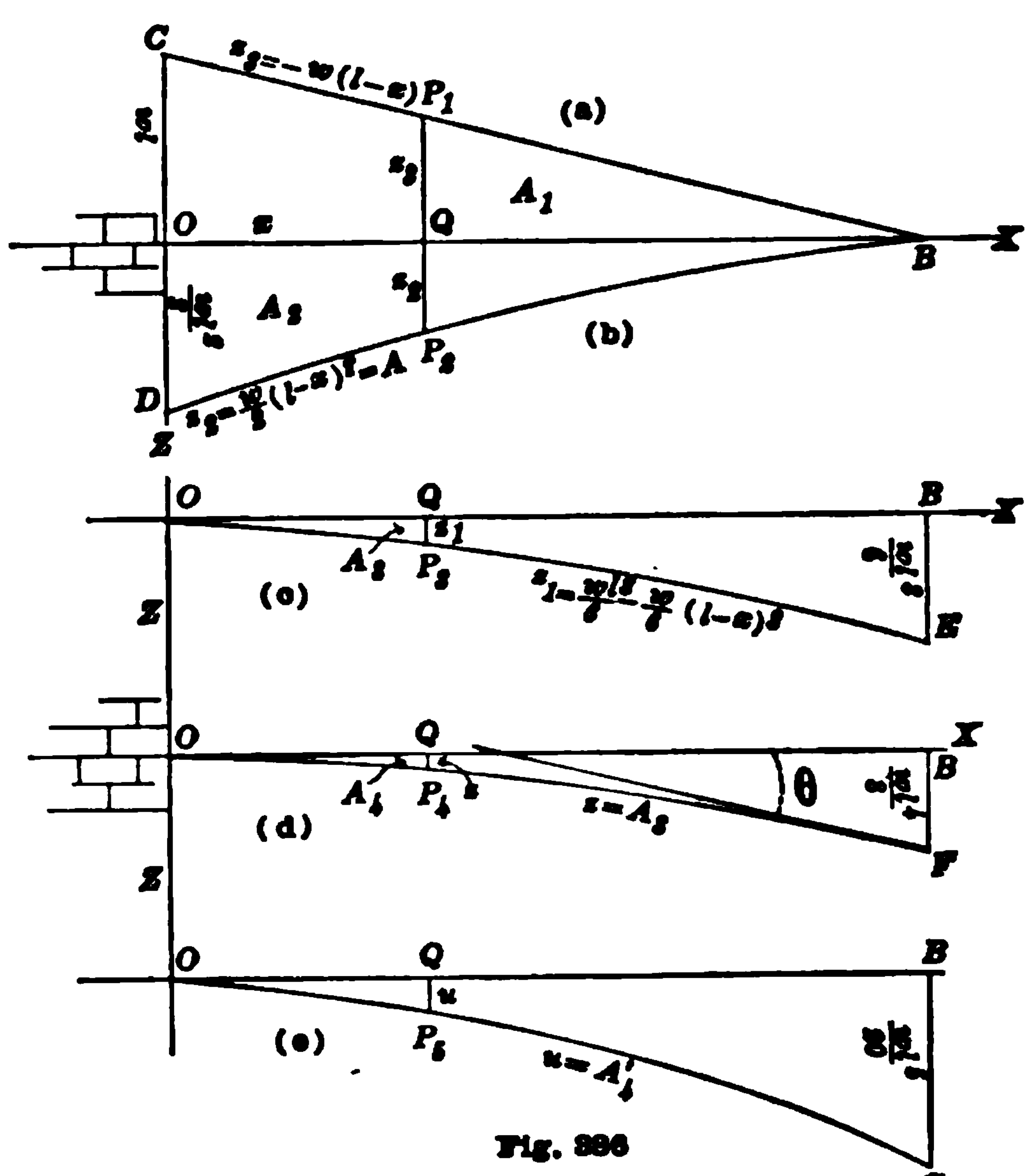


Fig. 386

since $M=0$ when $x=l$, and the area A_1 is the triangle BP_1Q which measures the ordinate z_2 in Fig. (*b*). The moment at O is the area of the entire shear triangle; hence $M_o=\dfrac{wl^2}{2}$.

Integrating again we get

$$EI\frac{dz}{dx}=-\frac{w}{6}(l-x)^3+K'=\frac{wl^3}{6}-\frac{w}{6}(l-x)^3=z_1$$

$$EI\frac{dz}{dx}=\frac{w}{6}(3l^2x-3lx^2+x^3).$$

The slope $\dfrac{dz}{dx}$ is zero when $x=0$, hence $K'=\dfrac{wl^3}{6}$.

Now $\dfrac{wl^3}{6}$ = the area of the entire spandril to the parabola BP_2DO, which is $\dfrac{1}{3}\cdot\overline{BO}\cdot\overline{OD}$; and $\dfrac{w}{6}(l-x)^3$ is the area of the part spandril BQP_2; hence $EI\dfrac{dz}{dx}=A_2=$ area of the figure QP_2DO, the area bounded by the moment ordinate at (x), and the moment ordinate where $\dfrac{dz}{dx}=0$.

Equation for Slope is represented by the curve OP_3E, Fig. (*c*), which is a cubic parabola; its greatest ordinate when $x=l$, is

$$EI\tan\theta_l=\frac{wl^3}{6}=\text{the entire moment area } BDO, \text{ Fig. } (b).$$

$$\tan\theta_l=\frac{wl^3}{6EI}$$

The angle θ_l is shown in Fig. (*d*).

Integrating again, we have

$$EIz=\frac{wl^3x}{6}+\frac{w}{24}(l-x)^4-K''.$$

When $x=0$, $z=0$; and $K''=-\dfrac{wl^4}{24}$, hence

$$EIz=\frac{wl^3x}{6}+\frac{w}{24}(l-x)^4-\frac{wl^4}{24}=\int_0^x z_1dx=\int_0^x\left[\frac{wl^3}{6}-\frac{w}{6}(l-x)^3\right]dx=A_3$$

which is the equation of the axis of the bent beam. Every ordinate to the curve measures the deflection of the curve at the point corresponding to the ordinate. The maximum deflection is found by making $x=l$, so that

Max. $\Delta=\dfrac{wl^4}{8EI}$ = the entire slope area OBE divided by EI.

401. The work done in bending the beam. We can carry our analysis one step further by multiplying our deflection z by wdx and integrating. The quantity wdx is an element of the distributed load, and the product $wzdx$ measures the *Potential Energy exerted* by the load at that point during the bending of the beam.

Hence $$d(P.E) = wzdx;$$

but only *one-half* of this potential energy is spent in bending the beam; the other half is spent in *over-coming a yielding support* which compels the beam to *come to rest* when the support vanishes. It follows that the *work done* on the beam, or rather *in* the beam, is

$$U^l_o = \frac{w}{2}\int_o^l zdx = \frac{w^2}{2EI}\int_o^l \left(\frac{l^3x}{6} + \frac{(l-x)^4}{24} - \frac{l^4}{24}\right)dx$$
$$= \frac{w^2l^5}{40EI}$$

as will be found in Chapter XXIII by a very different method.

402. Another relation of interest and value between ordinates and areas must not be neglected. The deflection, at the point (x), is measured numerically not only by the *Slope Area* A_3, but by the *moment* of the *moment area* A_2. Take the equations of **399** and Fig. 386. The deflection at B was found to be

$$BF = \frac{wl^4}{8EI}$$

The total area of the moment curve, Fig. (b), was the spandril

$$BDO = \frac{1}{3}\cdot\left(\frac{wl^2}{2}\cdot l\right) = \frac{wl^3}{6};$$

and its *moment* about B is

$$\frac{wl^3}{6}\cdot\frac{3}{4}l = \frac{wl^4}{8} = EI\cdot\Delta.$$

Again , and similarly, the *greatest slope* at F, Fig. (d), is

$$EI\tan\theta_l = \frac{wl^3}{6}.$$

which is equal to the *moment* of the *shear triangle BOC* about OC;

Finally, the greatest ordinate in the *Work diagram* (e), Fig. 386, is

$$EI\cdot U \div \frac{w}{2} = \frac{wl^5}{20}$$

which is equal to the *Deflection Area OBF*, Fig. (*d*), and to the moment of the *Slope Area OBE*, about the line, *BE*.

Similar propositions are true for the Slope, Deflection and Work at the point (x).

403. General proof of the proposition illustrated in the last section.

$$\text{Let } z = f(x)$$
$$z_1 = f'(x)$$
$$z_2 = f''(x)$$

be three consecutive members of a series of derived equations, and let their *loci* be represented by curves (*a*), (*b*), (*c*), in Fig. 387.

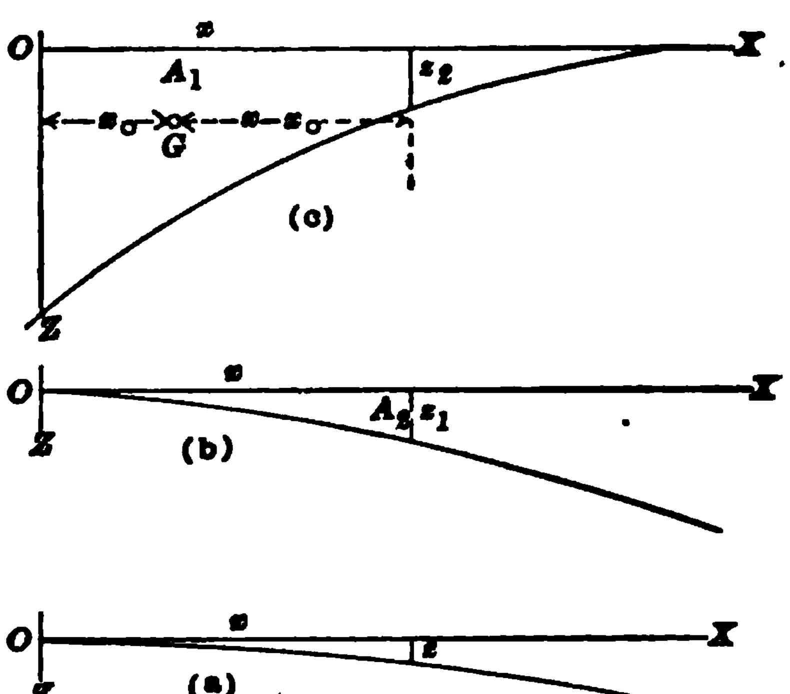


Fig. 387

$$z_1 = \int_o^x z_2 dx = A_1$$

$$z = \int_o^x z_1 dx = A_2$$

$$\text{Now } \int_o^x z_1 dx = \int_o^x A_1 dx$$

Integrating the second member "by parts,"

$$\int_o^x z_1 dx = A_2 = z = A_1 x - \int_o^x x dA_1$$
$$= A_1 x - A_1 x_o$$
$$= A_1(x - x_o).$$

Hence $z =$ Moment of A_1 about the ordinate z_2 at the point (x),

which proves the proposition used in the last section.

404. A second proof. It may be more satisfactory to the student to *see* the *reasonableness* of the conclusion just reached, and accordingly the following proof is given in the case of the Deflection of a simple beam under a uniform load.

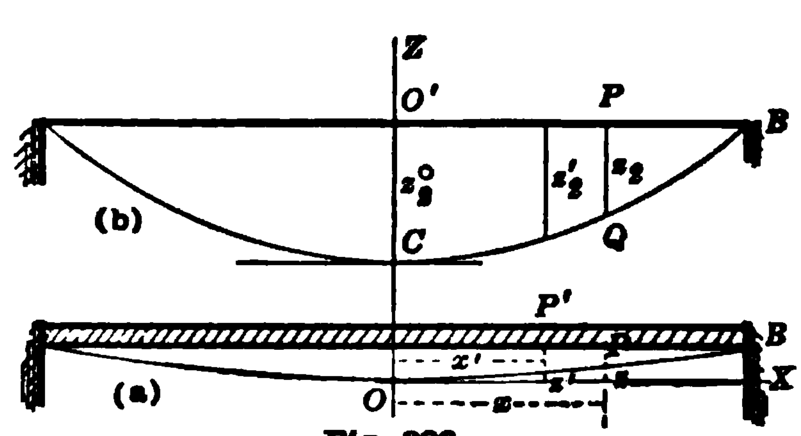


Fig. 388

The deflection at P, Fig. 388 (*a*), from the tangent at O where it is

horizontal, is due to an elementary bending at *every point* between O and P. Let x' be the abscissa of a point between O and P. The moment at P' is z_2', and the change of direction of the tangent for dx' is $d\theta$, and the deflection at *due* to that $d\theta$ at P' is $(x-x')d\theta$; so that

$$dz=(x-x')d\theta$$

Now, $\dfrac{dz}{dx'}=\tan\theta$ at every point,

and
$$\frac{d^2z}{(dx')^2}=\frac{d(\tan\theta)}{dx'}=\sec^2\theta\frac{d\theta}{dx'}.$$

But we are dealing with beams originally straight, and very nearly straight when bent; hence the difference between $\sec^2\theta$ and unity is negligible; hence we put $\sec^2\theta=1$, and write

$$\frac{d^2z}{(dx')^2}=\frac{d\theta}{dx'}$$

But
$$\frac{d^2z}{dx'^2}=\frac{M}{EI}=\frac{z_2'}{EI}=\frac{d\theta}{dx'}$$

$$d\theta=\frac{z_2'dx'}{EI}.$$

So that
$$dz=\frac{(x-x')z_2'dx'}{EI}$$

and
$$\Delta=z=\frac{1}{EI}\int_0^x(x-x')z_2'dx'$$

which is the *moment* of the *moment area* about the ordinate at (P).

This theorem is convenient if the moment curve is already drawn and the deflection is required. Take for example the simple beam and its shear and moment diagrams, shown in Fig. 388.

The deflection at B from the tangent at O, Fig. (a), is equal to the moment about B of the half parabolic segment $O'CB$, Fig. (b).

We have the *area* of the half-segment

$$\frac{2}{3}z_2{}^o\cdot\frac{l}{2}=\frac{wl^3}{24}.$$

The centroid of the surface is distant horizontally from B, $\dfrac{5}{8}\cdot\dfrac{l}{2}$.

Hence we have at once

$$\Delta=\frac{wl^3}{24}\cdot\frac{5l}{16}=\frac{5wl^4}{384EI},$$

as found in Chapter XIX.

405. Related Problems for beams under concentrated loads, or under a Distributed and one or more Concentrated Loads.

1. A *Cantilever* with a *single load* at the end. Let the student draw the shear and moment diagrams and show that

$$A_1 = W(l-x)$$
$$A_2 = W\left(\frac{l^2}{2} - \frac{(l-x)^2}{2}\right)$$

and that the maximum deflection is

$$\Delta = W\frac{l^3}{3EI}$$

and that the internal work done by the load in bending the beam to a state of rest is

$$U = \frac{W^2 l^3}{6EI}.$$

2. If the cantilever be bent by a uniform load, show: (*a*) That the moment area for the center is the *spandril* of a half segment of a parabola; (*b*) That the slope at the end is $\frac{Wl^2}{6EI}$; (*c*) that the deflection at the end is $\Delta = \frac{Wl^3}{8EI}$.

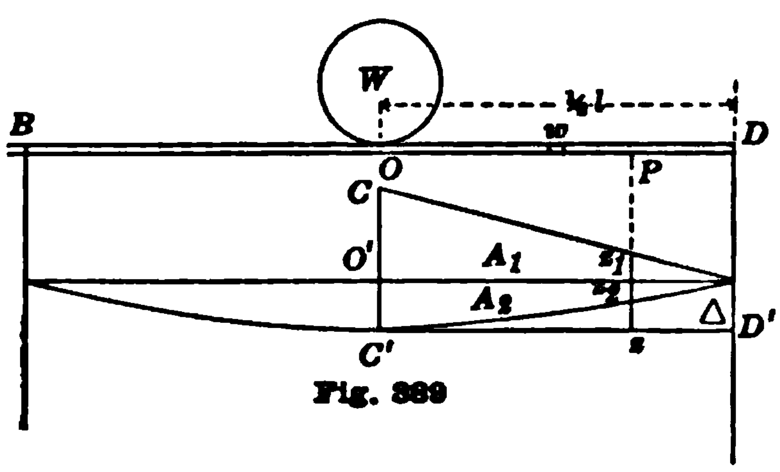


Fig. 389

The student should show in a drawing the shear and the moment areas, and use them in arriving at the above results.

3. A simple beam has a concentrated load W at the center, and a uniform load. (Fig. 389.)

(*a*) The *Moment* at the center due to W is $\frac{Wl}{4} = O'C$.

The *moment Area* up to D due to W is the triangle $O'CD' = A_1 = \frac{Wl^2}{16}$.

The *moment* of A_1 about an axis at D is

$$A_1 \times \frac{2}{3} \cdot \frac{l}{2} = \frac{Wl^3}{48}$$

This is $EI\Delta_1$, the deflection of the point D due to the load W, from the tangent at O.

(*b*) The Moment at the center due to the uniform load is

$$O'C' = \frac{wl^2}{8}$$

The moment *Area* up to the point D is the half parabolic segment

$$O'C'D' = A_2 = \frac{2}{3}\cdot\frac{wl^2}{8}\cdot\frac{l}{2} = \frac{wl^3}{24}.$$

The Moment of the moment *Area*, about an axis at D is

$$\frac{5}{8}\cdot\frac{l}{2}\cdot A_2 = \frac{5wl^4}{384} = EI\Delta_2$$

Hence $$\Delta = \Delta_1 + \Delta_2 = \frac{l^3}{EI}\cdot\left(\frac{W}{48}+\frac{5wl}{384}\right)$$

which results agree with the table **375**, if l is put for $\frac{l}{2}$.

406. When there are two or more concentrated loads. Let a simple supported beam carry three concentrated loads. (Fig. 390.)

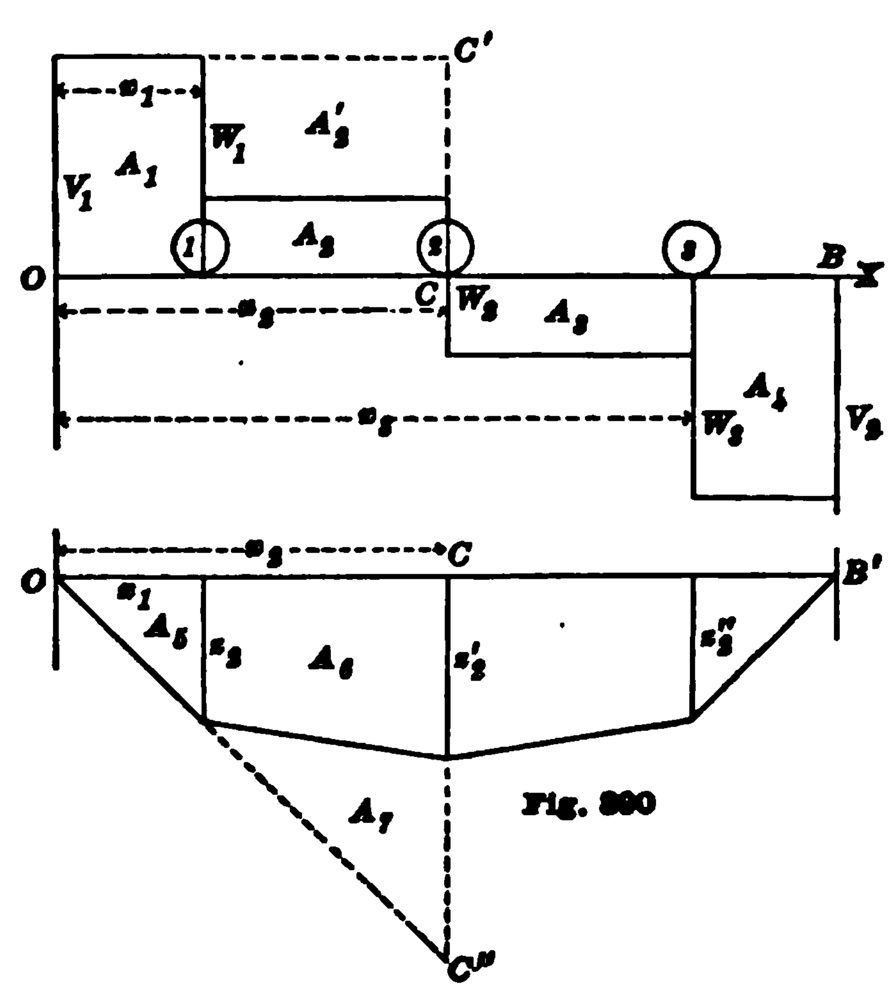


Fig. 390

The moment at W_1 is

$$z_2 = \int_0^{x_1} S dx = V_1 x_1 = \text{Area } A_1$$

The moment at W_2 is

$$z_2' = \int_0^{x_1} S dx + \int_{x_1}^{x_2} (V_1 - W_1) dx$$
$$= A_1 + A_2$$

The moment at W_3 is

$$z_2'' = A_1 + A_2 + \int_{x_2}^{x_3} (V_1 - W_1 - W_2) dx$$
$$= A_1 + A_2 - A_3$$

An area below the X-axis is of course negative.

The deflections at W_1 and W_2, by the rule derived above, are as follows:

Deflection at W_1 = [Mom. of A_5 about z_2] $\div EI$.

$$= \frac{A_5}{EI}\cdot\frac{x_1}{3} = \frac{V_1 x_1^3}{6EI}$$

As measured from the tangent at W_1.

Deflection at W_2 = [Mom. of A_5 and A_6 about z_2'] $\div EI$

$$= \frac{V_1 x_1^2}{2EI}\left(x_2 - \frac{2}{3}x_1\right) + \frac{V_1 x_1}{2EI}(x_2 - x_1)^2 + \frac{(V_1 - W_1)(x_2 - x_1)^3}{6EI},$$

$$= \frac{V_1 x_2^3 - W_1 (x_2 - x_1)^3}{6EI}.$$

A careful examination of the figure shows that the term $\frac{V_1x_2^3}{6EI}$ measures the deflection of the point O (as compared with the tangent at C) *due to the reaction* V_1; that $CC'' = +A_2+A'_2 = V_1x_2$, and that the moment of the triangle OCC'' about CC'' is $\frac{V_1x_2^3}{6EI}$.

Moreover the negative term $\frac{W_1(x_2-x_1)^3}{6EI}$ is the *negative* deflection, due to the *negative* shear area A'_2, and the *moment* of the *negative* area A_7.

These results illustrate a very important principle which must be clearly stated:

When a straight beam, originally horizontal, is given a partial load, its deformation is so small that the addition of another partial load produces an additional deformation or stress, just as large as it would have produced had the beam been wholly unloaded.

Hence when there are two or more partial loads on a beam, we can compute the shear, moment and deflection at the point (x) for each load separately, and get the algebraic sum of the results.

For example: If a cantilever has both a uniformly distributed load and at the end a concentrated load, the deflection at the end is

$$\frac{wl^4}{8EI} = \frac{Wl^3}{3EI}.$$

It will of course be noted that there will, in the case of a beam supported at both ends, be a component of V_1 for each partial load, but every component will have the same co-efficient so that they can be added at once as in the example above with three loads;

$$V_1 = V' + V'' + V'''.$$

A similar statement can be made with reference to the moments at the fixed ends if there are any; their action upon moments and deflections along the beam can be represented separately. This must be examined carefully.

407. Take a simple beam having moments at the ends, and *no loads.*

For the sake of generality, let the moments be unequal, and let M_1 have the greater magnitude. Let V_1' and V_2' be the necessary supports, but let there be no proper *load* on the beam.

The moment at (x) is $$M = V_1'x - M_1 \qquad (1)$$

If $x=l$, the moment V_1l-M_1 is *balanced* by M_2, hence

$$V_1'l-M_1+M_2=0$$

$$V_1'=\frac{M_1-M_2}{l}.$$

Similarly, when $x=0$, the Moment $-M_1$ is balanced by $M_2-V_2'.l$; hence

$$M_2-V_2l-M_1=0$$

$$V_2'=-\frac{M_1-M_2}{l}=-V_1'$$

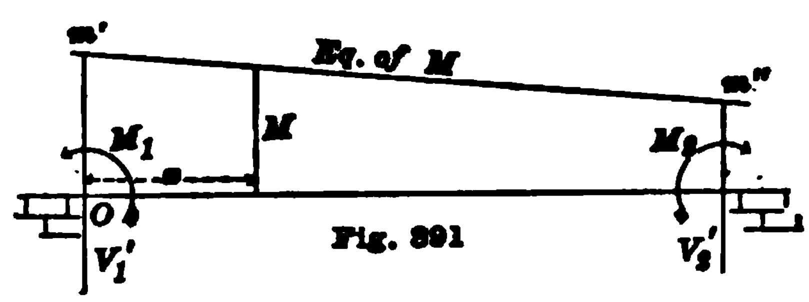


Fig. 291

This is as should have been expected, since the *greater* moment M_1 would *lift* the end B were it not chained down.

Substituting for V_1' in (1) we get the final equation for moments

$$M=\frac{(M_1-M_2)x}{l}-M_1=-\frac{M_2x}{l}-\frac{M_1(l-x)}{l}$$

which shows that M is always negative.

If $M_1=M_2$, we see that $V_1=0=V_2$ and that M is always numerically equal to the common value of M_1 and M_2.

The moment area is the trapezoid $\frac{M_1+M}{2}\cdot x$. Of course the beam is bent and the deflection at (x) from the tangent at O is measured by the moment of the trapezoid about the ordinate at x. That is

$$\Delta=\frac{x^2}{6EI}(2M_1+M_2).$$

If $M_1=M_2$, M is constant, and numerically equal to M_1 and M_2, and the elastic curve of the beam is the arc of a circle, and the deflection for the beam at one end compared with the tangent at the other end is

$$\Delta=\frac{l^2M}{2EI}.$$

This agrees with the algebraic results got by integration supposing the tangent at one end be taken as the axis of X.

$$EI\frac{d_2z}{dx^2}=M(\text{constant})$$

$$EI\frac{dz}{dx}=Mx+0$$

$$EIz=\frac{Mx^2}{2}+0.$$

If $x=l$, $$z=\frac{Ml^2}{2EI}, \text{ as above.}$$

It must *not* be forgotten that all of these results are true only when the bent beam is so nearly straight that ds is practically equal to dx.

408. The student must not suppose that all the uses of the equilibrium and moment polygons have been illustrated, as he will find if, after finishing this book, he takes up the work of Professor Turneaure already referred to. This discussion closes with a short quotation from the scholarly Dean.

"A thorough understanding of the graphical method of analysis here explained will often enable the desired values of deflection or angular change to be written out by inspection. It is a convenient method of analysis for continuous girders, arches and many other problems involving the deflection of beams."—p. 6, Part II. *Statically Indeterminate Structures and Secondary Stresses. Modern Framed Structures.*

INFLUENCE LINES.

409. It is not the intention of the author to deal with the numerous and important uses of *Influence Lines*, in connection with the various effects produced by moving loads upon a beam or truss. In a book intended for all students of Engineering and Architecture, only a few definitions can be given and illustrated. A full discussion of their many uses belongs to that department of civil engineering known as Bridge Engineering.

Thus far in this book graphical representations of shear, bending moments and deflections, have been shown by an ordinate, at a point on a beam, or truss, produced by forces acting at other points; the student will recall shear diagrams, moment diagrams and equilibrium polygons, and the curves of elastic beams.

An influence line is quite a different thing. It shows the effects at a *certain fixed point* of a beam or truss, by a load (or deforming force) occupying successively *every possible point on the structure*, and this effect is shown, *not* by an ordinate at the fixed point, but by ordinates drawn successively thru the points where the load acts. Suppose the influence line is to show the *bending moment at C* caused by a load P moving from B to A along the beam AB. Fig. 392.

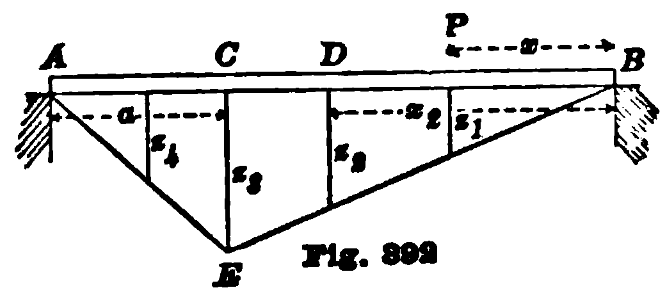


Fig. 392

Let the numerical value of P at the point x be *one* (lb. or ton).

Then the *support* at A is $\frac{x}{l}P=\frac{x}{l}$, and the *moment at* C is $M=\frac{ax}{l}\cdot$ At P drop an ordinate $z_1=\frac{ax}{l}\cdot$ When the load P reaches D, the ordinate becomes $z_2=\frac{ax_2}{l}\cdot$

When P reaches C, the ordinate becomes $z_3=\frac{ax_3}{l}\cdot$

It is evident that the ordinates end in a straight line BE whose equation is $z=\frac{ax}{l}\cdot$ After passing C, the ordinate z_4 ends in a second straight line EA, whose equation is $\frac{l-a}{l}(l-x)=z$, which becomes $\frac{a'}{l}x'=z$, if we let $l-a=a'$ and $l-x=x'$. The two straight lines which contain BE and AE are readily drawn as their respective intercepts on the support lines are a and a'. The broken line BEA is the *influence line* showing the moment at C for a unit load at every point from B to A or from A to B. If instead of a unit load, the load had W units, the moment at C would have been $M_C=Wz=\frac{Wa}{l}x$. Had there been two or three moving loads, between B and C, the moment at C would have been the sum of the moments due to the loads separately, thus:

$$M_C=W_1z_1+W_2z_2+W_3z_3$$

$$=\frac{a}{l}[W_1x_1+W_2x_2+W_3x_3]$$

Had there been loads between C and A_1, they would have increased the moment at C by the quantity $\frac{a'}{l}\Sigma Wx'$.

Hence, generally, $$M_C=\frac{a}{l}\Sigma Wx+\frac{a'}{l}\Sigma Wx'.$$

Example.

Suppose $l=24$ feet, and $a=6$ feet, and $a'=18$ ft.

Let $W_1=600$ lbs., $x_1=8$ ft., whence $z_1=2$, $W_1z_1=1200$

$W_2=1000$, $x_2=15$, whence $z_2=\frac{15}{4}$, and $W_2z_2=3750$

$W_4=800$, $x'_4=3$ ft. whence $z_4=\frac{9}{4}$, and $W_4z_4=1800$

$$M_C=\Sigma Wz=6750 \text{ foot lbs.}$$

It is a simple matter to find by trial the maximum moment at C, as the three loads $W_1W_2W_4$ pass over the beam. When W_2 reaches C, W_4 is on the support at A and causes no moment, and we have

$$x_1=11, \text{ and } z_1=\frac{11}{4}, \quad W_1z_1=1650$$

$$x_2=18, \text{ and } z_2=\frac{9}{2}, \quad W_2z_2=4500$$

$$M_C=6150$$

Again suppose W_4 is at C, and the distances between loads remain as before.

$$x_1=\ 5,\ z_1=\frac{5}{4}, \qquad W_1z_1=750$$

$$x_2=12,\ z_2=3, \qquad W_2z_2=3000$$

$$x_4=18,\ z_4=\frac{9}{2}, \qquad W_4z_4=3600$$

$M_C=7350$, which is clearly the maximum.

With accurate drawings the value of any z may be measured with sufficient accuracy.

410. An influence line may represent the extreme fiber stress at a fixed point as a load or loads pass over the beam. To take the simplest case, suppose the beam to be prismatic. Then, since $M=\frac{pI}{c}$, the stress varies directly as M.

That is $\quad \frac{pI}{c}=\frac{Wax}{l}$, and henee p (at C) $=\frac{ca}{Il}\cdot Wx.$

If there be several loads on the same side of the point C (Fig. 332).

$$p=\frac{ca}{Il}\Sigma(Wx)$$

If a part of the loads are on the other end of the beam

$$p\,(\text{at } C)=\frac{c}{Il}\left[a\Sigma(Wx)+a'\Sigma(Wx')\right]$$

in which x' is measured from the left-hand support, and $a'=l-a$.

411. A continuous load. The use of Influence Lines for the Moment at a fixed point (at P) due to a continuous load. (Fig. 393).

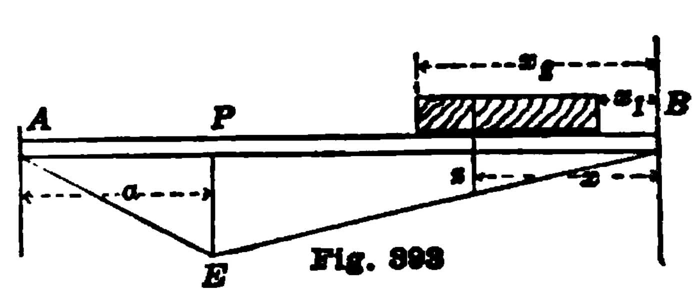


Fig. 393

Let w be the intensity of the load

per foot. AEB is the influence line for a unit load at the point x; $z = \frac{a}{l}x$. For a load wdx the moment at P is $dM = \frac{aw}{l}xdx$. For the whole load

$$M = \frac{aw}{l}\int_{x_1}^{x_2} xdx = \frac{aw}{2l}(x_2^2 - x_1^2)$$

This reduces to

$$M = \frac{a}{l}\,\frac{w(x_2-x_1)}{2}\left(x_2+x_1\right) = \frac{aW}{l}\cdot x_0$$

in which x_0 is the distance of the center of gravity from B, and M is the moment at P.

If the load be continuous over the entire span, integrate on the right from 0 to a', and on the left from 0 to a; thus:

$$M\,(\text{at } P) = \frac{aw}{l}\int_0^{a'} xdx + \frac{a'w}{l}\int_0^{a} x'dx'$$

$$= \frac{w}{2l}\,[aa'^2 + a'a^2]$$

$$= \frac{waa'}{2}$$

The area of the triangle ABE is

$$\text{area } ABE = \frac{l}{2}\cdot\frac{aa'}{l} = \frac{aa'}{2}$$

hence the moment at P is equal to the area of the influence triangle for P multiplied by the load per linear unit.

412. Influence line for shear. Fig. 394. With a unit load at x, the shear at P is the same as the support at A, viz.: $\frac{x}{l} = z$. It increases uniformly as x increases. When next to P it becomes $\frac{a'}{l}$. As it passes P it changes sign to $1 - \frac{a'}{l} = \frac{a}{l}$, and the broken line $BEE'A$ is the influence line of shear at the point P.

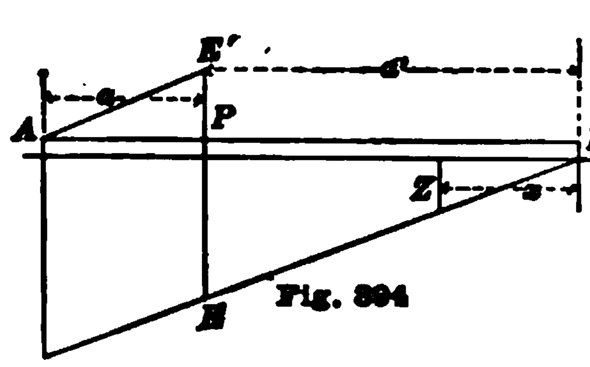


Fig. 394

The shear at P due to several loads on the beam, Fig. 395 is,

$$S(\text{at } P) = W_1z_1 + W_2z_2 - W_3z_3 - W_4z_4.$$

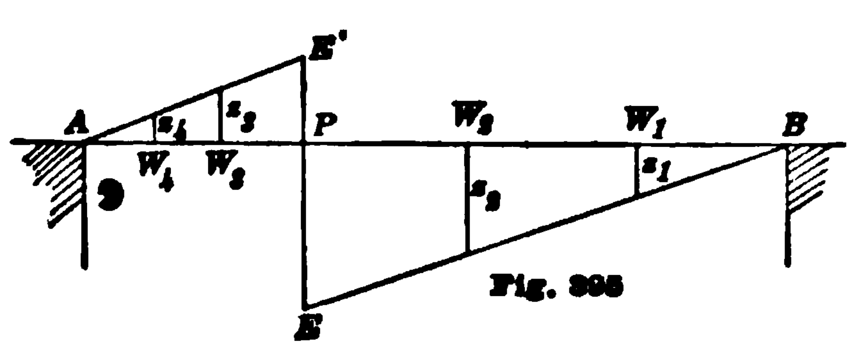


Fig. 395

413. Deflection influence lines. The deflection of a point P produced by a unit load at any

other point Q is given for a simple prismatic beam and is found by the method used in **366** to be

$$\Delta_P = -\frac{a(l-x)}{6EIl}(2lx - x^2 - a^2)$$

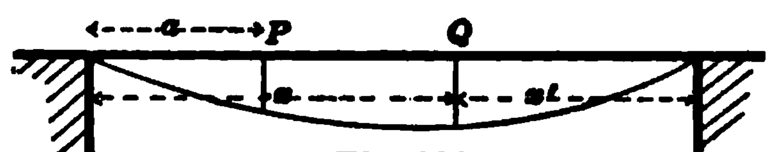


Fig. 396

This is the deflection influence line for the point P for a unit load.

"Deflection influence lines are of special value in determining stresses in redundant members and reactions of swing bridges, arches, and other structures having redundant supports."*

CHAPTER XXI.

Shearing Stress. Its Distribution in Beams and Shafts.

414. The distribution of shearing stress in beams. Thus far, we have considered stresses on different planes intersecting *at a point* within a body which was acted upon by other bodies, in a region of uniform stress. But, as we go a finite distance from point to point, we find conditions are different and the state of stress is different. Some of the problems of *varying internal stress* are now to be considered and solved.

Shearing stresses in beams. In the last chapter we found that the total shearing stress at a right cross-section of a beam was equal to the algebraic sum of the forces parallel to the plane of section, acting on one of the segments into which the section divides the beam.

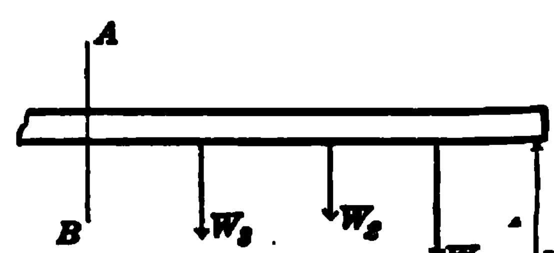


Fig. 397

Thus, if one portion of the beam is acted upon by the support V_1, and the loads W_1, W_2, W_3, the shearing stress at the section AB is

$$V_1 - \Sigma W. = S. \tag{1}$$

We must now find how it is *distributed* over the surface of the section.

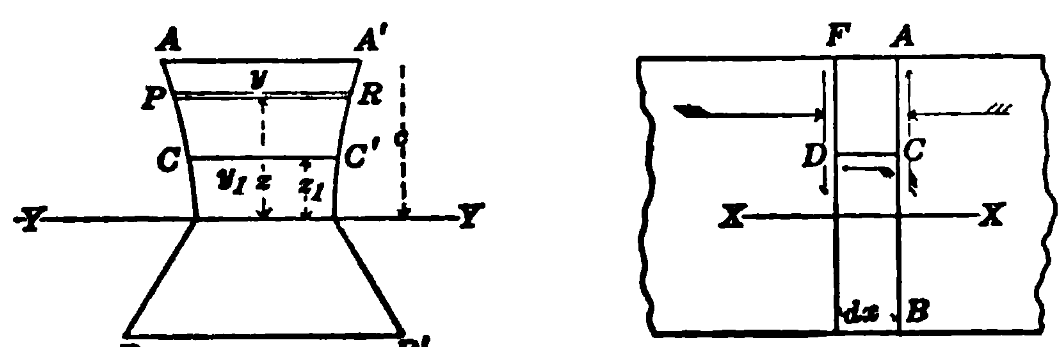


Fig. 398

Suppose we pass a horizontal plane CD, cutting the vertical section AB. Now, whatever the shearing stress in the plane AB is at C, we know by (**196**) that the shearing stress is

* Professor F. E. Turneaure.

the same in the plane CD at C, since the planes are perpendicular. So we will first find the shearing stress in the plane CD at the point C.

To that end, we will pass a third plane parallel to AB, distant dx from it. The *horizontal* actions upon the block $ACDF$ balance. These actions are: the normal stresses on the faces AC and FD, and the shearing stress on the rectangle $CD \times CC'$. Hence we have the equation,

$$\text{Normal stress on the face } AC + \text{Shearing stress on the face } CD. \qquad (2)$$
$$= \text{Normal stress on the face } FD.$$

The normal stress on the face AC is found by integration. Let the element $PR = ydz$, Fig. (b), be a strip of the cross section. The intensity of normal stress is az, and the amount of stress on the element is $azydz$. To get the total normal stress on $ACC'A'$, we integrate between the limits z_1 and c; hence

$$\text{Normal stress on } ACC'A' = \int_{z_1}^{c} ayzdz = a\int_{z_1}^{c} yzdz. \qquad (3)$$

When we go to the other face of our cross-sectional block, we find a different state of stress (arising from a different bending moment), and a has changed to $a+da$ so that our expression for the normal stress on $FDD'F'$ is

$$\text{Normal stress on } FDD'F' = (a+da)\int_{z_1}^{c} yzdz \qquad (4)$$
$$= a\int_{z_1}^{c} yzdz + da\int_{z_1}^{c} yzdz$$

and our equation becomes, after canceling the integral $a\int_{z_1}^{c} yzdz$

$$da\int_{z_1}^{c} yzdz = q(CC' \times dx) = qy_1dx, \qquad (5)$$

in which q *is the intensity of shearing stress* over the horizontal surface y_1dx, and hence at C in the vertical face AB. Now $M = aI$; hence $da = \frac{dM}{I}$; and since $dM = Sdx$, we have

$$qy_1dx = \frac{Sdx}{I}\int_{z_1}^{c} zydz$$

or

$$q = \frac{S}{y_1 I}\int_{z_1}^{c} zydz \qquad (6)$$

It may be well to recall the fact that the bending moment at the section AB (see Fig. 397) is due to the resultant $V_1 - \Sigma W = S$ acting at some distance from AB. If that distance be increased by dx, we have an increase in M amounting to Sdx.

415. Maximum intensity of shear. The general value of q at the distance z from the neutral axis in a *rectangular beam*, is, since $y = y_1 = a$ *constant*,

$$q = \frac{S}{I}\int_z^c z\,dz = \frac{c^2 - z^2}{2}\cdot\frac{S}{I}$$

It is evident from the last equation that the integral is a maximum when $z_1 = 0$, and vanishes at the top, when $z = c$. Hence

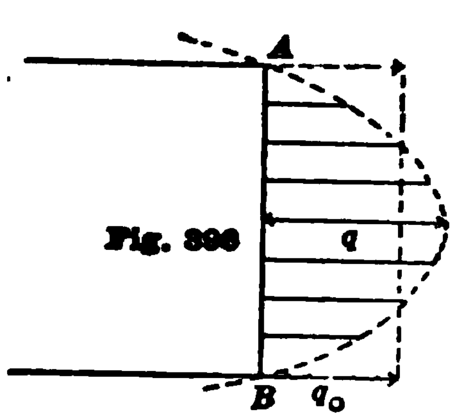
$$\text{Max. } q = \frac{S}{I}\int_0^c z\,dz = \frac{S}{I}\,\frac{c^2}{2} = \frac{Sh^2 12}{8bh^3} = \frac{3}{2}\,\frac{S}{bh} = \frac{3}{2}\,\frac{S}{A}$$

$$\text{Max. } q = \frac{3}{2} q_o$$

where q_o is the *average intensity* of stress on the face AB.

Hence in *every beam* the shearing stress is *zero at the top and bottom* and a *maximum at the neutral axis.* In the case of a rectangular beam the maximum shear is $\frac{3}{2}$ times the average.*

Fig. 398



If the intensity q, which really represents *tangential* stress on the plane AB, be represented by ordinates *perpendicular* to AB, the above equation is the equation of a parabola, the ordinates to which represent the intensity of shearing stress at points of the section.

Examples. 1. Given a steel cantilever beam 20 feet long, 2 inches wide, and 12 inches deep. It is loaded with a continuous load of 200 lbs. per foot, including its own weight. Find the maximum shearing stress in the beam.

2. Prove that the maximum intensity of the shearing stress in the cross-section of a *solid cylindrical beam* is $\frac{4}{3} q_o$, *i. e.* $\frac{4}{3}\cdot\frac{S}{A}$.

416. Shear in an *I*-beam. If, instead of a plate of uniform thickness, we had had an I-beam with a web, $10'' \times \frac{3}{8}''$, in Ex. 1. of

* As every *stress* has its *strain* or *deformation*, and as there is no vertical stresss and therefore no vertical strain at the top and bottom of a beam, it follows that there is no deflection due to shear, according to our theory of persistent cross-section planes. But such persistent planes are inconsistent with the internal deformation produced by shear; hence in the last analysis our results must be regarded as approximations, tho exceedingly close ones.

the last section, and if it be assumed, as is usually done, that the entire shear is distributed uniformly over the web-section, the intensity would have been assumed to be 1666.7 lbs. per sq. inch.

If the entire cross-section of an I-beam, or plate girder is considered, Fig. 400, the integral breaks up into two (or more) parts, as follows:

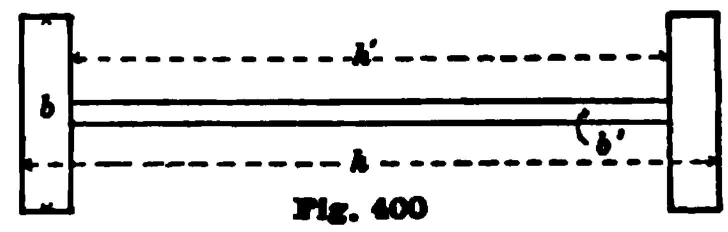


Fig. 400

$$\text{Max. } q = \frac{S}{b'I}\left[b'\int_0^{\frac{h'}{2}} z\,dz + b\int_{\frac{h'}{2}}^{\frac{h}{2}} z\,dz\right]$$

$$= \frac{S}{I}\left(\frac{h'^2}{8} + \frac{b}{b'}\cdot\frac{h^2-h'^2}{8}\right)$$

$$\text{Max. } q = \frac{S}{8b'I}(b'h'^2 + bh^2 - bh'^2)$$

The value of I is

$$I = \frac{b'h'^3 + bh^3 - (b-b')h'^3}{12} = \frac{2b'h'^3 + b(h^3 - h'^3)}{12}$$

hence $$\text{Max. } q = \frac{3}{2}S\left(\frac{b'h'^2 + bh^2 - bh'^2}{2b'^2h'^3 + bb'(h^3 - h'^3)}\right)$$

In the case of I-Beams and Channels of large size, the *slope of the flanges* should be taken into account in getting I. See Note p. 121.

If in accordance with usual practice, the web, $b'h'$, is assumed to take *all the shear*, the second integral in (9) must be omitted, and I be reduced to $I = \dfrac{b'h'^3}{12}$, we then have

$$\text{Max. } q = \frac{Sh'^2}{8I} = \frac{3}{2}\cdot\frac{S}{b'h'} = \frac{3}{2}\cdot\frac{S}{A} \text{ as before.}$$

If it be desired to see how close this last approximation is, let the student take the plate girder shown in Fig. 151.

Taking the *web alone* he will have $b' = \frac{7}{8}$ inch, $h' = 60$ inches, hence $A = 52\frac{1}{2}$ sq. inch, and

$$\text{Max. } q = \frac{3}{105}\cdot S = S\ 0.029 \text{ for the web alone.}$$

The average shear for the web is $\dfrac{S}{b'h'} = \dfrac{S}{\frac{7}{8}\times 60} = S\ 0.0197.$

It will be sufficiently exact to consider *each angle* as equivalent to an additional plate, so that each flange shall be held to consist of a solid rectangle 14"x3⅛". the value of h remaining 66¼".

The rest of the work is purely numerical if one uses the formulas given above.*

417. The shearing stress is usually so small that unless the web is very thin, it may be neglected in the case of iron and steel, in which the shearing strength is about equal to the tensile strength. In some woods, the shearing strength in planes parallel with the grain is very low, and should rarely be trusted.

The thrust of a strut or rafter, connected with a horizontal timber near its end, Fig. 401, should not be left to the uncertain shearing strength of the planes AB, cd and ef. A chance split or check may render the pushing out of the block, cfA, an easy matter. The pin P, or better still, the bolt K should always be added. (See Fig. 401.)

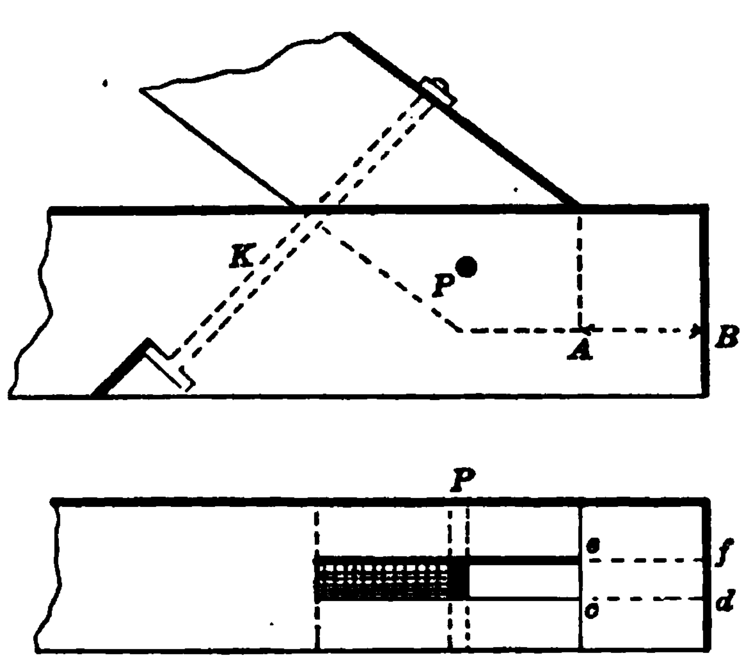


Fig. 401

The dangerous character of posts made by nailing together pieces of joice will be pointed out later when we are discussing long columns; such posts lack shearing strength and stiffness.

When beams are supported at the ends, the total shear near the support is equal to the supporting force; *i. e.*, $S=V$. The webs of some beams need re-inforcing at such points.

418. The Modulus of Shearing Stress. Before one can understand what is meant by "allowable" shearing stress, and the "elastic limit" as applied to shear, he must see clearly how the deformation produced by shearing stress may be measured, and what is meant by the **Modulus of Shearing Stress.**

While *measurements* are most readily made by twisting a cylindrical shaft, of known length and radius, the exact meaning of the *Modulus* can best be seen from an ideal test such as may now be described.

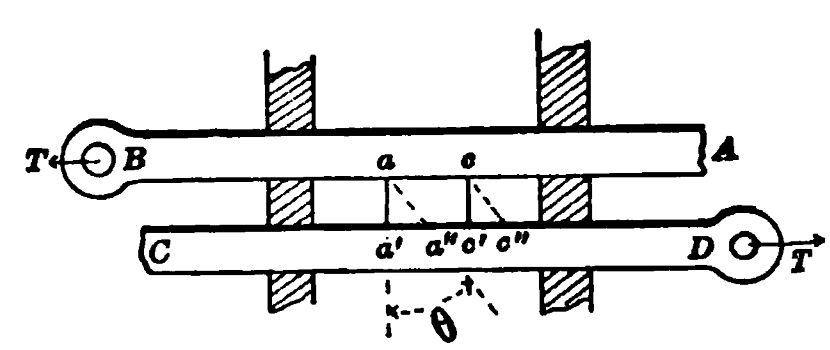


Fig. 402

Fig. 402 represents two eye-bars, AB and CD, rectangular in cross-

* "In plate iron [steel] girders generally, it is sufficiently accurate for practical purposes to consider the *whole bending moment* M at any vertical section as borne by the upper and lower ribs, flanges or plates; and the *whole shearing stress* S by the *vertical web;* and also to consider the resistance of each of the horizontal ribs as concentrated at the centre of gravity of its section." (Rankine, Applied Mechanics, p. 367.)

section, free to slide parallel to each other in slots of a rigid frame. Between the bars there is a block of steel which, by some process, is so thoroly brazed or welded to the eye-bars that the union of the bars and the block is perfect.

Now suppose tension is applied at B and D as indicated by the arrows. The block is at once deformed; the edges aa' and cc' are no longer perpendicular to the bars; there is a displacement $a'a''$, which, by delicate instruments, may be measured. It is found that the displacement or distortion $a'a''$ varies as the tension T varies, and that *it is proportional* to T. If the tension, per square inch of the face ac, is not excessive, the block recovers its original shape when the tension is taken off.

Up to a certain point, known as the **Elastic Limit** the displacement is proportional to the intensity of the shearing stress, which is

$$q = T \div \text{area face } ac$$

Since aa' is constant for different values of T, the displacements vary as does the values of $\tan\theta$; that is

$$\frac{q_1}{q_2} = \frac{\tan\theta_1}{\tan\theta_2}$$

With an ideal case where the elasticity is sufficiently near perfect, we may suppose that q_2 is great enough to make $a'a'' = aa'$, so that $\tan\theta_2 = 1$, the stress q_2 then takes a new letter and a new name; it is written E_s and is called the Modulus of Shearing Stress for the material of the block.

The above proportion then becomes

$$\frac{q}{E_s} = \frac{\tan\theta}{1}$$

$$q = E_s \tan\theta$$

$$E_s = q \cot\theta$$

The angle θ is readily measured on the surface of a twisted cylinder. The value of E_s (sometimes called the *Coefficient of Rigidity*) varies for steel from 9,000,000 lbs. per sq. inch to 12,000,000 lbs.

The Elastic Limit of Shearing Stress.

Although the strength of steel in resisting a destructive shear is very great, its *elastic* limit is low. When under test the block or shaft fails to recover its original shape if relieved, the *elastic limit* is reached, and yet, tho the shaft may have taken a permanent twist, it may be as strong as ever, and elastic as before.

The *elastic limit* for steel is put in the vicinity of 4,000 lbs. per sq. inch, *beyond which, as a working stress, q should not go.*

419. The distribution of shearing stress in a shaft. Suppose a shaft is transmitting energy from a pulley or gear wheel at one end to a pulley or gear wheel at the other end without loss due to friction, or outside resistance. Let M be the driving moment at one end and also the resisting moment at the other end. See Fig. 403.

The moment M is the same at every cross-section. If two sections are taken at A and B, distant dx, the disk, dx in thickness, is acted upon by both segments of the shaft. No. 1 is turning it to the left by shearing stress distributed over the entire surface of the circle at A. No. 2 is resisting the turning by shearing stress over the entire surface of the circle at B.

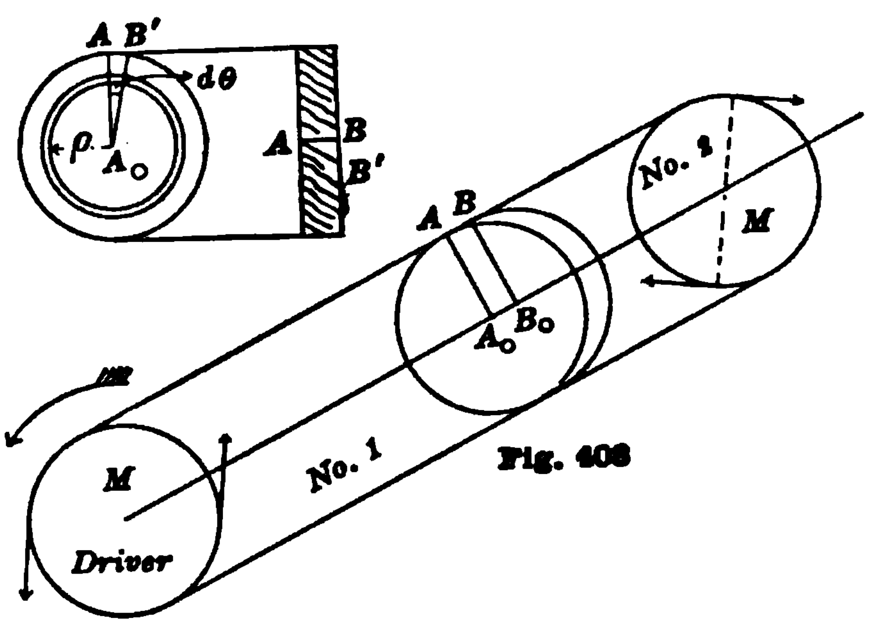


Fig. 403

The driving moment has the same magnitude as the resisting moment, hence there is no change in the angular velocity of the disk. The effect of the shear upon the longitudinal fibers in the disk is to *deform* them, as compared with their positions when the shaft was transmitting no moment. Under stress the fibers are no longer parallel to the axis. In fact, the shaft is *twisted,* not only in the disk AB, but thruout, and every fiber is a part of an helix, the steepness of which depends upon its distance from the axis.

If AA_0 and BB_0 were two parallel radii when the shaft was not in action, they show a relative angular displacement when the shaft is in action, and if BB_0 be *projected* upon the plane of AA_0 we have an elementary angle $d\theta = AA_0B'$. The *strain* of the fibers (and therefore the shearing stress) is proportional to their distance from the axis of the shaft. If a be the intensity of the stress at a unit's distance from the axis, the intensity at the distance ρ is $a\rho$, which is the same at all points of an elementary ring $d\rho$ in width. The shearing stress on the surface of that ring is $(a\rho)\ (2\pi\rho d\rho) = 2\pi a\rho^2 d\rho$, and the moment of that stress about the axis is $(\rho)\ 2\pi a\rho^2 d\rho$, so that $dM = 2\pi a\rho^3 d\rho$,

and
$$M = \int_0^r dM = \frac{\pi r^4}{2} \cdot a. \qquad (1)$$

So that the moment of twisting stress in a cylindrical shaft is

$$M = \frac{\pi r^4}{2} \cdot a = aI_p$$

in which I_p is the "polar moment of inertia," and a is the intensity of stress at distance unity from the axis, or the *rate* of the varying stress. The intensity of stress at the surface is ar.

If $q=ar$ be the *greatest working stress* allowed for the material, then

$$M=\frac{\pi r^3}{2}q$$

is the limiting moment the shaft can transmit; or it is the *measure of its strength.*

Had the shaft been a thick cylinder, the moment would have been

$$M=\int_{r_1}^{r}2\pi a\rho^3 dd\rho=\frac{\pi(r^4-r_1)^4}{2r}q$$

420. The economy of hollow shafting. Let us compare the strength of a solid shaft with the strength of a hollow shaft with an external diameter *n times as great*, but with the *same net amount* of material, using the same value of q. Fig. 404. Let x be the internal radius of the tube.

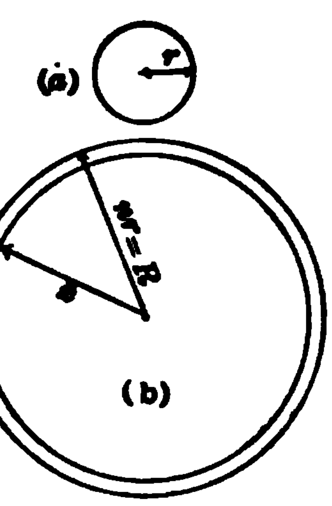


Fig. 404

Let M_1 be the twisting strength of the solid shaft:

$$M_1=\frac{\pi r^3 q}{2}. \tag{1}$$

Let M_2 be the twisting strength of the hollow shaft:

$$M_2=\frac{\pi(n^4r^4-x^4)}{2}\cdot\frac{q}{nr}.$$

Since

$$\pi r^2=\pi(n^2r^2-x^2)$$

$$x^2=(n^2-1)r^2$$

$$n^4r^4-x^4=2n^2r^4-r^4=r^4(2n^2-1)$$

$$M_2=\frac{\pi r^3}{2}\left(2n-\frac{1}{n}\right)q; \tag{2}$$

and

$$\frac{M_2}{M_1}=\frac{2n^2-1}{n}=2n-\frac{1}{n}$$

which means, that the hollow shaft, tho having the same material (and weight) as the solid one, is $\frac{2n^2-1}{n}$ times as strong; that is,

if $n=2$, it is $3\frac{1}{2}$ times as strong,
if $n=3$, it is $5\frac{2}{3}$ times as strong,
if $n=4$, it is $7\frac{3}{4}$ times as strong,
if $n=10$, it is nearly 20 times as strong, and so on.

The great economy of hollow shafting is not so remarkable as its superior stiffness.

421. The angle of torsion. It was seen above, when we considered the fibers of the thin disk with faces AA_0 and BB_0, Fig. 403, that when in action under stress there was a deformation more or less of all the fibers, and a relative angular displacement between the two faces, which we called $d\theta$, an element of the **Angle of Torsion.** Now, if the *angle* of torsion was $d\theta$, the actual displacement of the fibers on the surface was $rd\theta$, where the shearing stress was q. By the law of proportion for elastic material

$$\frac{q}{E_s} = \frac{rd\theta}{dx}$$

in which dx was the original length of the material, and E_s is the *Modulus of Shearing Stress.* From the last equation, we get, if l be the length of the shaft,

$$\int_0^\theta d\theta = \frac{q}{E_s r}\int_0^l dx$$

$$\theta = \frac{ql}{E_s r} \qquad (3)$$

which is the Angle of Torsion for the length l of a solid shaft.

Let us now compare the *stiffness* of a hollow shaft with that of a solid shaft of the same weight, if the exterior radius of the tube is n times that of the solid cylinder, *the twisting moment being the same.*

The **Stiffness** of a shaft is inversely as the angle of torsion, and hence is measured by $\frac{1}{\theta} = \frac{E_s r}{ql}$. It thus appears that the stiffness is proportional to r so long as the extreme fiber stress is the same. If then the radius is made nr, it will be n times as stiff, even tho the twisting moment is $\frac{2n^2-1}{n}$ times as great. If only an *equal moment* is to be transmitted thru the tubular shaft, the fiber stress q_2 will be $\frac{qn}{2n^2-1}$ which is found by making $M_1 = M_2$. If this be substituted for q, and nr be put for r in (3), we have θ_2 for the angle of torsion of a hollow shaft; hence the ratio of the stiffness of the tube to that of the solid shaft, while transmitting the *same moment*:—

$$\frac{\text{Stiffness of Tube}}{\text{Stiffness of Solid Cylinder}} = \frac{\frac{1}{\theta_2}}{\frac{1}{\theta_1}} = \frac{\frac{E_s r}{ql}}{\frac{E_s nr}{ql\frac{n}{2n^2-1}}} = 2n^2-1 \qquad (4)$$

or an **actual stiffness is $2n^2-1$ times as great.** If $n=3$, $2n^2-1=17$.

Every shaft when in action springs more or less, and if such springing is considerable, it is *dangerous*. Should a gear, tooth break, or a belt snap-the resilience (or jumping back) might do great harm. The stiffness of a shaft, if it be a long one, may be of more importance than the strength.

422. The relation of the modulus of shearing stress to that of tensile stress. Not only is the *definition of* E_s closely *analogous* to that of the Modulus of tensile and compressive stress, but E_s is related to E_t in another way which is worth considering.

It was said above that every fiber in a shaft, which is longitudinal when the shaft is idle, becomes a helix when the shaft is in action. Take a fiber on the surface of a cylindrical shaft which we will assume is perfectly elastic without limit, and suppose that, while l and r are constant, the twist reaches so great a deformation that the slope of the helix is 45°, so that for a length l, the deformation is also l. If the surface of the shaft showing the deformed fiber be developed, it will appear as in figure 405.

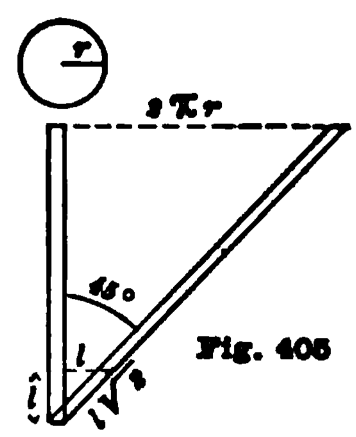


Fig. 405

Every rectangular solid fiber has been stretched diagonally by the *two equal shears*, until its length, which was l, is now $\sqrt{2}$, or it has been *stretched* 0.414 l.

Now E_t would stretch the same fiber l; hence $E_s = E_t$ 0.414.

If $E_t = 29{,}000{,}000$ lbs. (per sq. inch), $E_s = 12{,}000{,}000$ lbs.± which is a fair approximation to the value of E_s as found in other ways.

Having now a value of E_s, the value of θ can be found in a practical example.

423. Problem. We are required to design a steel shaft which is to transmit 240 H-P while making 180 revolutions per minute. The shearing stress must not exceed 4,000 lbs. per square inch, and the spring of the shaft which is 24 feet long must not exceed 5°. The shaft may be a tube with the material one-half an inch thick. Required x, the *external radius.*

The *data*, put into algebraic form, are:

$$l = 24 \text{ feet} = 288 \text{ inches.}$$
$$q = \text{or} < 4{,}000 \text{ lbs.}$$
$$\theta = 5^\circ \text{ or less.}$$
$$R.\ P.\ M. = 180. \quad (\text{Rev. per Min.}) = N,$$
$$\text{H-P} = 240.$$

If x = external radius, $x - \frac{1}{2} = x_1$, inner radius.

The definition of Horse Power was given on page 298; and on page 308 it was shown that the *work done* by a moment M (or the resistance offered) in one revolution of a shaft was $2\pi M$, Hence in N revolutions per min., the work done per min. is $2\pi MN$, The number of Horse Power is therefore

$$\text{H-P} = \frac{2\pi MN}{33{,}000},$$

and

$$M = \frac{\text{H-P}33{,}000}{2\pi N} \text{ foot-lbs.}$$

If inch-pounds is wanted we multiply by 12, and substitute for H-P, and N, and make $\pi = \frac{22}{7}$, and get the

Twisting Moment $M = 84{,}000$ inch-lbs.

The twisting moment is

$$M = 84{,}000 \text{ inch-pounds.}$$

The formula $\quad M = \dfrac{\pi(x^4 - x_1^4)}{2} \cdot \dfrac{4000}{x}$ gives

$$\frac{42x}{\pi} = x^4 - \left(x - \frac{1}{2}\right)^4 = 2x^3 - \frac{3}{2}x^2 + \frac{x}{2} - \frac{1}{16}$$

$$\frac{147}{11} = \frac{14}{\pi} = \frac{7 \times 14}{22} = 2x^2 - \frac{3}{2}x + \frac{1}{2} - \frac{1}{16x}$$

Dropping for the present the term $\frac{1}{16x}$ which is *very small,* we have

$$x^2 - \frac{3}{4}x = \frac{283}{44}$$

$$x = \frac{3}{8} + \sqrt{\frac{283}{44} + \frac{9}{64}}$$

$$x = 2.94 \text{ nearly.}$$

$$2x = 5.88.$$

Probably the nearest value in the market is $2x = 6$ inches.

Let us now see what the diameter must be if the angle of torsion is limited to 5°. The formula is $\theta = \dfrac{ql}{E_s x}$.

Here we must bear in mind that the limiting factor may be θ, not q, and it is quite possible that in order to have a shaft which is stiff enough,

we may be forced to use a shaft unnecessarily strong, in which case, as M is still the same, the extreme fiber stress is less than q. Accordingly, we will eliminate the uncertain fiber stress by substituting the value of $\frac{q}{x} = \frac{E_s\theta}{l}$ in the equation for the moment.

$$M = \frac{\pi}{2}\left[x^4 - \left(x - \frac{1}{2}\right)^4\right]\frac{q}{x} = \frac{\pi E_s\theta}{2l}\left[x^4 - \left(x - \frac{1}{2}\right)^4\right]$$

We find the value of the arc (unit radius) of $5°$ by the equation

$$(\text{arc of } A°) = \frac{\pi}{180}A_0 = \frac{22.A_0}{7(180)}.$$

Hence $$\text{arc of } 5° = \frac{\pi}{36} = \frac{11}{126}$$

Hence $$x^4 - \left(x - \frac{1}{2}\right)^4 = \frac{84{,}000 \times 576 \times 7 \times 126}{22 \times 12(10)^6 \times 11} = 14.5$$

$$2x^3 - \frac{3}{2}x^2 + \frac{x}{2} - \frac{1}{16} = 14.5$$

$$x^3 - \frac{3}{4}x^2 + \frac{x}{4} = 14.44.$$

This cubic equation gives us no trouble, for if we try $x = 3$ inches, we see that it is much too large to satisfy the equation. Hence we see that the external diameter of a shaft of the *requisite stiffness*, need be only about $5\frac{1}{2}$ inches. But it is clearly not strong enough, for $q = \frac{x\theta E_s}{l}$ is $x.(3{,}637)$ which is far beyond 4,000.

Hence we see that the dominating feature in the specifications is the value of q, which must not exceed 4,000 lbs.

Accordingly, the external diameter must be 6 inches, and the internal diameter 5 inches.

Example 1. Kent* says: "A 10-inch hollow shaft with an internal diameter of 5 inches, weighs 25 per cent less than a solid 10-inch shaft, but its strength is only $6\frac{1}{4}$ per cent less." Let the student check these figures.

Ex. 2. *Problem.* The shaft of a steam turbine makes 600 Rev. per Min., and transmits 1800 H-P. The shaft is hollow with an *inner* diameter of six inches. If q is limited to 4,000 lbs., what is the external diameter? What the angle of torsion if $l = 120$ feet?

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* See Mechanical Engineers' Pocket Book. P. 1109.

424. The hollow shaft of the U. S. Battleship Delaware. Fig. 406.

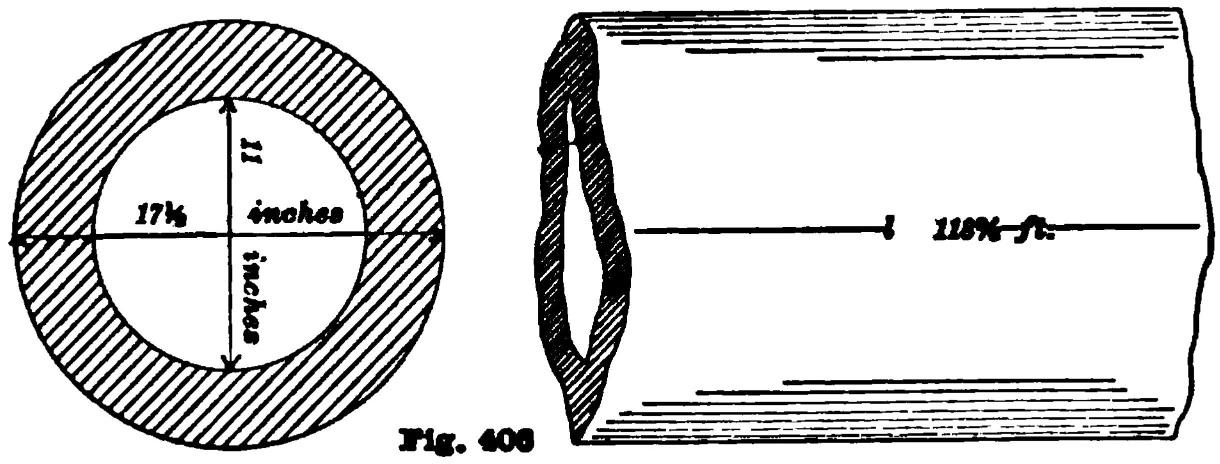


Fig. 406

The steel shaft has an exterior diameter of 17½″, and an interior diameter of 11 inches. It is 118¾ feet long, is driven with a maximum of 130 R. P. M., and transmits when under full speed 12,500 H-P.*

The Thrust Block of this shaft is shown in Fig. 340, Chapter XVIII.

The student may calculate the value of q, the maximum shearing stress, and the total angle of torsion θ.

425. The torsion balance, or the rotating pendulum. We saw from the formula in **423**,

$$\theta = \left(\frac{l}{E_s}\right)\left(\frac{q}{r}\right) = \frac{lM}{E_s I_p}, \left(\text{since } M = aI_p = I_p\frac{q}{r}\right)$$

that the angle of torsion is proportional to the twisting moment for a given shaft or wire. This relation holds, however, only when the shearing stress q is within the elastic limit which must be determined by experiments.

Suppose a vertical steel wire, or slender rod, has its upper end firmly fixed in a rigid support, and its lower end securely attached to a solid body having the axis of the rod as an axis of symmetry.

If now the suspended body be turned by an external couple so as to twist the rod thru the angle θ, without otherwise disturbing it, and then be released, the moment at the upper end acting thru the elastic material in the rod will turn the body with an increasing angular velocity until the twist is reduced to zero. From that point, the energy stored in the revolving body will twist the rod in the opposite direction against a resisting moment at the upper end of the rod, until the angular velocity is reduced to zero, and the angle of torsion is again θ. The instant it stops, if free from the influence of external forces, it starts back again just as it started forward when first released. The weight of the body (*i. e.*, the pull of gravity) has no influence upon its motion, and if no energy is lost thru friction and heat, the body will continue to rotate back and forth like a swinging pendulum. In fact, it is a rotating pendulum, the energy being stored alternately in the twisted rod, and

* The author is indebted for these data to the courtesy of Mr. H. I. Cone, Engineer-in-Chief, U. S. N. Bureau of Steam Engineering.

in the rotating solid. (In the case of a swinging pendulum, the energy was stored alternately in a *higher position* of the swinging body, and in the body itself when it was at its lowest point.)

If when first twisted the rod is held so that both θ and the magnitude of the moment of the twisting couple can be measured on a graduated scale, the apparatus becomes a *balance* by which bodies can be weighed. A *Torsion Balance,* with a graduated scale on a dial is easily constructed and no illustration is necessary.

426. The analysis of the rotating pendulum, or "balance wheel" is as follows:

Let CO be the fixed position of a radius of the suspended body when at rest with no twist in the vertical rod C. Let CA be the position of CO after the initial twist, the angle being θ_1. Let CP be the position of the same radius t seconds after the body has been released. The angle of torsion is now θ, and the *unbalanced moment* now acting is

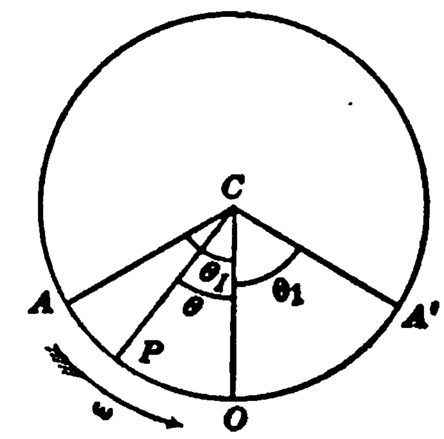


Fig. 407

$$M = \frac{E_s I_p \theta}{l} \tag{1}$$

in which I_p is the polar moment of Inertia of a cross-section of the rod.

The angular velocity is now ω, and its *angular acceleration* $\frac{d\omega}{dt} = a$ is by (XV) $a = \frac{M}{I_0}$, in which I_0 means the moment of Inertia of the *suspended body,* with reference to the axis of the rod. As the entire coefficient of θ in the formula (1) is determined from known values of E_s and the dimensions of the rod, we designate $\frac{E_s I_p}{l}$ as k, so that we have

$$M = k\theta. \tag{2}$$

and the value of $a = \frac{k\theta}{I_0}$. [It must be remembered that I_p refers to the twisted rod or wire, while I_0 refers to the body it suspends.]

Using the well-known equation $\omega d\omega = a d\theta$,

we have
$$\omega d\omega = \frac{k\theta}{I_0}(-d\theta).$$

The negative sign is given $d\theta$ since θ decreases as t increases. Integrating between the limits shown

$$\int_0^{\omega} \omega d\omega = \frac{k}{I_0}\int_{\theta_1}^{\theta} -\theta d\theta = \frac{\omega^2}{2} = \frac{k}{I_0}\,\frac{\theta_1^2 - \theta^2}{2}.$$

The angular velocity, ω, is zero, when $\theta = \pm\theta_1$, hence the radius OA moves to OA'.

Since $\omega = \frac{d\theta}{dt}$, we have

$$dt = \sqrt{\frac{I_o}{k}} \cdot \frac{-d\theta}{(\theta_1^2 - \theta^2)^{\frac{1}{2}}}$$

If we integrate from the start to the first stop, we shall have the time of one oscillation, hence

$$\int_o^{t_1} dt = t_1 = \sqrt{\frac{I_o}{k}}\left(-\text{arc}\sin\frac{\theta}{\theta_1}\right]_{\theta_1}^{-\theta_1} = \pi\sqrt{\frac{I_o}{k}}$$

which shows that the time occupied in rotating from OA to OA', $\pi\sqrt{\frac{I_o}{k_1}}$, which is one oscillation, is independent of the initial angle θ. Hence, as a pendulum, it is *isochronous*, the time being independent of the amplitude θ_1.

This agrees with the straight pendulum and the cycloidal pendulum discussed in XIV. It may properly be called *harmonic oscillation*.*

The suspended body may be a disk of uniform thickness, a plain solid circular cylinder, or a compound of a cylinder and two solids of revolution. In the case of an irregular body, its I_o can be found experimentally with considerable accuracy by the use of the formula just found:—

$$I_o = \frac{kt_1^2}{\pi^2} = \frac{E_sI_p}{l\pi^2} \cdot t_1^2,$$

in which E_sI_p and l refer only to the *wire* and *rod* used, and t_1 is found by counting the oscillations made in a definite time. If the time be several minutes and the oscillations be slow, t_1 may be found with great accuracy. If it oscillates 800 times in 24 minutes and 44 seconds, $t_1 = \frac{(24\times 60)+44}{800} = 1.855$. For geometrical solids, the values of I_o may be found by means of the Table on p. 223.

Example. Let $l = 24''$, $E_s = 12{,}000{,}000$, $I = \frac{\pi r^4}{2} = \frac{\pi D^4}{32}$ in which $D = 2r = (0.125)''$, so that $k = \frac{(12{,}000{,}000)\pi(0.125)^4}{(32)(24)}$. Let R, the radius

* In calculating the value of I_p, no account was taken of the mass of the tiny suspension rod which was assumed to be relatively unimportant; first, because the rod itself was assumed to be very small, and secondly, because its share in the final I_p would have been still less worthy of counting.

of a disk $=4''$; the thickness, $(1.5)''$; w, the weight per cubic inch $=\frac{490}{1728}$, $g=32.2$, $m=\frac{W}{g}$, $I_0=\frac{mR^2}{2}=8m$.

Find t_1.

427. The combination of stresses in a shaft due to bending and torsion. A shaft, projecting as a cantilever beyond a bearing, may have upon its outer end a crank, or a pulley with either gears or a belt. In every such case the shaft is both bent and twisted at the bearing. The *amount* of bending and twisting is not now taken into account, beyond saying that the amount is small and does not sensibly affect the stresses we are considering. In the case of a crank, Fig. 408, both bending and twisting moments are at their greatest when the connecting rod has its greatest obliquity, β (Fig. 409), (with full steam pressure), which is when the center line of the rod is tangent to the crank circle.

In the case of the crank, Figs. 408 and 409, the *moment of torsion* is

$$M_t = ApR \sec \beta$$

in which A is the area of the face of the piston, p the *effective* steam pressure (eliminating friction). The *bending moment* at the effective edge of the bearing of the shaft is

$$M_b = Pl = Apl \sec \beta$$

in which l is the extension of the center of the crank pin beyond the edge of the bearing of the shaft.

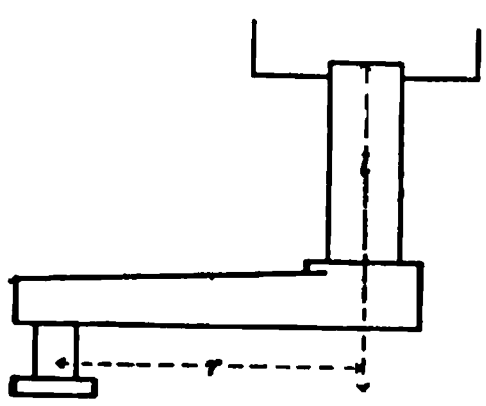


Fig. 408

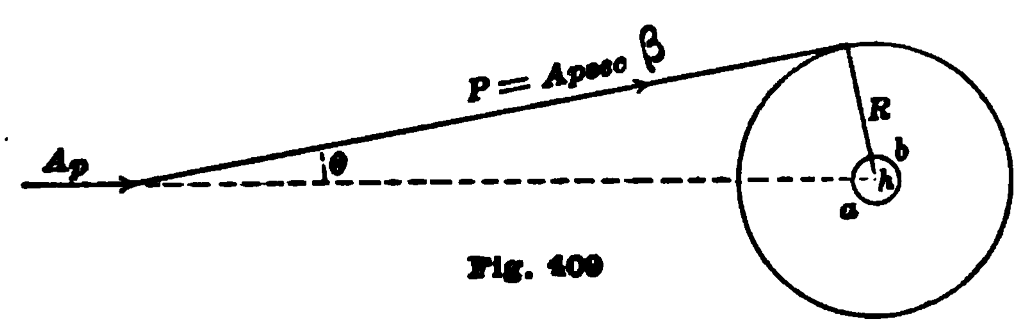


Fig. 409

For the belt pulley $M_t = (T_1 - T_2)R$ in which R must include half the thickness of the belt, and $M_b = (T_1 + T_2)l$, if the belt plies are parallel. If they are not parallelthe resultant tension is easily found.

For the gear-wheel, $M_t = P_p$, in which P is the thrust of teeth, and R is the perpendicular from the line of action of teeth to the center of shaft under examination. For the line of thrust see Fig. 322. The magnitudes of these factors are hard to determine in practice since the ideal direction found from an assumption of teeth perfect in shape and smoothness, is modified by both imperfect form and friction. Having found the two moments, let us consider the internal stress at a point

where the tensile stress produced by the bending, and the sheering stress produced by the torsion are both at their maximum; viz., on the surface of the shaft at a point on the circumference of the circle distant l from the plane of the twisting couple. Fig. 411.

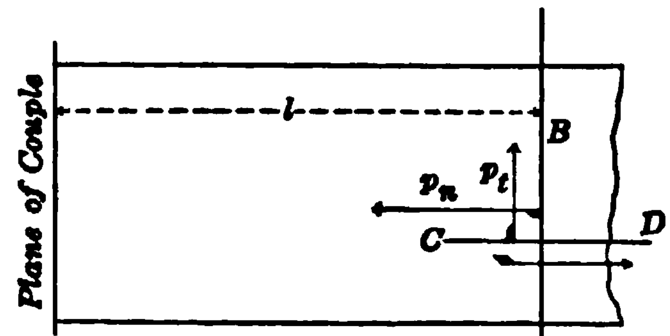


Fig. 411

Let AB be a cross-section of the shaft at the bearing, and let the intersection of the lines AB and CD be the point where the material is in greatest tension, due to bending.

The shearing stress near the circumference of AB has the maximum value which is the same over the entire surface of the shaft. The intensity of the tensile stress is found from the bending moment

$$M_b = \frac{\pi p_n r^3}{4} \therefore p_n = \frac{4M_b}{\pi r^3}$$

And P_t is found from the torque (1)

$$M_t = \frac{\pi q r^3}{2} \therefore q = \frac{2M_t}{\pi r^3}$$

Now consider a longitudinal section CD of the shaft. There is no normal stress on CD. The shearing stress along the line CD is of the same intensity as on the face AB: that is $q_1 = p_t$.

We have now the stresses on two planes perpendicular to each other, hence we find the *principal stresses* by **208** both graphically and algebraically.

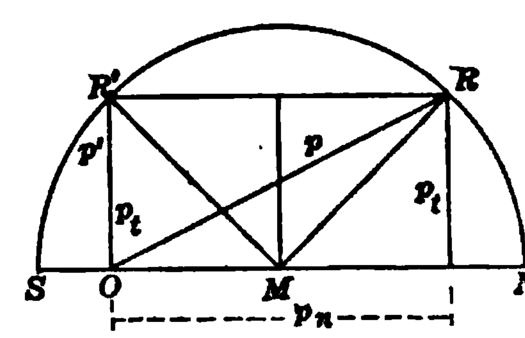


Fig. 412

The construction is shown in Fig. 412; OR' is the stress on CD of the last figure; OR is the resultant of p_n and p_t on the surface AB; the *maximum stress* is

$$p_x = OM + MR = ON$$

$$\left.\begin{aligned} p_x &= \frac{p_n}{2} + \sqrt{\left(\frac{p_n^2}{4} + p_t^2\right)} \\ &= \frac{2M_b}{\pi r^3} + \sqrt{\left(\frac{4M_b^2}{\pi^2 r^6} + \frac{4M_t^2}{\pi^2 r^6}\right)} \\ &= \frac{2}{\pi r^3}\left[M_b + \sqrt{(M_b^2 + M_t^2)}\right] \\ &= \frac{2M_b}{\pi r^3}\left[1 + \sqrt{1 + \left(\frac{M_t}{M_b}\right)^2}\,\right] \end{aligned}\right\} \quad (2)$$

The values of M_b, the bending moment, and of M_t, the twisting moment, have already been given,

$$\left.\begin{aligned} M_b &= Ap \sec\beta \times l \\ M_t &= Ap \sec\beta \times R \end{aligned}\right\}$$

This formula (2) suffices for finding the value of r, the radius of the shaft, when for p_x is put the limiting working stress allowed for that material.

CHAPTER XXII.

BEAMS OF UNIFORM STRENGTH.

428. The thoughtful student must have seen that a prismatic beam, loaded and supported as the beams thus far discussed have been loaded, has contained a large amount of material that was not needed under the conditions named. From end to end the beam was strong enough to stand the maximum bending moment, tho that strength was *needed* at only one (or two) points. It would seem to be the part of economy to design the beam at every point just strong enough to sustain the bending moment at that point. The result would be a tapering beam, like the trunk of a tree, like a fishing rod, like the mast and spars of a ship, the blade of a knife, a flagstaff, etc. Such modifications are countless in nature, and in primitive constructions by untaught men. Of late years in all large constructions, similar modifications are made by engineers and architects. Nevertheless, the use of machinery, in the formation of prismatic bars of steel and pieces of lumber, make such forms cheaper than tapering forms with less material, so that economy of labor often overbalances economy of material in simple constructions.

The phrase "*uniform strength*" is used "technically"; that is, the meaning differs from that ordinarily conveyed by the words. The technical meaning is, *equal liability to break at all points*; in other words, the *extreme fiber stress* is the *same* at every cross-section. If p_1 be the extreme fiber stress, and c its distance from the neutral axis of a section, we have the familiar formula for the (arithmetical) equality of *bending moment*, and the *moment* of *internal stress*:

$$M = \frac{p_1}{c} I \quad \therefore \quad p_1 = M \div \frac{c}{I}.$$

Now if p_1 is to be constant at all sections while M varies, the quantity $\frac{I}{c}$ must vary with M. Suppose all cross-sections of a beam are rect-

angles, bh, so that $I=\frac{bh^3}{12}$, and $M=\frac{bh^2}{6}p_1$ If bh^2 is to vary with M as we go from point to point, that variation can be affected by:

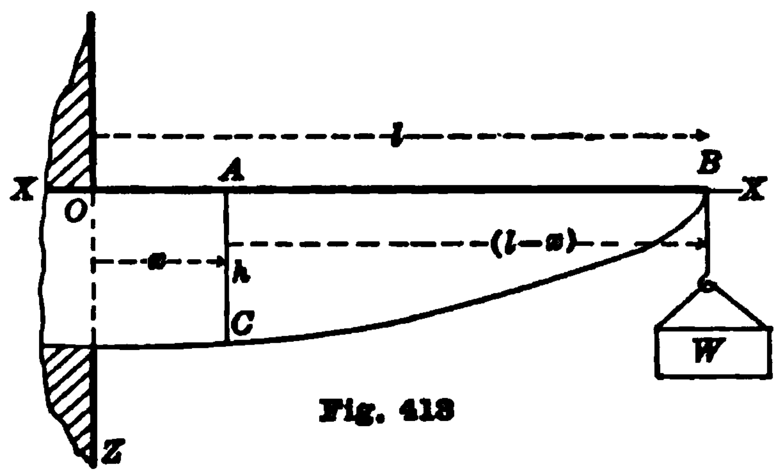


Fig. 413

Keeping b constant while h varies;
Letting h be constant while b varies;
By having both h and b vary.

429. CASE 1. **Cantilever beam of "uniform strength"**; Rectangular sections b, constant; load at the end. Fig. 413.

$c=\frac{h}{2}$.

$$M_x=\frac{2p_1}{h}\frac{bh^3}{12}=\frac{p_1bh^2}{6}=W(l-x)\therefore h^2=\frac{6W}{p_1b}(l-x) \tag{1}$$

This gives the required value of h for every section x, and if h is measured up (or down) from OX we have the outline of the beam.

The curve CB is a parabola. The factors l, x, h and b must have the same unit of length, and p_1 and W the same unit of force. The unloaded beam may have a horizontal top or a horizontal bottom, as desired, or the ordinate may be made equally up and down, the outline is still a parabola.

The **deflection** of the cantilever of "uniform strength" with constant width, under a concentrated load at the end is found as follows:—

As I is no longer constant the equation we start with is

$$E\frac{d^2z}{dx^2}=\frac{M}{I}=a=\frac{2p_1}{h} \tag{2}$$

The quantity p_1 is the extreme fiber stress which is constant, and h is found from the equation already derived above.

$$h=\sqrt{\frac{6W}{p_1b}}(l-x)^{\frac{1}{2}}$$

Hence

$$E\frac{d^2z}{dx^2}=2p_1\sqrt{\frac{p_1b}{6W}}\cdot(l-x)^{-\frac{1}{2}} \tag{3}$$

Integrating we have

$$E\frac{dz}{dx}=-4p_1\sqrt{\frac{p_1b}{6W}}\cdot(l-x)^{\frac{1}{2}}+H \tag{4}$$

$$Ez=+\frac{8}{3}p_1\sqrt{\frac{p_1b}{6W}}\cdot(l-x)^{\frac{3}{2}}+Hx+K \tag{5}$$

Letting $x=0$ in (4) we have

$$H=4p_1\sqrt{\frac{p_1bl}{6W}}.$$

Putting $x=0$ in (5) we have

$$K=-\frac{8}{3}p_1l\sqrt{\frac{p_1bl}{6W}}$$

and putting $x=l$ in (5) we have the deflection at B,

$$\Delta=\frac{1}{E}\left(4p_1l\sqrt{\frac{p_1bl}{6W}}-\frac{8}{3}p_1l\sqrt{\frac{p_1bl}{6W}}\right)$$

$$\Delta=\frac{4p_1l}{3E}\sqrt{\frac{p_1bl}{6W}} \tag{6}$$

If h_o is the depth of the beam at the support, we have

$$h_o=\sqrt{\frac{6Wl}{p_1b}},\text{ hence }\sqrt{\frac{p_1bl}{6W}}=\frac{l}{h_o},\text{ and since }Wl=\frac{2p_1}{h_o}I_o,$$

the deflection becomes

$$\Delta=\frac{4p_1l^2}{3Eh_o}=\frac{2Wl^3}{3EI_o} \tag{7}$$

The deflection of a prismatic cantilever beam with a cross section bh_o, loaded as above, was found to be $\frac{Wl^3}{3EI}$.

It thus appears that the reduced beam havin*g* the same base as the prismatic beam, the same length, and the same load, *deflects twice as much.* We may therefore say that, compared with the prismatic beam: *the beam* of "uniform strength," tho *just as strong is only half as stiff.*

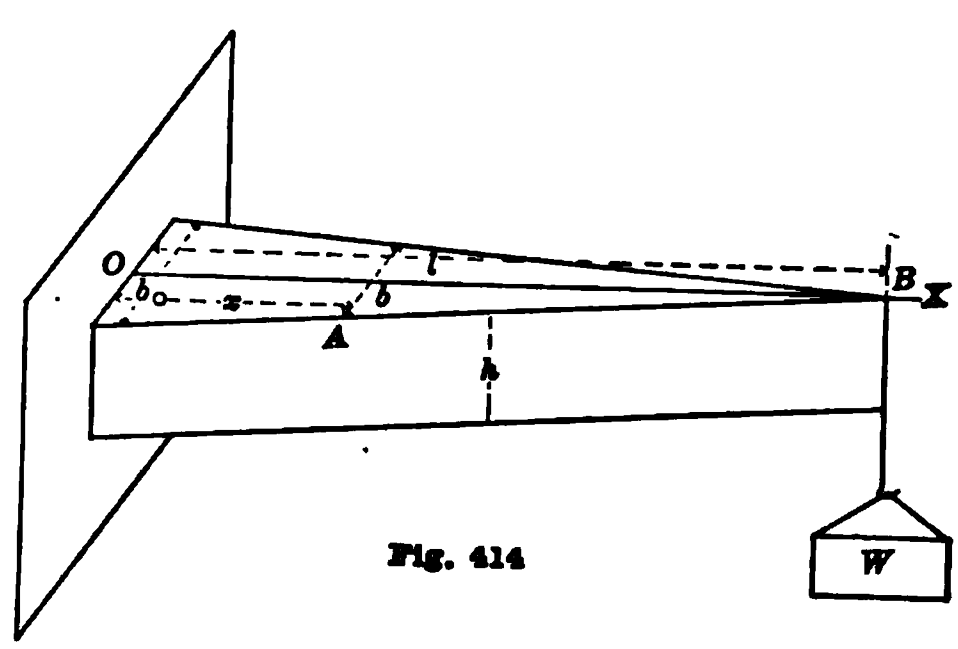


Fig. 414

430. Case 2. **Uniform strength and constant depth.** Here we have a differently shaped beam. In this case we have for the section at A,

$$M=W(l-x)=\frac{p_1bh^2}{6}$$

hence

$$b=\frac{6W}{p_1h^2}(l-x)\text{...(a straight line)} \tag{1}$$

which shows that b is proportional to $(l-x)$, and that $b_0 = \frac{6wl}{p_1h^2}$. The wedge-shaped beam is shown in Fig. 2.

The deflection is readily found since is $\frac{2p_1}{h} = \frac{M}{I}$ constant, and we have the four equations:

$$E\frac{d^2z}{dx^2} = \frac{2p_1}{h}$$

$$E\frac{dz}{dx} = \frac{2p_1x}{h} + (H=0)$$

$$Ez = \frac{p_1x^2}{h} + (K=0)$$

$$\Delta = \frac{p_1l^2}{Eh} = \frac{Wl^3}{2EI_0} \tag{2}$$

Cantilevers of Uniform Strength Compared with a Prismatic Beam.

431. A comparison of these three beams, having equal bases at support, and equal lengths and loads, is full of suggestion. The student must not forget, however, that the beams are ideally perfect in form and quality, but without weight, and that no account is made of shearing stress, not even on the end of No. 3. (Fig. 415)

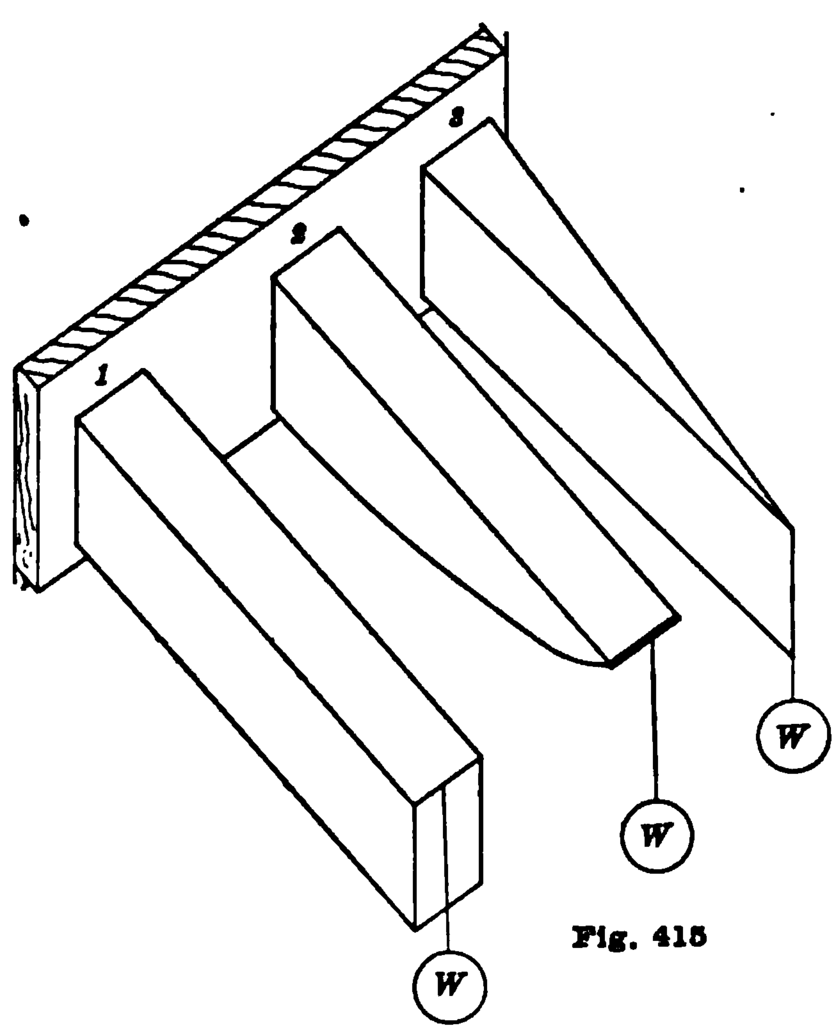


Fig. 415

In real beams the slight addition of material in the body of the beams and on the ends of No. 2 and 3, do not sensibly affect the deflections or p_1. The stress p_1 is the *safe working stress*, not the breaking stress.

Remarks:—

(*a*) Under the same load W they are *equally strong*.

(*b*) The *volume* (and hence the weight) of No. 2 is *two thirds* that of No. 1.

(*c*) The *volume* of No. 3 is *one half* of No. 1.

(*d*) The *deflection* of No. 2 is *twice as great* as the deflection of *No. 1*.

(*e*) The deflection of No. 3 is $\frac{3}{2}$ *as great as No. 1*.

(*f*) The total shear, V, is W at all sections in all the beams, and hence is excessive on the knife edge of No. 3; hence it should be tenon-shape, not a sharp edge.

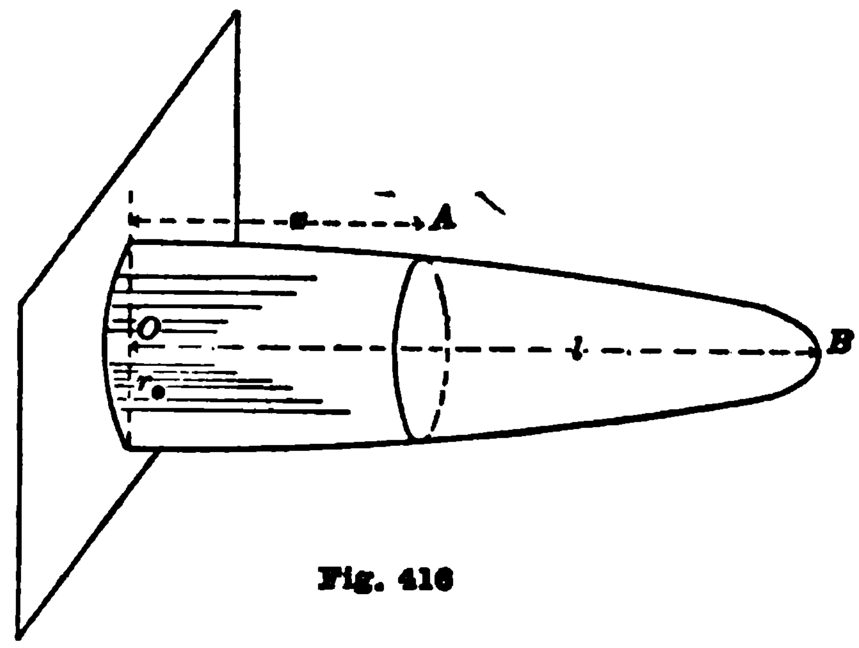


Fig. 416

432. Both h and b vary. A beam of uniform strength may have *square* or *circular* cross-sections. We will discuss the latter, Fig. 416. The length is l, the load at the end is W, and the section at A is a circle. Let $h=2r$. Then

$$M=W(l-x)=aI=\frac{p_1}{r}\,\frac{\pi r^4}{4}=\frac{p_1\pi r^3}{4}$$

hence
$$r^3=\frac{4W}{\pi p_1}(l-x) \tag{10}$$

which is the equation of a cubic parabola, and the beam is a "cubic paraboloid of revolution."

To find its deflection we have

$$E\frac{d^2z}{dx^2}=\frac{M}{I}=a=\frac{p_1}{r}=\frac{p_1(l-x)^{-\frac{1}{3}}}{\left(\frac{4W}{\pi p_1}\right)^{\frac{1}{3}}}=p_1\left(\frac{\pi p_1}{4W}\right)^{\frac{1}{3}}\cdot\ (l-x)^{-\frac{1}{3}}$$

$$E\frac{dz}{dx}=-\frac{3p_1}{2}\left(\frac{\pi p_1}{4W}\right)^{\frac{1}{3}}(l-x)^{\frac{2}{3}}+H$$

$$=\frac{3p_1}{2}\left(\frac{\pi p_1}{4W}\right)^{\frac{1}{3}}\left[l^{\frac{2}{3}}-(l-x)^{\frac{2}{3}}\right]$$

$$Ez=\frac{3p_1}{2}\left(\frac{\pi p_1}{4W}\right)^{\frac{1}{3}}\left[l^{\frac{2}{3}}x+\frac{3}{5}(l-x)^{\frac{5}{3}}-\frac{3}{5}l^{\frac{5}{3}}\right]$$

$$\Delta=\frac{3p_1}{2E}\left(\frac{\pi p_1}{4W}\right)^{\frac{1}{3}}\left(\frac{2}{5}l^{\frac{5}{3}}\right)=\frac{3p_1l^2}{5Er_0}=\frac{3Wl^3}{5EI_0}. \tag{11}$$

Since
$$\frac{p_1}{r_0}=\frac{M}{I_0}=\frac{Wl}{I_0}.$$

Let the student compare this deflection with that of a solid cylinder with the same base, length, and load.

433. The student should beware the notion that an approximate beam of "uniform strength" and constant breadth can be made by superposing joists or sheets of steel plates as shown in the figure, bolted tightly together at A only. Fig. 417.

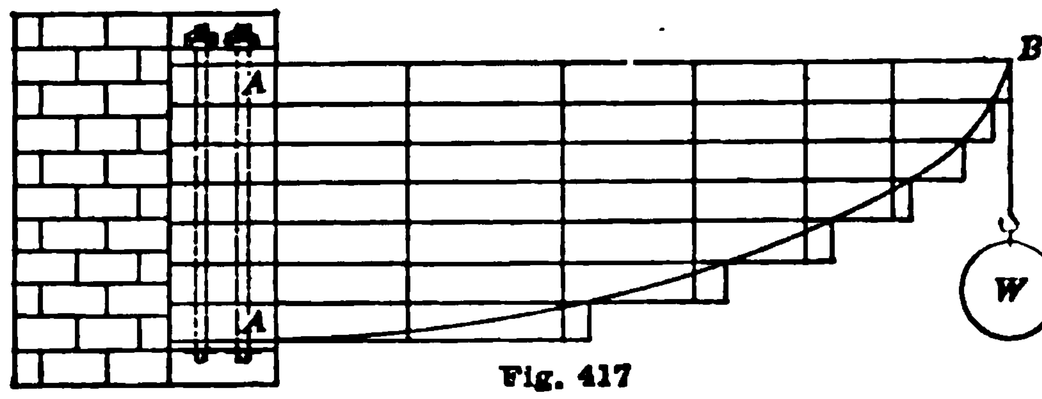


Fig. 417

Prismatic elements thus placed together do not form a united beam, but a series of nearly independent beams of differing lengths. There is no legitimate horizontal shear, except that which arises from friction as they bend, and friction is too uncertain and too small to be counted upon. If a load were hung at the end, all would bend, and there would be sliding at every surface of contact. It is well worth while for the class to build in the laboratory such a combination of boards with planed surfaces. Bolt them together securely at A, and draw several vertical lines on the side at different points, as shown in the figure; then place a reasonable heavy load at B and note the deflection and the breaking of the lines originally vertical. Then remove the load and rebuild the combination by spiking the pieces together *thoroly* one by one; then apply the *same* load and note the deflection and the deformation.

434. Cantilever beams of "uniform strength" under uniform loads, compared with a prismatic beam with same base, length and load. Fig. 418.

Maximum fiber stress the same in all.

No. 1, prism; No. 2, plane wedge; No. 3, wedge with parabolic sides.

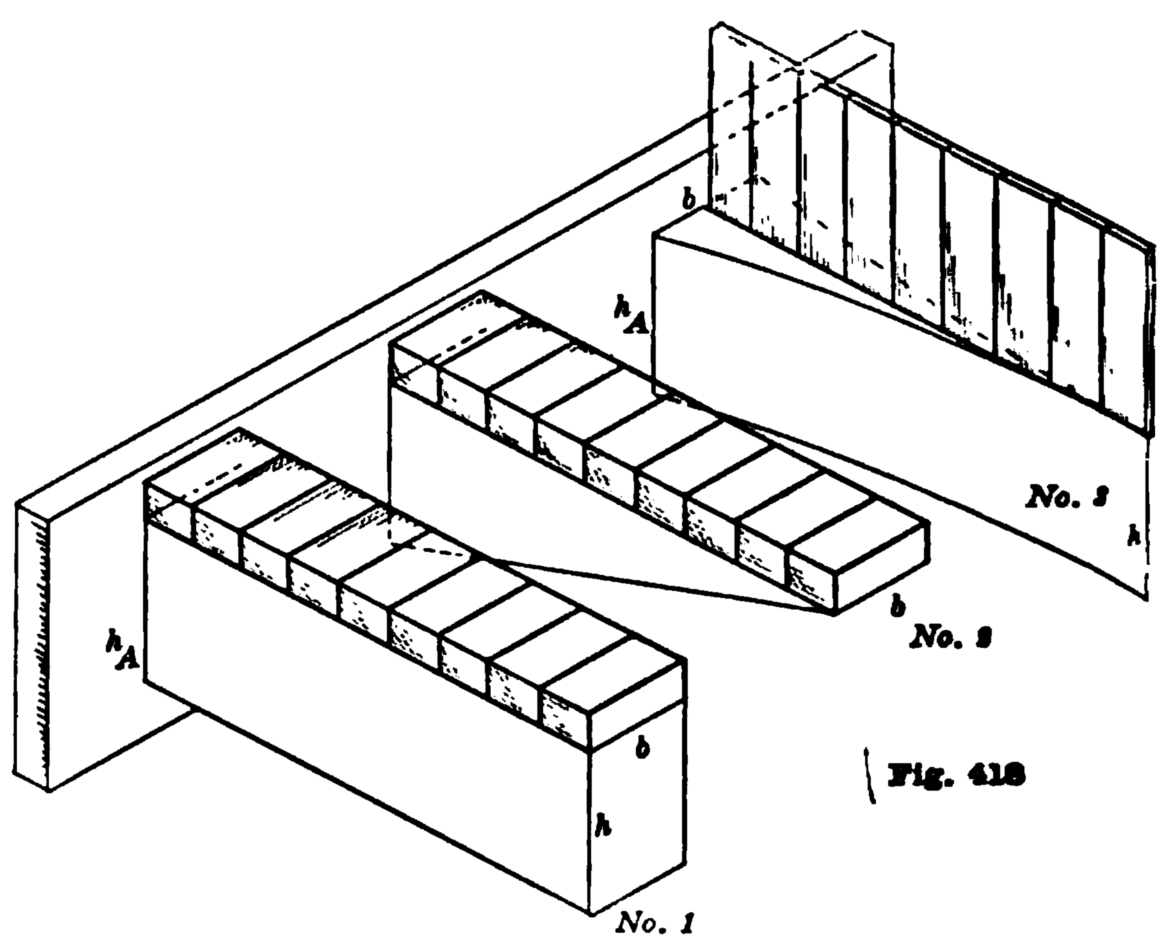


Fig. 418

Volume of No. 1 $= (bhl) = V$.

Volume of No. 2 $= \dfrac{1}{2}(bhl)$. Volume of No. 3 $= \dfrac{1}{3}(bhl)$.

Deflection of No. 1 $\quad \Delta_1 = \dfrac{1}{2}\dfrac{p_1 l^2}{Eh}$

Deflection of No. 2 $\Delta_2 = \dfrac{2p_1l^2}{Eh} = 4\Delta_1$

Deflection of No. 3 $\Delta_3 = \dfrac{p_1l^2}{Eh} = 2\Delta_1 = \dfrac{1}{2}\Delta_2$

The student should check the above results.

435. A beautiful example of beams of uniform strength but minimum weight is found in the spars of a sailing vessel. In a square-rigged vessel, the bending force may come from any direction; so like a flagstaff the cross-sections must be circular, and they must taper much like the cantilever in Fig. 416. The actual proportions of ships spars are probably the result of experience, not of mathematical theory. In this case, as in hundreds of others, correct theory always confirms the best practice.

436. It is so in natural forms.

Nature builds cantilevers of "uniform strength" in the shape of tapering trunks of trees, which are solid, and in bamboo poles and wheat straws which are hollow. The wind blows from any horizontal direction, so the cross-sections are circles or rings, of varying radii. If, like a palm or a tall southern pine, the load is c o n c e n t r a t e d (that is, if the foliage which catches the wind is bunched) at the top, it is a cantilever with a load at the end. Fig. 419. If W is the total force of the *wind* (in lbs.), and h the distance from the ground to the center of wind pressure, the bending moment at any height is $M = W(h-z)$. The value of I for a radius x is $I_z = \dfrac{\pi x^4}{4}$, and a is $\dfrac{p_1}{x}$.

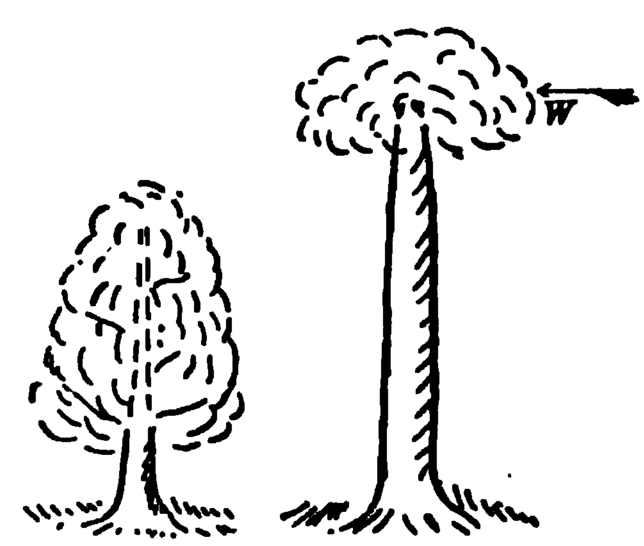


Fig. 419

Hence $$M = aI = \frac{p_1\pi x^3}{4} = W(h-z)$$

and $$x^3 = \frac{4W(h-z)}{\pi p_1}$$

which is the equation of a cubic parabola with the vertex at the top of the tree.

Hence the effective radius at the base is $R = \sqrt{\dfrac{4Wh}{p_1\pi}}$, in which p_1 is the fiber stress just under the bark, on the assumption that the internal stress varies uniformly (that is, that E is constant for all radii). So much for the ideal tree. Actual measurements show a fair approxima-

tion on the part of real trees to the ideal form of a cubic-paraboloid of revolution.

If the tree stands alone on the level plane, it is more often shrouded with foliage nearly to the ground, so that it approximates a cantilever with a uniform load, and the trunk tapers more rapidly, in accord with theory, which is, that it approaches a common parabola in out line.

$$x^2 = \sqrt{\frac{2W}{\pi f}}\,(h-z)$$

In the case of a growing bamboo, we see an economic cantilever of "uniform strength" in the shape of a conoidal tube.

437. A simple beam of uniform strength under a varying load. Loads in actual practice are rarely uniformly distributed, but in designing a beam or girder, a certain possible (tho improbable) maximum load must be assumed and the beam or girder must carry it with safety, not merely once, but indefinitely. The weight of the girder itself is a part of the load, and tho its final dimensions (and weight) are matters to be found, the design must surely cover the assumed weight. It will be instructive to consider first a simple solid beam with end supports, *uniform width*, but *varying height* so adjusted that under a *uniform load*, the extreme fiber stress will be the same at all cross-sections. Fig. 420 shows the beam, and the load is assumed.

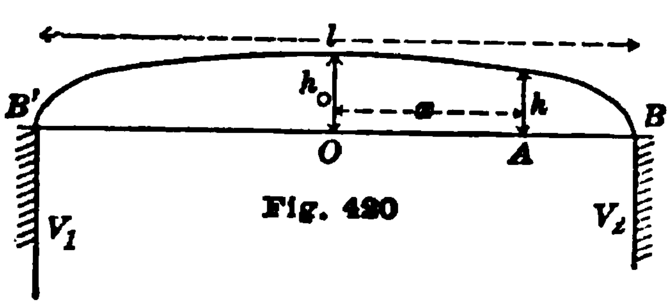


Fig. 420

The moment at A is

$$M = V_2\left(\frac{l}{2} - x\right) - w\left(\frac{\left(\frac{l}{2} - x\right)^2}{2}\right) = \frac{p_1 b h^2}{6}$$

$$\frac{x^2}{p_1 b} + \frac{h^2}{3w} = \frac{l^2}{4p_1 b},$$

which is the equation of an ellipse having the semi-axes

$$\frac{l}{2}, \text{ and } h_o = \left(\frac{3w}{p_1 b}\right)^{\frac{1}{2}} \frac{l}{2}.$$

438. 1. If the maximum load is fixed at the center, the bounding outline will consist of two intersecting parabolas, each similar to Fig. 413, with the cantilever inverted. The student can for himself see why this is so.

2. If a concentrated load W *rolls across* a simple beam, show that, if the extreme fiber stress at the section where the load is, is to be always the same, the outline must be a semi-ellipse whose semi-axes are $\frac{l}{2}$ and $\sqrt{\frac{3Wl}{2bp}}$.

439. Steel girders of uniform strength. Built-up Girders for Bridges conform closely to the requirements for uniform strength. The girder shown in Fig. 421 is a fine example of a good design. The photograph shows it as it is being lifted from railway cars to its masonry supports. The girder as here shown is 101 feet long from the anchor bolt at the west end to the center of the group of eight rollers at the east end. The depth is uniformly 9 ft. 2 inch. The web plates are $109\frac{3}{4}$ inches wide and $\frac{1}{2}$ inch thick, excepting the end plates which are

Fig. 421. Steel Girder of "Uniform Strength", on Chouteau Avenue, St. Louis.

$\frac{5}{16}$ of an inch thick for the purpose of taking the maximum of shear which, as has been seen, occurs at the ends.

The flanges are built of plates 20 inches wide and $\frac{3}{4}$ inch thick, with two heavy angles for each, the entire length. At the center of the span extending over a length of 46 feet 6 inches there are five plates each $\frac{3}{4}$ inch thick. The number of plates falls off towards the ends to 4, then to 3, and then to 2 at the edge of the second panel. In the upper flange, which withstands compression, the two plates extend to the end of the girder. In the lower flange, which withstands tension, The number of plates is reduced to *one* in the first panel. A close examination of the photograph will reveal some of these details, tho

the thoro riveting of plates and angles is only partially visible. The girder is designed to carry a working load (that is, the maximum "*live,*" and the maximum "*dead,*" load *simultaneously* of 2 tons per lineal foot.*

440. Providing for the horizontal shear. The importance of providing for the shear so that two or more superposed joice or plates may act as a unit, and not as individual members, is readily seen, if several dry, planed boards are placed and loaded, as shown in the figure, and the deflection noted. Then bolt or nail the boards together, using washers and nuts well set up, or stiff cut nails, and repeat the loading and note the stiffness. If the inner surfaces are lubricated (with tallow, or thick grease of any sort) the difference will be more marked. If equal loads are placed at P_1 and P_2, and the central weight be omitted, no nailing between P_1 and P_2 is necessary to make it stiff. Why is this?

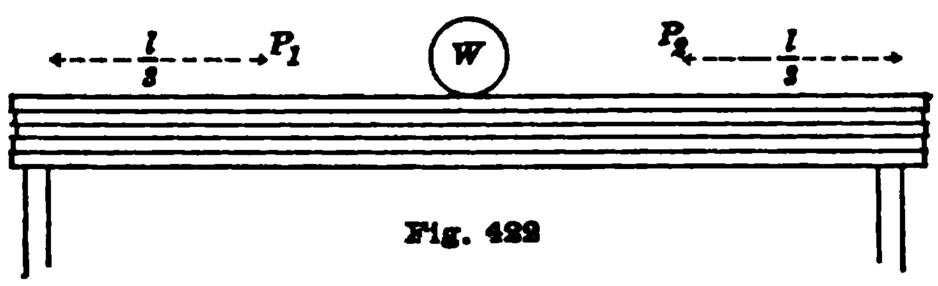


Fig. 422

Continuous Girders.

441. The *Thorem* of *Three Moments* for prismatic beams under uniform loads for each span. Altho the spans may be of different lengths, and have loads of different intensity, the girder is continuous with a constant moment of inertia. This theorem establishes an important relation between the three consecutive M_0's. The pier tops are on the same level. S_1 and S_2 are not the supports V_1 and V_2, but they are the *shears* acting on the girders AB and BC respectively.

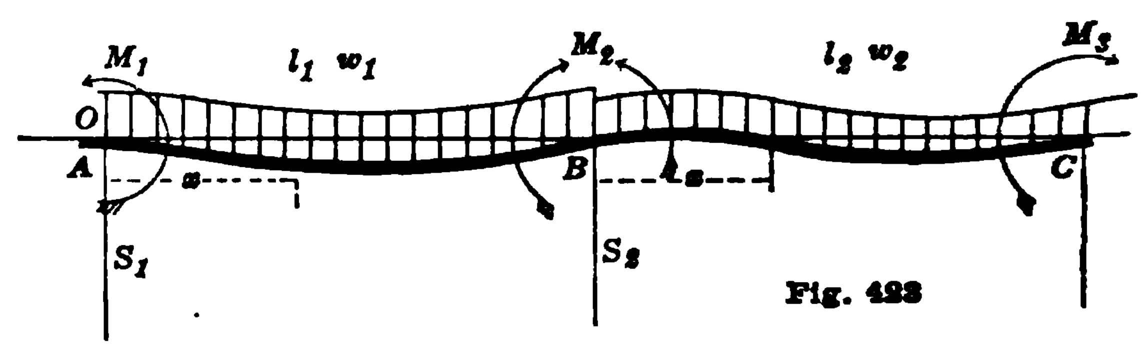


Fig. 423

For the girder AB we have

(1) $$EI\frac{d^2z}{dx^2}=M_x=S_1x-M_1-\frac{w_1x^2}{2}$$

(2) $$EI\frac{dz}{dx}=S_1\frac{x^2}{2}-M_1x-\frac{w_1x^3}{6}+H$$

(3) $$EIz=S_1\frac{x^3}{6}-\frac{M_1x^2}{2}-\frac{w_1x^4}{24}+Hx+(K=0)$$

* The author is indebted to Mr. J. C. Travilla, Street Commissioner of the City of St. Louis, for the photograph and a set of blue prints of the details.

In (1) when $x=l_1$, we have

$$M_x=S_1l_1-M_1-\frac{w_1l_1^2}{2}$$

This moment is *balanced* by M_2, hence

$$S_1l_1-M_1-\frac{w_1l_1^2}{2}+M_2=0$$

(4) $$S_1=\frac{M_1-M_2}{l_1}+\frac{w_1l_1}{2}$$

In (3) $z=$zero when $x=l_1$, hence

(5) $$H=\frac{M_1l_1}{2}+\frac{w_1l_1^3}{24}-\frac{l_1(M_1-M_2)}{6}-\frac{w_1l_1^3}{24}=\frac{l_1}{6}(2M_1+M_2)-\frac{w_1l_1^3}{24}$$

Proceeding in exactly the same way for the span BC, remembering that the action of M_2 *on it* is left-handed, while the action of M_3 is right-handed, we have:

(6) $$EI\frac{d^2z}{dx_2}=M_x=S_2x-M_2-\frac{w_2x^2}{2}$$

(7) $$EI\frac{dz}{dx}=S_2\frac{x^2}{2}-M_2x-\frac{w_2x^3}{6}+D$$

(8) $$EIz=S_2\frac{x^3}{6}-\frac{M_2x^2}{2}-\frac{w_2x^4}{24}+Dx+(G=0)$$

(9) $$S_2=\frac{M_2-M_3}{l_2}+\frac{w_2l_2}{2}$$

(10) $$D=\frac{l_2}{6}(2M_2+M_3)-\frac{w_2l_2^3}{24}$$

But the slope of AB at B as given by (2) when $x=l_1$, is the same as the slope of BC as given by (7) when $x=0$.

Hence

(11) $$S_1\frac{l_1^2}{2}-M_1l_1-\frac{w_1l_1^3}{6}+H=D.$$

Substituting for S_1 from (4), H from (5), and D from(10); we have the equation required:

(12) $$M_1l_1+2M_2(l_1+l_2)+M_3l_2=\frac{1}{4}(w_1l_1^3+w_2l_2^3)$$

which is known as "The Theorem of Three Moments."

COROLLARY I. If there are but two spans and the ends at A and C are *not fixed*, then $M_1 = 0 = M_3$ and

$$M_2 = \frac{w_1 l_1^3 + w_2 l_2^3}{8(l_1 + l_2)}.$$

Moments, Slopes and Deflections can now be found at will.

COROLLARY II. If the ends are fixed, there are two additional equations between constants obtained from the facts that $\frac{dz}{dx} = 0$ for $x = o$ in Eq. (2), and for $x = l_2$ in Eq. (7). Those two new equations will suffice for finding all the moments.

COROLLARY III. If there are three spans, with the end moments zero, we have two moments to find. Applying our general formula to M_1, M_2 and M_3, and again to M_2, M_3 and M_4, we have two equations for finding M_2 and M_3.

COROLLARY IV. If there are n spans with the ends free on the first and last supports, there will be $n-1$ groups of three consecutive M's, which will suffice for a complete solution.

COROLLARY V. If the three piers of a girder continuous over two spans are found after erection to be out of level, let a line touching the first and second piers be *assumed* to be *horizontal*, and let the *measured deflection* of No. 3, + or −, be entered as the value of z in Eq. (8) when $x = l_2$. This will give an additional constant term in the value of D, but equations (2), (3), (5), and (6) will suffice for finding all the constants including M_2.

REMARKS. A formula for Three Moments with concentrated loads can be derived by the methods already used, and also for piers *out of level*, but they are seldom necessary. Continuous girders are rarely used if foundations are liable to settle.

The student who has thoroughly mastered Chapters XIX, XX, and XXI is well prepared to take up an advanced work on "Framed Structures" dealing with Bridge Trusses, Arches and Steel Buildings.

442. Re-enforced concrete. When steel rods are combined with concrete in beams, columns, etc., the steel is so placed and designed as to take all the tension when there is any, leaving the concrete to take only compression. The reason is obvious, for while concrete has tensile strength when sound, and is elastic within narrow limits, its elastic limit is very low, and its soundness cannot be taken for granted. Accordingly its tensile strength is ignored in the best practice.

Steel and concrete in beams. The following example illustrates the way in which problems may arise, and the method of solving them.

The steel rods (corrugated, twisted, on smooth, round or rectangular) are imbedded in the concrete, near the surface along the tensile side of the beam. The arrangement is fairly shown in the figure.

It is assumed that the concrete above the neutral axis is subjected to compressive stress varying uniformly to the surface where it is f_c. The steel bars or rods are supposed to have uniform stress, whose intensity is f_s. Whatever may be the condition of the concrete on the steel side of the neutral axis, it is assumed to have no tensile stress, but has sufficient shearing strength.

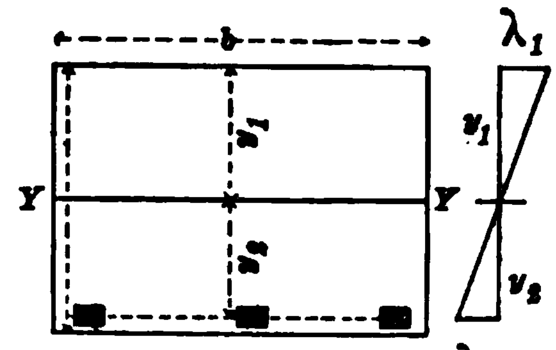


Fig. 494

The position of the neutral axis, which generally is *not* in the middle of the section, is found from two conditions:—

1. The algebraic sum of the normal stresses is zero.
2. A plane cross-section under no stress remains a plane when the beam is under stress.

Let A be the total steel area. The first condition gives the equation.

$$f_sA=\frac{1}{2}f_cy_1b \tag{1}$$

The second condition requires that the *unit strain* in the outer layer of concrete is to the *unit strain* in the steel, as y_1 is to y_2. Let E_c be the modulus of Elasticity of the concrete; then the *unit strain* in the extreme fiber is $\lambda_1=\frac{f_c}{E_c}$. Similarly the unit strain in the steel is $\lambda_2=\frac{f_s}{E_s}$.

Hence
$$\frac{\lambda_1}{\lambda_2}=\frac{y_1}{y_2}=\frac{f_c}{f_s}\cdot\frac{E_s}{E_c} \tag{2}$$

Let
$$\left.\begin{aligned} &y_1+y_2=d;\\ &\frac{E_s}{E_c}=n;\\ &\frac{A}{bd}=p;\\ \text{and}\quad &y_1=kd;\\ \text{then}\quad &y_2=(1-k)d. \end{aligned}\right\} \tag{3}$$

Combining (1), (2) and (3) and solving for k, we have

$$k=\sqrt{2pn(pn)^2}-pn \tag{4}$$

so that y_1 and y_2 are known if d is known.

The Moment of Resistance is equal to the moment of one of the two forces given in (1), multiplied by the lever-arm, which is the distance

from the center of the steel area, to the "center of action" of the concrete; that is, $y_2 + \frac{2}{3}y_1$. Hence we have

$$\text{Moment of Resistance} = Af_s\left(y_2 + \frac{2}{3}y_1\right) = \frac{1}{2}f_c b y_1\left(y_2 + \frac{2}{3}y_1\right) = M. \quad (5)$$

In which M is the bending moment of external forces.

Let
$$y_2 + \frac{2}{3}y_1 = d\left(1 - \frac{k}{3}\right) = jd.* \quad (6)$$

From (3), (5) and (6) we get the useful formulas:

$$\left.\begin{aligned} f_s &= \frac{M}{pjbd^2} \\ f_c &= \frac{2M}{jkbd^2} \\ p &= \frac{A}{bd} = \frac{1}{2}\cdot\frac{f_c}{f_s}\cdot\frac{1}{m\cdot\frac{f_s}{f_c}+1} \end{aligned}\right\} \quad (7)$$

The numerical value of n is quite generally taken as 15.

In good practice f_s rarely exceeds 15000 lbs. per square inch; and f_c is generally not greater than 640 lbs.

In shallow beams or slabs h is generally one inch greater than d; that is:

$$\left.\begin{aligned} &h = d + 1 \\ \text{For deep girders}\quad &h = d + 2''.5 \end{aligned}\right\} \quad (8)$$

In the case of beams or slabs continuous over several supports, the larger portion of steel rods may also be continuous, shifting from the bottom at A to the top near the point of "no moment" (for uniform load) with a single small bar along the bottom by

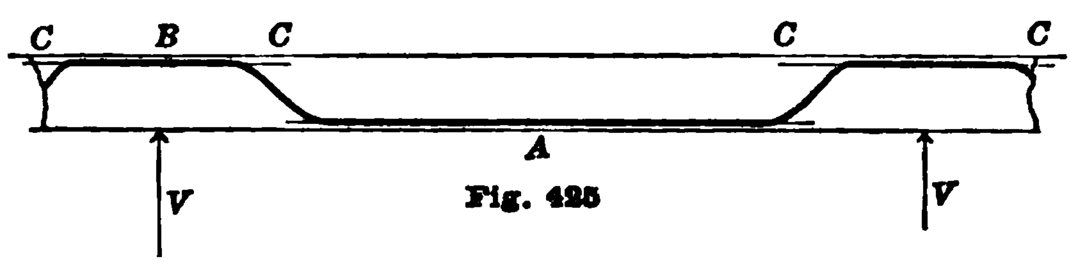


Fig. 425

A and extending beyond C and C; and a *larger* straight rod along the top by B, also extending beyond C on each side. The extensions provide for live loads, which change the positions of C and C, and give opportunity for greater resistance at B.

In the case of deep girders, the re-enforcing bars are sometimes bent as shown in Fig. (426) and woven into a quite rigid shape by a net

* The special nomenclature used in this section is that recommended by a Joint Committee of American Engineers.

Fig. 426

work made of small and comparatively flexible rod or wire, ready to be lifted into its place in a temporary box or mould and flooded with concrete.

443. Retaining walls and columns. When steel is used in foundation and retaining walls, the bars or rods are so placed as to resist by tension a tendency to bend thru the action of lateral pressures. In the case of a cylindrical column the bars are placed as shown in Fig. 427. As the degree of rigidity of steel is about 15 times as great as that of the concrete, under a well-centered uniform load, the steel is sustained laterally by the concrete, and supports a load per sq. inch, about fifteen times as much as the concrete. It is only under very eccentric loads, and excessive lateral forces that the tensile strength of the steel is needed. It is customary to surround the bars of steel in a column by a helical coil of slender steel rod, the helix having a small pitch. This rod is continuous from bottom to top of the column and is rigidly fastened at each end. While this coil of steel adds nothing *directly* to the strength of the column, it adds much to the strength and safety of the column by supporting the bars and by restraining the concrete against a tendency to expand laterally, and to crack longitudinally under heavy pressure. It is thus clear that the *pitch* of the coil should be small, and that the rod should have sufficient strength to prevent visible expansion and surface cracking.

Fig. 427

Extensive experiments upon columns thus re-enforced and restrained have been made with various grades of concrete and various proportions of steel.

The formulas in use are largely emperical, and the reader is referred to the Reports of Tests by Engineers, and the Hand Books of Specialists.

CHAPTER XXIII.

Work in a Bent Beam. Doctrine of Least Work. Deformation. Displacements.

444. Resilience. Reference has been made to the danger arising from the backward spring or jump called Resilience, of a twisted

shaft when the resisting moment is suddenly removed. This back-action is due to the Energy stored in the shaft, as energy is always stored in elongated, compressed or twisted elastic material. If the deformation is within the limit of perfect elasticity, the stored energy is measured by the work done by the external force (or forces) which caused the deformation. A bent beam may be deformed by an external force of gradually increasing intensity. It takes time for an external force to stretch or compress a rod, or bend a beam; and for a couple to twist a shaft or rod. Hence, there is motion, acceleration and velocity; but if the force is applied so slowly that no sensible (measurable) momentum or kinetic energy is produced, then the *work done* by the deforming force is measured by the distance the point of action has moved *multiplied by one-half the final magnitude of the force.*

Suppose a flexible cantilever is bent by the vertical pressure of one's finger. The pressure starts from zero. The least touch causes the beam to *start.* As the pressure increases the deflection increases.* When the pressure reaches the magnitude P, the deflection becomes $\Delta = \frac{Pl^3}{3EI}$, see (**375**). The average value of the pressure is $\frac{P}{2}$; hence the "work done," or the energy stored in the beam (due to the bending) is $\frac{P}{2} . \Delta$. Hence

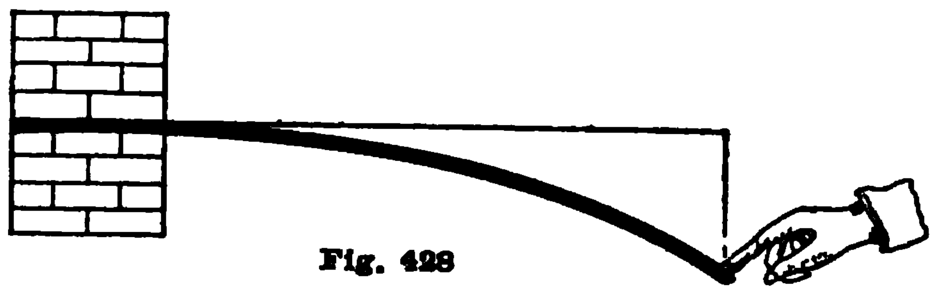


Fig. 428

$$U = \frac{P^2l^3}{6EI}.$$

This is the measure of the *Resilience.* If now the finger be suddenly removed, the beam will fly back, and if it meets with small resistance, it may attain a high velocity. If it encounters the inertia (or mass) of a small body, like a marble, which may be substituted for the finger, it may throw it high in the air.

445. It may be interesting to consider, for a moment, just where and how the energy is stored in the beam. If the beam is rectangular, all the fibers in the upper half of the beam are stretched, and all in the lower half are compressed. The lengthening (or shortening) of a fiber will vary with its distance from the neutral plane of the beam, and with its distance from the end where P is applied. It will be shown later (see **447**), that the work done in stretching and shortening the fibers of a cantilever beam by a load P at the end is

$$U = \text{Work} = \frac{P^2l^3}{6EI} \text{ which is just the same as}$$

$$\Delta\frac{P}{2} = \frac{P}{2}\cdot\frac{Pl^3}{3EI} = \frac{P^2l^3}{6EI}$$

from which the deflection could be found.

It is well again to call attention to the difference between a *slow* application of a load and a *sudden* one. Suppose we have two equal cantilevers with equal buckets at their ends.

One bucket is empty; the other is full of sand (or shot), but an overhead wire at A prevents the load from acting on the beam.

Now allow a small stream of sand (or shot) to fill the first bucket and note the resulting deflection of the beam. Next, arrange an automatic recorder (a pencil attached to the end of the beam with its point against a vertical card) so that the deflection of the second fully loaded bucket on its *first fall* may be accurately recorded. Then cut the overhead wire at A and let the load come suddenly on the beam.

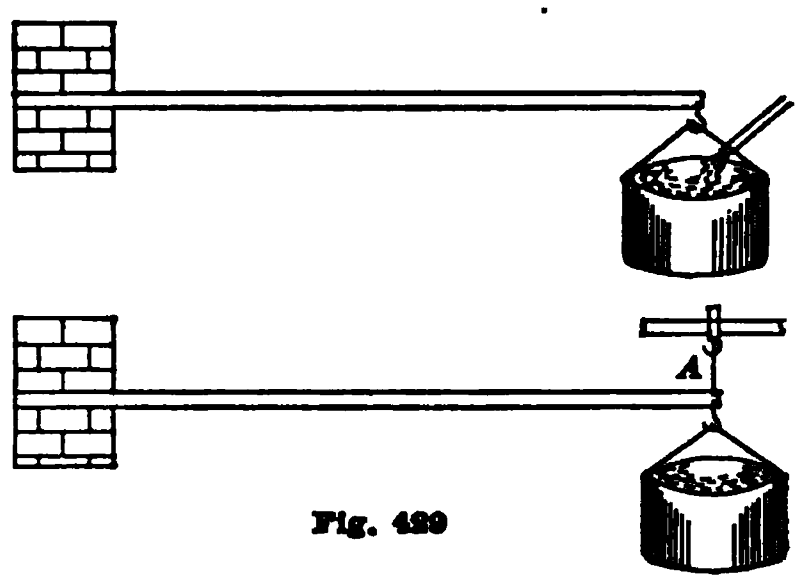


Fig. 429

The deflection will greatly exceed the deflection of the first beam, but will not be twice as great. Had the beam been *ideal*, that is, without mass but strong and elastic, the second deflection would have been just twice the first. Instead of cantilevers, simple beams with the loads in the center might have been used with similar results.

446. Had the load been dropped upon the end of the beam from a height h, the deflection and maximum pressure would have been still greater. This is readily shown. The elastic beam acts as does the coiled spring upon which a heavy weight falls, or against which a moving car bumps.

The "force" of the spring must be known by dividing an applied force (or weight) by the consequent extension, compression, or deflection. We have seen that a certain W will cause a certain deflection Δ, if placed gently on the end of a cantilever, thus

$$\Delta = \frac{Wl^3}{3EI}, \text{ which is supposed known;}$$

hence the "force" of that springing beam is $p = \dfrac{W}{\Delta}$.

Now let W fall from a height h upon the end of the beam. The beam will deflect a distance Δ, and the work done by gravity is $W(h+\Delta_1)$. This work is equal to the energy stored in the springing beam, which is $\frac{\Delta_1^2 p}{2}$. Hence the equation

$$W(h+\Delta_1) = \frac{\Delta_1^2 p}{2} = \frac{\Delta_1^2 W}{2\Delta}$$

so that

$$\Delta_1 = \Delta\left(1+\sqrt{\frac{2h}{\Delta}+1}\right)$$

and the maximum pressure on the beam is, since $P = \frac{\Delta_1}{\Delta} \cdot W$,

If $h=0$,	$P=2W$ as before found.
If $h=\Delta$,	$P=\Delta$, $p=W(2.73)$.
If $h=\frac{3}{2}\Delta$,	$P=3W$.
If $h=12\Delta$,	$P=6W$.

and so on.

An experiment will deeply impress the wisdom of the practical maxims: "Don't drop things." "Let heavy loads come on slowly"; whether dumb-bells or locomotives.

The student should not forget that the above discussion ignores the mass of the beam and all energy lost thru the crushing of material and molecular vibration; but the formulas are sufficiently close to the *real* effects of falling bodies, and suddenly applied rolling loads, to be very valuable. A swiftly moving locomotive has been known to wreck a truss-bridge which it has crossed safely at low speed; and a tumbling stone has demolished a staging which it has carried without harm, when at rest.

447. The "work" done (energy exerted) in bending a beam. It has been pointed out frequently that when a beam is bent, one side of it is stretched and the other is shortened. We shall now find the relation of the work done during such deformations to the bending moments actually applied to the beam. First, we must recall the fact that when a spring (or fiber) is *stretched* a length λ by a tension which begins at zero, and increases up to F, the average tension is $\frac{1}{2}F$, and the "work" done is $\frac{1}{2}F \times \lambda = \frac{\lambda F}{2}$.

Take a thin element of the beam by parallel cross-section planes. AB and CD, distant from each other dx. Before bending, the length of all longitudinal fibers in the element was dx; after bendnig they are more or less lengthened on one side of the neutral axis, and compressed on the other side, the faces of the element still being planes. The elongation or strain is proportional to the stress. If a is the stress intensity at a unit's distance from the neutral axis, the stress at a strip across the face of the element, which is z distant, will be az, and its elongation (or strain) $d\lambda$ is found by the proportion

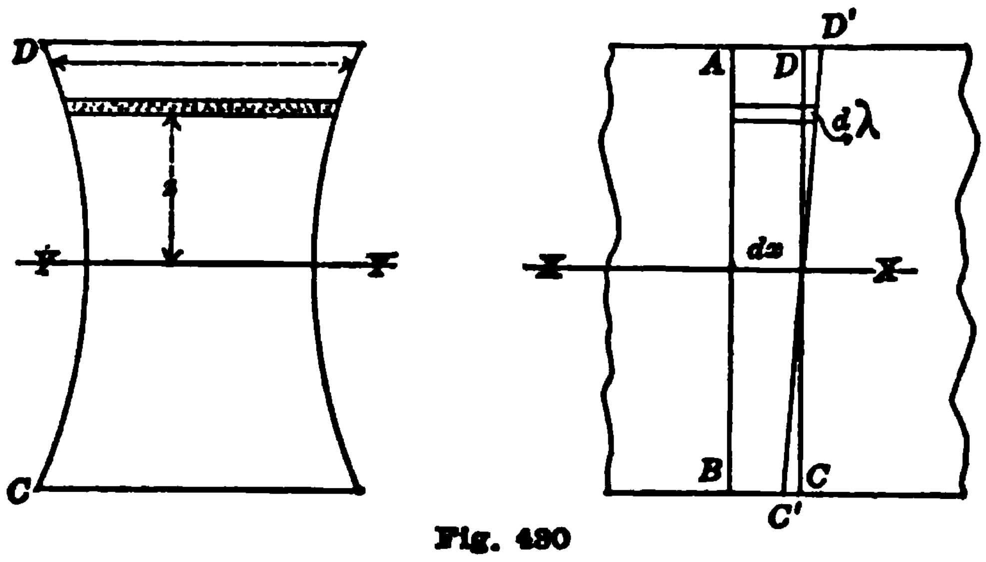


Fig. 490

$$\frac{az}{E} = \frac{d\lambda}{dx}; \text{ hence } d\lambda = \frac{azdx}{E}.$$

Now the amount of stress on the strip ydz is $azydz$, and the work done in stretching that strip is

$$\tfrac{1}{2}(ayzdz)d\lambda = \frac{a^2z^2ydzdx}{2E}.$$

From the familiar relation $M = aI$, where M is the bending moment at the section, we have $a^2 = \frac{M^2}{I^2}$, which in the above gives the second differential

$$d^2U = \frac{M^2z^2ydzdx}{2EI^2}$$

on a strip of the thin layer dx. It should be noted that this differential is never negative. Integrating for z so as to include the whole section face while all but z and y are constant, we have, since $\int z^2(ydz) = I$,

$$dU = \frac{M^2dx}{2EI}$$

and the *Total Work in the beam* $U = \frac{1}{2}\int_0^l \frac{M^2dx}{EI}$.

If E and I are constant, we have

$$U = \text{Total "Work"} = \frac{1}{2EI}\int_0^l M^2dx$$

which is the formula for finding the work stored in a bent prismatic beam.

448. Example illustrating the use of the formula. Suppose a prismatic beam fixed at one end, and carrying a uniform load has a support R at the other end applied by an overhead spring balance.

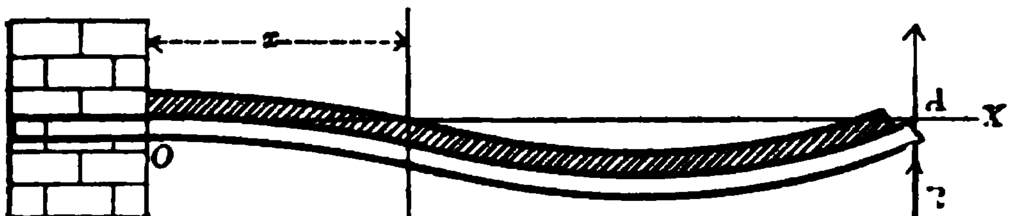


Fig. 431

Find the "Work" done by the load and R in bending the beam. Fig. 431.

$$M_x = R(l-x) - \frac{w(l-x)^2}{2}$$

$$M^2 = R^2(l-x)^2 - Rw(l-x)^3 + \frac{w^2(l-x)^4}{4}$$

$$\int_0^l M^2dx = \left[-\frac{R^2(l-x)^3}{3} + \frac{Rw(l-x)^4}{4} - \frac{w^2(l-x)^5}{20} \right]_0^l$$

$$= \frac{R^2l^3}{3} - \frac{Rwl^4}{4} + \frac{w^2l^5}{20}.$$

Hence $$U = \text{Work} = \frac{2EI}{l^3}\left(\frac{R^2}{3} - \frac{Rwl}{4} + \frac{w^2l^2}{20}\right)$$

This is seen to depend upon R, and must vary with R, and it is evident that there is some value of R which will make the "Work" a minimum. Hence we will put $\frac{dU}{dR} = 0$, and we get

$$O = \frac{2}{3}R - \frac{wl}{4}$$

and $$R = \frac{3}{8}wl$$

which we found to be the value of R in **365** when the tangent at the fixed end touched the top of the support; in other words, that the beam was *unbent* when *unloaded*. If R were $> \frac{3}{8}wl$, the beam would be bent up above the axis of X; If $R < \frac{3}{8}wl$, the end would be below.

In either case more work would be put into the beam than there need be. We thus see that the *Work is a minimum when there is no deflection at the end.*

449. A second illustration shows how to deal with a beam which consists of two different elastic curves. Let the beam given above have a concentrated load in the center. Fig. 432.

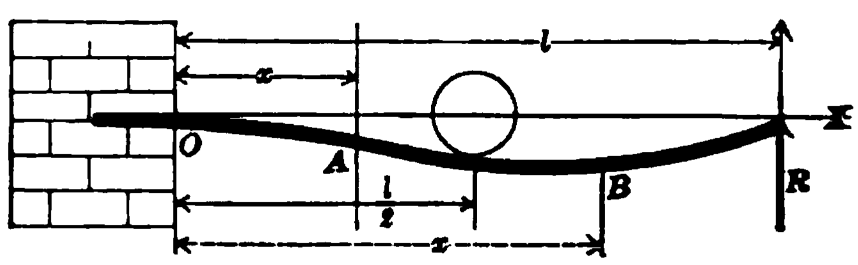


Fig. 432

$$M_A=R(l-x)-W\left(\frac{l}{2}-x\right); \qquad M_B=R(l-x).$$

$$(M_A)^2=R^2(l-x)^2-2RW\left(\frac{l^2}{2}-\frac{3}{2}lx+x^2\right)+W^2\left(\frac{l}{2}-x\right)^2,$$

$$(M_B)^2=R^2(l-x)^2.$$

Our integral now is $\displaystyle\int_0^{\frac{l}{2}}(M_A)^2dx+\int_{\frac{l}{2}}^{l}(M_B)^2dx:$

$$\int_0^{\frac{l}{2}}(M_A{}^2)dx=l^3\left[\frac{7R^2}{24}-\frac{5RW}{24}+\frac{W^2}{24}\right];$$

$$\int_{\frac{l}{2}}^{l}(M_B{}^2)dx=l^3\left[+\frac{R^2}{24}\right]$$

Hence $$U=\frac{l^3}{2EI}\left[\frac{R^2}{3}-\frac{5RW}{24}+\frac{W^2}{24}\right]$$

This, as before, depends on R. Putting $\dfrac{dU}{dR}=0$, we have

$$O=\frac{8R}{12}-\frac{5W}{24}$$

$$R=\frac{5}{16}W \text{ as on page 345,}$$

when the unloaded beam was horizontal.

If we substitute for R its value just found, we find the "Work" to be

$$U=\frac{l^3W^2}{2EI}\cdot\frac{7}{768}$$

Since the work done by the weight W in descending the distance Δ is $\dfrac{W\Delta}{2}$, we have

$$\frac{W}{2}\cdot\Delta=\frac{7W^2l^3}{2EI\cdot768}$$

$$\Delta=\frac{7Wl^3}{768EI}=\frac{Wl^3}{109.7EI}$$

which can be checked by making $x = \frac{l}{2}$ in eq. (3) p. 342, and solving for z. (This is not the *greatest* deflection. See **375**).

In **445**, the author promised to derive an expression for the work done in *bending a cantilever beam, with a load at its end.* See **Fig.** 428.

$$M = W(l-x), \qquad M^2 = W^2(l-x)^2$$

$$\int_o^l M^2 dx = -\left.\frac{W^2(l-x)^3}{3}\right]_o^l = \frac{W^2l^3}{3}$$

Hence the
$$\text{“Work”} = \frac{W^2l^3}{6EI},$$

as was assumed, and from which the deflection was found by dividing by $\frac{W}{2}$.

450. The "doctrine of least work." Just as all thru this book it has been assumed that when work was done by an unbalanced force or an unbalanced couple upon a body without loss from friction, heat, or non-reversible actions, the Total Energy was preserved (known as the doctrine of the "Conservation of Energy"); so it is assumed without positive proof that when an elastic body is caused to pass from one state of rest to another state of rest, the *Work done* or the *Energy Transferred is a minimum.*

This doctrine is of great value in the solution of problems which appear to be indeterminate, in consequence of redundant members; and consequently it is much used in the analysis of Internal Work of beams and trusses. It would be beyond the scope of this work to go far into this subject, but two examples of the solution of (so-called) indeterminate problems will suffice to point out the method.

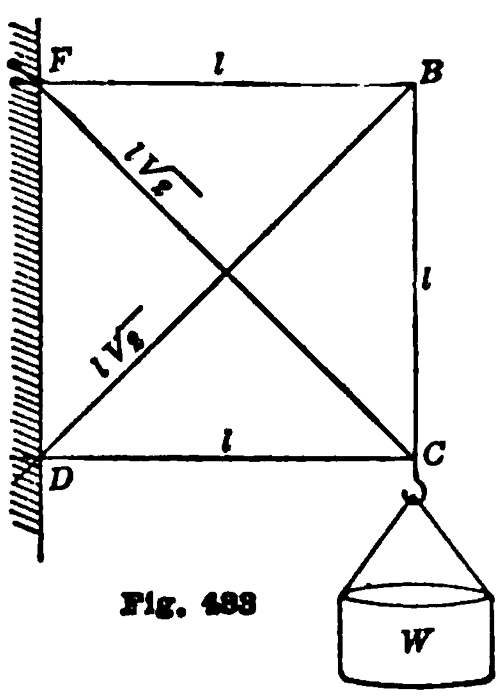


Fig. 433

451. The solution of a so-called indeterminate problem. A *square frame* hangs upon two pins connected with a *rigid* vertical wall and carries a load as shown in Fig. 433. The diagonals are not connected with each other. All joints are by smooth pins. The weights of the bars are neglected.

There seem to be three redundant bars, as the load with a long cord can be carried by FB and BD alone; or by FC and CD alone, assuming that DB and DC are capable of resisting compression

without buckling. All bars are prismatic and the workmanship is ideally perfect. Under such conditions, every bar does its share in sustaining the load, and suffers change in its length. The "work done" upon every member must now be found in terms of one unknown action or force.

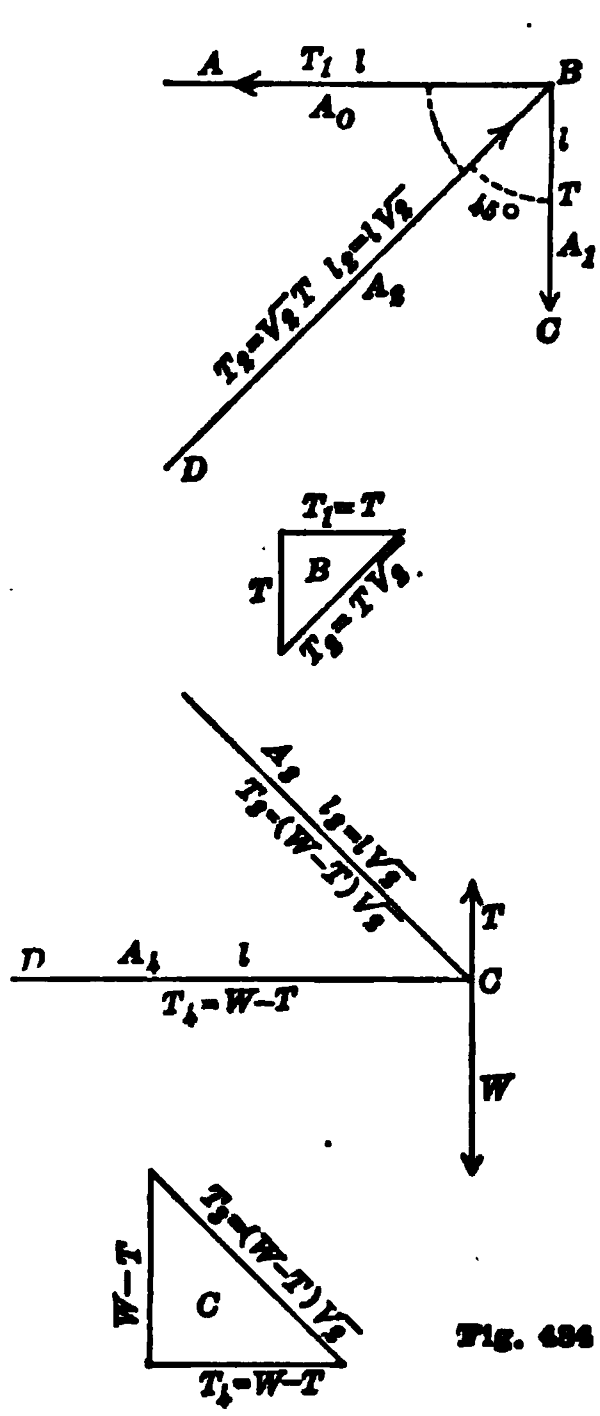


Fig. 434

Consider the frame after the load has been applied, the deformation has taken place, and the frame is again at rest.

Let T be the tension in the bar BC. The static triangle of the pin B, gives the total stresses in FB and BD. The cross-sections of FB and BC are the same. The cross-section of BD is A_2. It is assumed that the deformations do not sensibly change the static triangles, since W can be indefinitely small. Taking next the pin C and its static triangle, we have the stresses T_3 and T_4 as is readily seen.

Now to find the total work done in deforming the bars, we find λ for each. The elongation of FB is $\lambda = \dfrac{Tl}{A_1E}$, and the work done in it is

$$\tfrac{1}{2}T\lambda = \left(\frac{T^2l}{A_1}\right)\cdot\frac{1}{2E}$$

Hence in FB

$$U_0 = \frac{T^2l}{2A_1E}$$

In BC

$$U_1 = \frac{T^2l}{2A_1E}$$

In BD

$$U_2 = \frac{T_2^2l\sqrt{2}}{2A_2E} = \frac{T^2l\sqrt{2}}{A_2E}$$

In FC

$$U_3 = \frac{T_3^2l_3}{2A_3E} = \frac{(W-T)^2l\sqrt{2}}{A_3E}$$

In DC

$$U_4 = \frac{T_4^2l}{2A_4E} = \frac{(W-T)^2l}{2A_4E}$$

$$\text{Total Work } U = \frac{l}{2E}\left(\frac{T^2}{A_1} + \frac{T^2}{A_1} + \frac{2T^2\sqrt{2}}{A_2} + \frac{2(W-T)^2\sqrt{2}}{A_3} + \frac{(W-T)^2}{A_4}\right)$$

Now T must be of such a value as to make U a minimum. Hence we must have

$$\frac{dU}{dT}=0$$

Differentiating, we have after reducing,

$$\frac{T}{A_1}+\frac{2T\sqrt{2}}{A_2}+\frac{T}{A_1}-\frac{2(W-T)\sqrt{2}}{A_3}-\frac{W-T}{A_4}=0;$$

Hence

$$T=\frac{\frac{2\sqrt{2}}{A_3}+\frac{1}{A_4}}{\frac{1}{A_0}+\frac{1}{A_1}+\frac{2\sqrt{2}}{A_2}+\frac{2\sqrt{2}}{A_3}+\frac{1}{A_4}}\cdot W$$

which determines T and all the stresses are found from the static triangles.

Having found T we can find U; and from U we find the *vertical displacement* of C from the Eq. $\Delta=\frac{2U}{W}$.

If all bars have the same cross-section,

$$T=\frac{1+2\sqrt{2}}{3+4\sqrt{2}}\cdot W=\frac{13-2\sqrt{2}}{23}\cdot W=W(0.44)$$

If (a more rational supposition),

$$A_1=A_0 \qquad A_2=2\sqrt{2}A_0$$

$$A_3=A_0\sqrt{2} \qquad A_4=2A_0,$$

we shall have
$$T=\frac{5}{13}W.=W(0.38)$$

452. A second and very similar problem will now be solved in *two ways*, for the purpose of showing their relation, and the ease with which problems formerly held to be indeterminate can be solved.

As before, the frame is a rectangle attached by pins to supports in a rigid vertical wall. The weight of the bars is neglected and no allow-

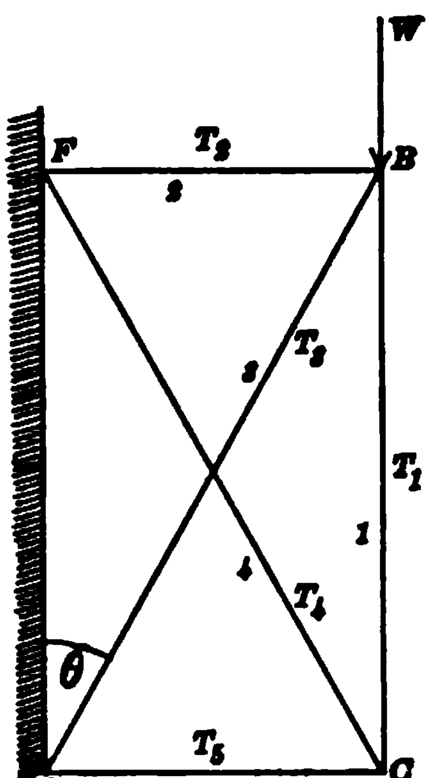


Fig. 435

ance is made for friction as the motion is inconsiderable. Fig. 435 shows the five bars with the load W placed above the pin B. The bars are prismatic and are numbered from 1 to 5; their lengths are l_1 to l_5 and their cross-sections are A_1 . . . . . A_5. A part of the load which we will call T_1 is transmitted from pin B to pin C, so that $W-T_1$ is supported by bars 2 and 3; meanwhile the load T_1 is supported by the bars 4 and 5. Two static triangles, one for B and one for C, will give the stresses (in terms of T_1) in the several bars, and from those stresses, the changes in length of all the bars are found, whether by lengthening or shortening. The student should show that:

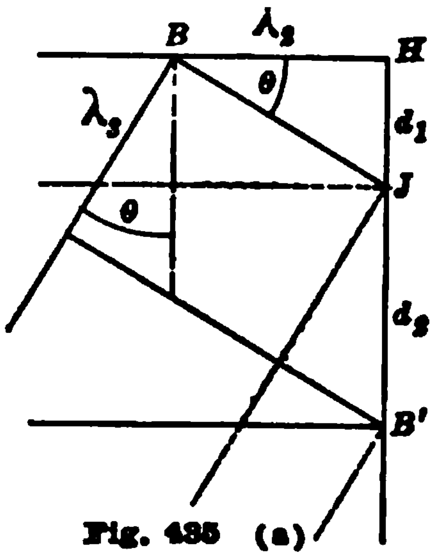


Fig. 435 (a)

$$\lambda_1 = \frac{l_1 T_1}{A_1 E}$$

$$\lambda_2 = \frac{l_2(W-T_1)\tan\theta}{A_2 E}$$

$$\lambda_3 = \frac{l_3(W-T_1)\sec\theta}{A_3 E}$$

$$\lambda_4 = \frac{l_4 T_1 \sec\theta}{A_4 E}$$

$$\lambda_5 = \frac{l_5 T_1 \tan\theta}{A_5 E}.$$

With such deformations it is plain that the total work done is

$$U = \frac{1}{2}\left(\lambda_1 T_1 + \lambda_2 T_2 + \lambda_3 T_3 + \lambda_4 T_4 + \lambda_5 T_5\right)$$

$$= \frac{T_1^2}{2E}\left(\frac{l_1}{A_1} + \frac{l_4 \sec\theta}{A_4} + \frac{l_5 \tan\theta}{A_5}\right) + \frac{(W-T_1)^2}{2E}\left(\frac{l_2\tan\theta}{A_2} + \frac{l_3\sec\theta}{A_3}\right)$$

Differentiating with respect to T_1 and making $dU=0$, we have after simplifying,

$$0 = T_1\left(\frac{l_1}{A_1} + \frac{l_4\sec\theta}{A_4} + \frac{l_5\tan\theta}{A_5}\right) - (W_1 - T_1)\left(\frac{l_2\tan\theta}{A_2} + \frac{l_3\sec\theta}{A_3}\right)$$

whence

$$T_1 = \frac{W\left(\dfrac{l_2 \tan\theta}{A_2} + \dfrac{l_3 \sec\theta}{A_3}\right)}{\dfrac{l_1}{A_1} + \dfrac{l_2 \tan\theta}{A_2} + \dfrac{l_3 \sec\theta}{A_3} + \dfrac{l_4 \sec\theta}{A_4} + \dfrac{l_5 \tan\theta}{A_5}}.$$

and since all lengths, angles and cross-sections are known, T_1 is known, and all the other T's, and hence U. The work was done by the descent of W slowly and gently applied. Hence $\frac{W}{2}\Delta = U$, and $\Delta = \frac{2U}{W}$, the deflection of B is found.

453. The student must not be disturbed by the necessity of drawing static triangles, when all the forces are unknown. The vectors T_1 and $W - T_1$ may have *any length.* The triangle is to show *relative* not *absolute* values. It may be said that the *scale* remains to be found when the value of T_1 has been actually determined by the formula.

The following examples will serve a double purpose; securing familiarity with the method; and affording a comparison of designs.

Problems to be solved by the method of least work.

1. Fig. 436 shows a composite bridge truss which carries a single load P at the middle lower pin; a part of the load T is transferred by a tie-bar to the pin at C. The length of a vertical member is taken as unity. The span is *four* (4) units. The cross-sections are either A_0 or a multiple of A_0. The component trusses are drawn separately. A stress diagram for each should be drawn by the student, and then the value of T be found.

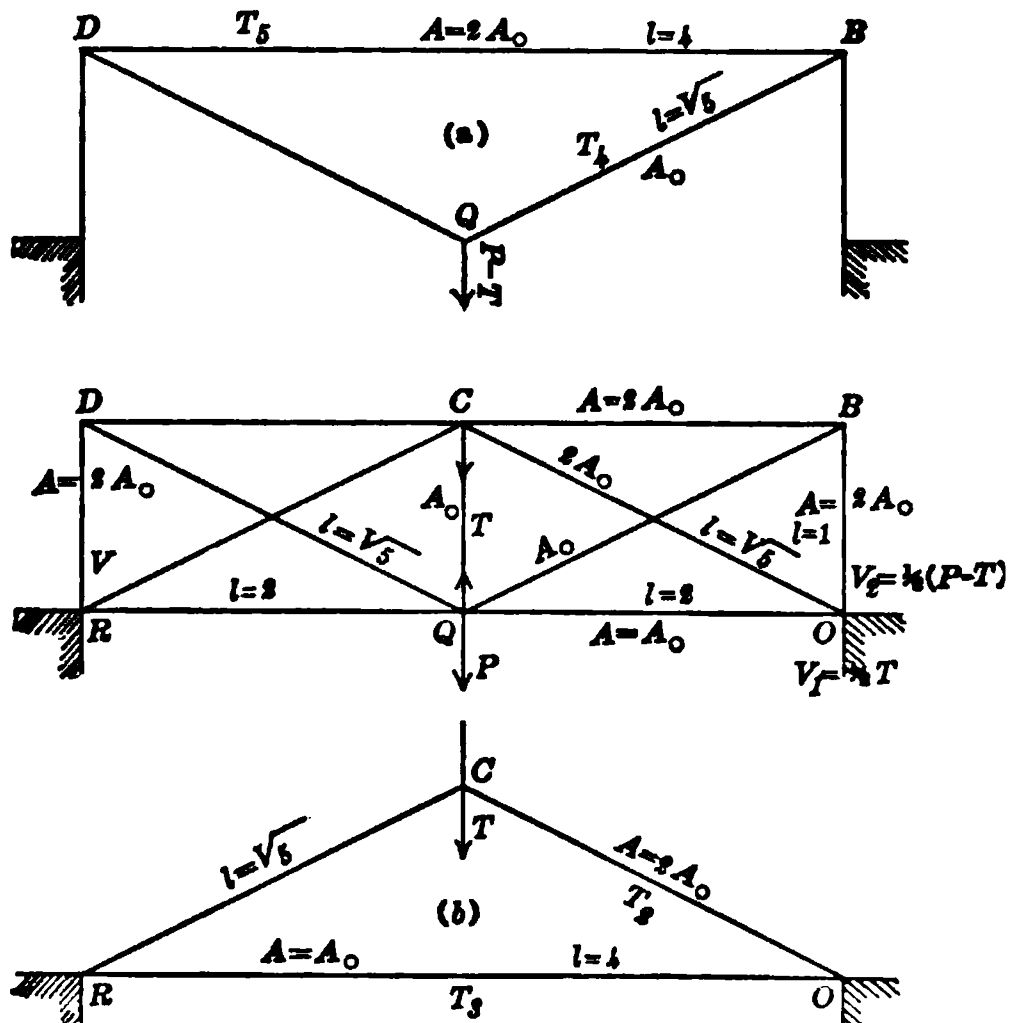


Fig. 436

Ans. $T = P.\dfrac{10\sqrt{5}+9}{15\sqrt{5}+29}.$

2. Rebuild the above truss, as shown in Fig. 437, reducing the two chords to one in which the *resultant* stress is small,

being a *difference* between a tension and a compression. The load W is placed upon a common pin at O. Find T.

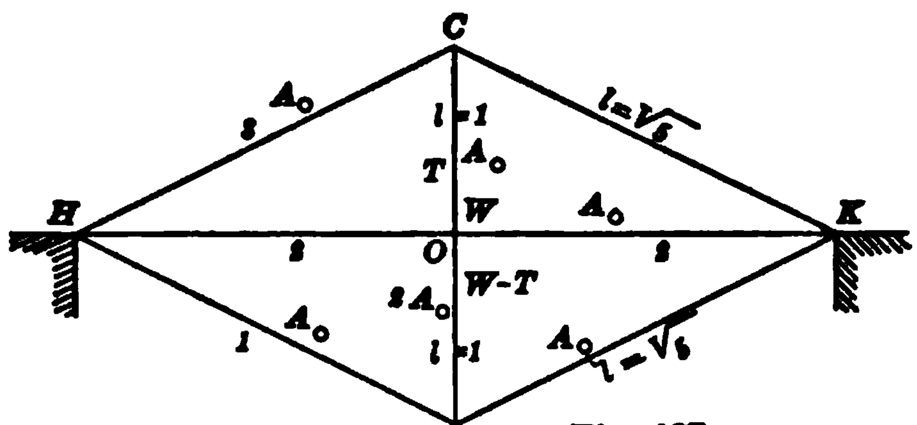


Fig. 437

3. Another compound truss is fully shown by Fig. 438. Altho the load W is placed upon the center pin in the lower chord, a part of it is transferred to the pin at O. The drawing of the truss is lettered, and the complete stress diagram is drawn with an *assumed* value of T_1. The student is to find the correct value, the cross-sections of bars being relatively as indicated in the drawing.

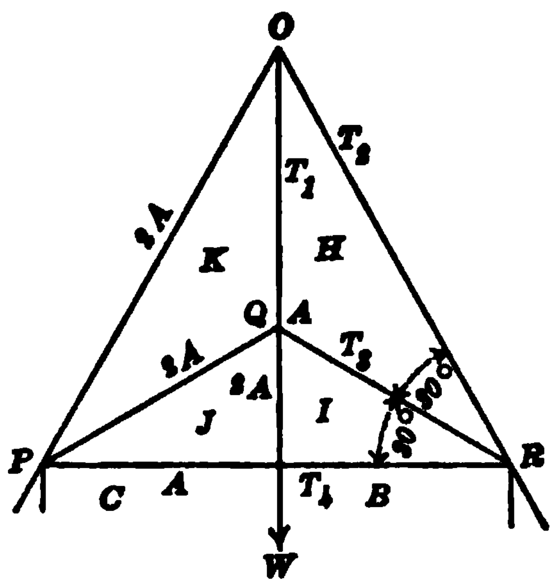


Fig. 438

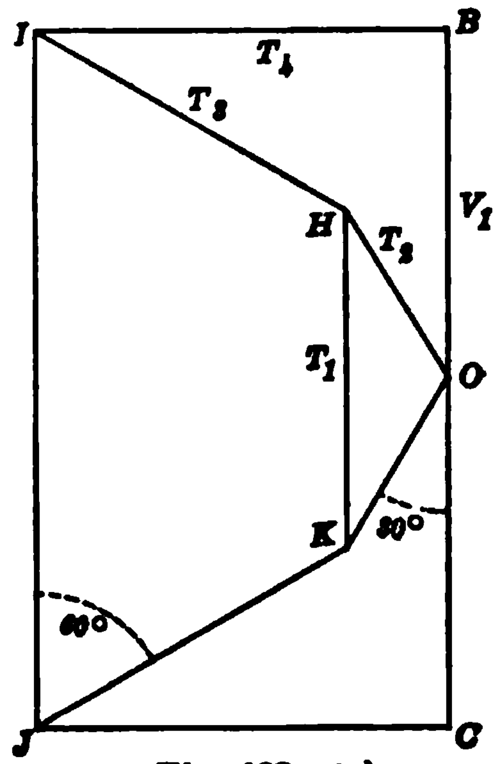


Fig. 438 (a)

454. A solution by consistent deformations. The second method of solving the problem of **452** is based upon the idea that the physical displacements of two pins, one in each component truss of a composite frame or truss are *consistent* with the *deformation of the bar connecting them.* Hence, this method of determining the stresses in redundant members by their displacements, may properly be called *the method of Consistent Deformations.* See Figs. 435 and 435 (a).

The two pins whose vertical displacement are to be found are B and C. The absolute values of all the changes in length from λ_1 to λ_5, in terms of constants and the unknown value of T_1 are here determined as found and given in **452**. The displacement of B, due to the elongation λ_2 and the compression λ_3, is visibly shown on a large scale in Fig. 435 (a). Both bars must revolve infinitessimally about pins at their fixed ends, so that B is moved to B', and the vertical displacement is

$$HB' = HJ + JB'$$

The component HJ is due to λ_2, and is equal to $d_1 = \lambda_2 \tan\theta$.
The component JB' is *then* due to λ_3, $\qquad d_2 = \lambda_3 \sec\theta$.
Hence the total vertical displacement of the pin B is

$$\Delta_B = \lambda_2 \tan\theta + \lambda_3 \sec\theta.$$

Going now to the pin C, we find by similar reasoning

$$\Delta_C = \lambda_4 \sec\theta + \lambda_5 \tan\theta.$$

Now the distance from B' to C' must be less than was the distance from B to C, since the bar No. 1 has *shortened* as it transferred the load which depressed the lower pin. Hence the difference between Δ_B and Δ_C *must be* λ_1; so that the **Consistent Equation** is

$$\lambda_2 \tan\theta + \lambda_3 \sec\theta - (\lambda_4 \sec\theta + \lambda_5 \tan\theta) = \lambda_1.$$

Every term is a simple function of T_1, and when the values of all the λ's are substituted, the value of T is found to be

$$T_1 = W \cdot \frac{\dfrac{l_2 \tan\theta}{A_2} + \dfrac{l_3 \sec\theta}{A_3}}{\dfrac{l_1}{A_1} + \dfrac{l_2 \tan\theta}{A_2} + \dfrac{l_3 \sec\theta}{A_3} + \dfrac{l_4 \sec\theta}{A_4} + \dfrac{l_5 \tan\theta}{A_5}}.$$

This value of T_1 is exactly the same as that found on page 424, by assuming that the *work* of *deformation* was a minimum. In fact this second method of solution may be said to offer a confirmation, if not an indirect proof of the *Doctrine of Least Work.*

455. Deflections found by displacements. It is convenient at this point that the displacement (deflection) of any pin P in an irregular (or regular) frame, whose position is determined by bars whose deformations are known, be found graphically and analytically by an extension of the above simple method.

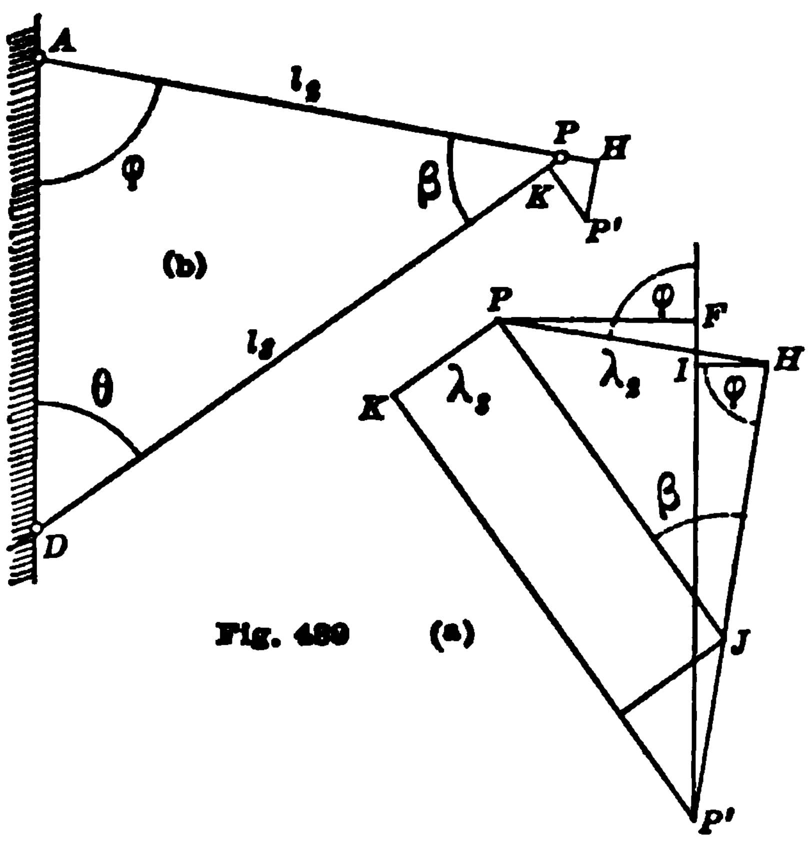


Fig. 439

Let, Fig. 439, the position of P be determined by the bars l_2 and l_3 whose deformations are known. AD is assumed to be vertical and the angles θ, ϕ and β are known. By the method of the last section the total displacement due to the two deformations $+\lambda_2$ and $-\lambda_3$, is PP'.

Analytical expressions for the vertical and horizontal components of PP' are found from the enlarged figure (a). The vertical component of PP' is

$$\begin{aligned} FP' = FI + IP' &= \lambda_2 \cos\phi + (HJ + JP') \sin\phi \\ &= \lambda_2(\cos\phi + \cot\beta \sin\phi) + \lambda_3 \operatorname{cosec}\beta \sin\phi \end{aligned}$$

and the horizontal component is

$$PF = \lambda_2(\sin\phi - \cot\beta\cos\phi) - \lambda_3 \operatorname{cosec}\beta \cos\phi.$$

If two additional bars l_4 from P, and l_5 from D, determine the position of another pin Q in the *same plane* (*b*), and if the deformations λ_4 and λ_5 should be found from known stresses,—the displacement of Q relative to P and D could be found just as PP' was found. If the enlarged figure for QQ' were so placed that Q coincides with P'; the graph then would show at a glance the displacement of Q due to the deformation in all four bars.

The method may be then extended to a third pin R depending for its position upon two new bars, l_6 connected with P' and l_6 connected with Q'. In this manner all the pins of a truss may be included and the displacement of each found.

The analytical method is less simple, especially if the triangles of the truss are unlike.

Example. Find the *displacements*, relative to the horizontal plane thru BD, of the pins P, Q, R and S, of the loaded equiangular truss shown in Fig. 440, the stresses and deformations in all bars having first been found by a complete stress diagram, as in Chapter XI.

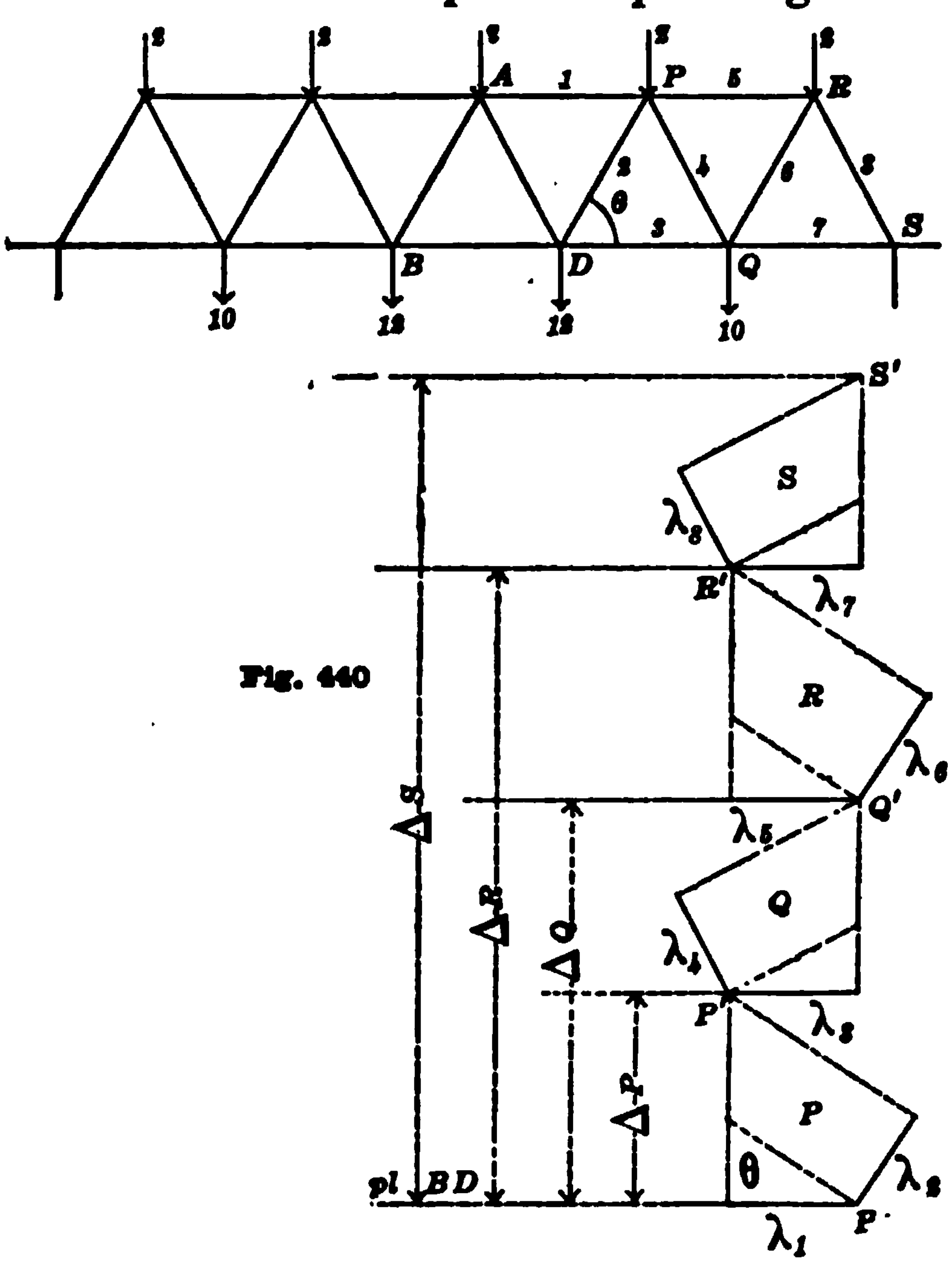


Fig. 440

Let the bars be numbered as shown, and let λ denote the elongation or compression (absolutely) as the case may be. The pins A and D are assumed to be fixed.

The graph P shows the displacement of P relative to A and D.
The graph Q shows the displacement of Q relative to P and D.
The graph R shows the displacement of R relative to Q and P.
The graph S shows the displacement of S relative to R and Q.
The total vertical displacements are ΔP, ΔQ, etc.

456. Analytically. The solution is very simple. See Fig. 410.

From graph P we get $\Delta P = \lambda_1 \cot\theta + \lambda_2 \operatorname{cosec}\theta$.

From graph Q we get $\Delta Q = \Delta P + \lambda_3 \cot\theta + \lambda_4 \operatorname{cosec}\theta$.

From graph R we get $\Delta R = (\lambda_1 + \lambda_3 + \lambda_5)\cot\theta + (\lambda_2 + \lambda_4 + \lambda_6)\operatorname{cosec}\theta$.

From graph S we get

$$\Delta S = \Sigma\lambda \text{ (in chords) } \cot\theta + \Sigma\lambda \text{ (in bracing) } \operatorname{cosec}\theta.$$

This method is based on the assumption that the angular changes in the bars of the truss are negligible, and that the chord BD is horizontal.

457. A Pratt Truss. Find Graphically and Analytically the displacements of P_1Q_1, etc., in respect to horizontal plane thru D. Stresses and therefore elongations and compressions are known.

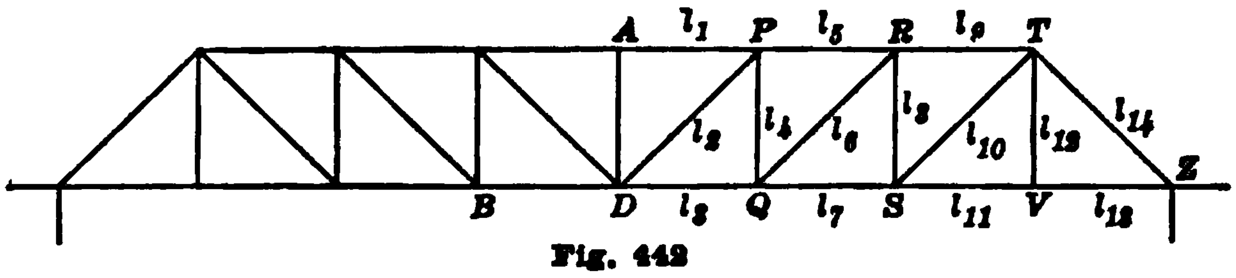


Fig. 442

Ex. Let the student turn back to the frame of Fig. 435, in which the diagonals are said to be "*not connected.*" When the frame is unloaded, it is seen that the center points of the diagonals coincide, or are in jaxta-position. Now let him find the displacement of each central point, and thereby determine whether or not the points still coincide.

CHAPTER XXIV.

Miscellaneous Problems Involving Uniformly Varying Stresses.

458. Pressures under foundations. It is usually assumed (tho the approximation to the truth is not close) that the surface action below a foundation, or at the base of a column or chimney, is a distributed force either uniform or uniformly varying. When the force is due to the weight of material which is arranged symmetrically about a vertical axis, the distribution is uniform and the center of action is at the centroid of the surface, and the intensity of the pressure is $p_0 = \dfrac{W}{A}$, where W is the total load, and A is the area of the surface of action.

When, however, the load is not symmetrically placed about the axis of the foundation or column, as in the case of a derrick supporting a heavy load, or of a column carrying an eccentric load, or in the case of a chimney or tower acted upon by a strong wind,—the distribution is not uniform, and the center of action is not at the centroid of the surface. These typical cases will be examined separately.

459. Fig. 448 represents a derrick rigidly secured to a short cylinder of masonry, and lifting a heavy weight. The supporting surface of action is a circle AB, whose radius is r. Let the load being lifted be W_1 the weight of derrick be W_2, assumed concentrated at the center; and the average specific weight of the foundation be w. Let the depth or altitude of the cylinder of masonry be h, and the horizontal length of the derrick arm be l.

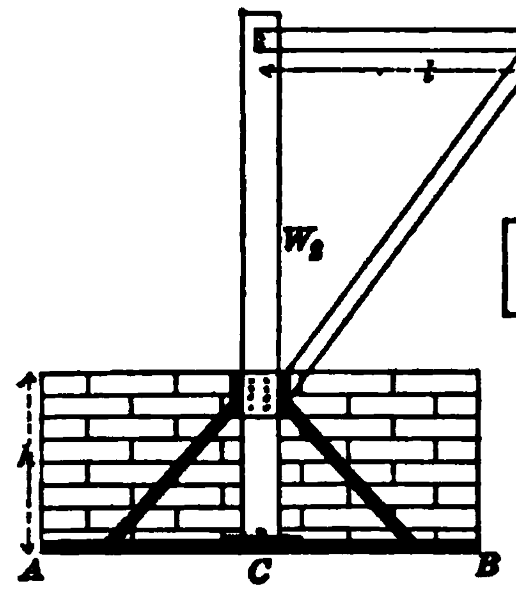


Fig. 448

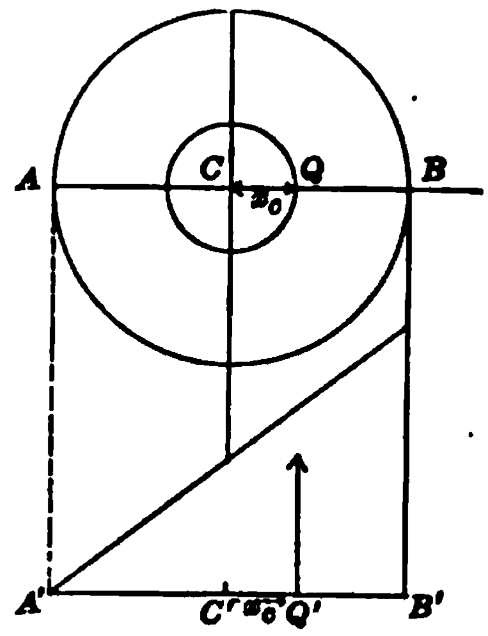


Fig. 448 (a)

It is of course understood that the derrick must be safely supported, when it is carrying its maximum load at the extreme end of the derrick arm, in whatever direction the arm is turned.

The vertical action of the substratum on the circular surface of action AB must balance the resultant of vertical force of gravity $W_2+\pi r^2hw=W_0$ acting thru C, and the force W_1 action at a distance l from C.

The resultant load is $R=W_1+W_0$ and the overturning moment is $M_c=W_1l$.

Hence the center of pressure is distant from the center $\frac{M_c}{R}$ or

$$x_0=\frac{W_1l}{W_1+W_0}$$

Now the supporting pressure must in amount equal R, and the center of the distributed support be thru the point Q.

It is evident at a glance that the moment W_1l tends to overturn the foundation and that the intensity of the support is greatest at B which is directly under the derrick arm. It is also reasonable to assume, if the supporting material be uniform, that the intensity will vary uniformly with the distance from the tangent at B. Experience teaches that for every material there is a limit, beyond which the intensity of pressure should not go. With the maximum load on the derrick, the maximum intensity at B occurs; call that intensity p_1. Assume for the moment that the pressure falls to the value p_2 at A. Then by inspection, we have for the total distributed pressure the value

$$P=\pi r^2p_2+\frac{1}{2}\pi r^2(p_1-p_2)=\pi r^2\left(\frac{p_1+p_2}{2}\right)$$

Hence by the necessary equality of R and P

$$\pi r^2\left(\frac{p_1+p_2}{2}\right)=W_1+W_o.$$

Now while there is no superior limit to the size and weight of the foundation which will safely support the derrick and its load, there is an inferior limit, and economy of construction demands the least size and weight which comports with safety and stability. The limiting condition is that while p_2 may be reduced to zero, it *must not be made negative* since the material under the foundation cannot be assumed to have any tensile action upon the foundation in question. Making $p_2=0$ we have the fundamental formula

$$\frac{\pi r^2 p_1}{2}=W_2+\pi r^2 hw+W_1. \tag{1}$$

Now the "center of pressure" for a uniformly varying pressure on a circular surface, when it is zero at a point on the circumference has already been found to be $r+\frac{r}{4}$ distant from the point of no pressure, $AQ=r+\frac{r}{4}$. (See **118** and **136**).

This center must lie in the line of action of R as found above.

Hence $$CQ=\frac{r}{4}=\frac{W_1 l}{W_2+\pi r^2 hw+W_1}$$

but by (1) this becomes $$\frac{r}{4}=\frac{2W_1 l}{\pi r^2 p_1}$$

or $$r=2\left(\frac{W_1 l}{\pi p_1}\right)^{\frac{1}{3}}=1.36\left(\frac{W_1 l}{p_1}\right)^{\frac{1}{3}} \tag{2}$$

and from (1) $$h=\frac{p_1}{2w}-\frac{W_2+W_1}{\pi r^2 w} \tag{3}$$

Thus the dimensions of the foundation are found. It is seen from the value of r, that it increases as p_1 becomes less, *i. e.* as the substratum is less firm and less unyielding. On the other hand, when the material is hard and unyielding, p_1 is greater and r is less. The center of pressure should never go outside the circle of radius $\frac{r}{4}$.

460. The great value of these results of analysis justifies a practical example.

Given the derrick arm l	$=27$ feet
Given $W_1=12$ tons	$=24000$ lbs.
Given $W_2=1000$	$=1000$ lbs.
Given w (stone reinforced by steel)	$=200$ lbs. per cub. ft.
Given $p_1=$	$=3000$ lbs. per sq. ft.

$$r=1.36\left(\frac{24000\times 27}{3000}\right)^{\frac{1}{3}}=\ 8.19\ \text{feet.}$$

$$h=\frac{3000}{400}-\frac{25000}{\pi(8.19)^2 200}$$

$$h=7\frac{1}{2}\ -\ 0.57=7\ \text{feet, nearly.}$$

461. A second method of analysis. Instead of limiting the center of pressure to the smaller circle with a radius $\frac{r}{4}$, a very common practice is to use the equation of moments about the point B, without regard to the limiting value of p_1. Hence

$$(W_1+W_2+\pi r^2 hw)r=fW_1 l$$

where f is a factor of safety.

The value of f could only be determined by experience, and experimental results would greatly depend upon the nature of the substratum on which the foundation rests. The above equation contains two variables, r and h. It is, however, customary to assume an arbitrary relation between the two, such as

$$h=\frac{4r}{5}.$$

This value substituted in the last equation gives an equation of the fourth degree in r:—

$$(W_1+W_2)r+\frac{4}{5}\pi r^4 w=fW_1 l$$

or

$$r^4+\frac{5(W_2+W_1)r}{4\pi w}=\frac{5fW_1 l}{4\pi w}.$$

The common value of f is $\frac{5}{2}$. If this value be used and the values of W_1, W_2, 1, and w given in the last example, the equation becomes

$$r^4+49.72r=3221.6$$

This equation can be solved by approximation and the result can be compared with the results found in the last section. This gives r = 7.32 feet nearly.

The previous result was 8.16, which shows that the constant f, which entered into the value of the second member of the above equation, was too small. If now, in our general equation, we put in the value of $r = 8.19$, we find the value of f, the factor of safety which should have been used, to be 3.75, instead of 2.5.

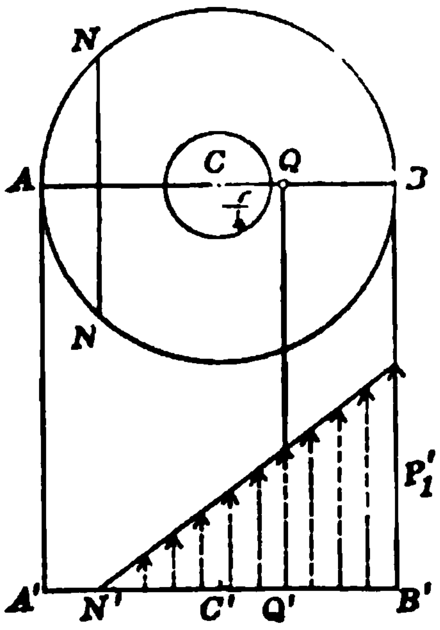


Fig. 444

With the value of r last found, namely, 7.32 feet, the center of pressure would have come outside the circle whose radius was $\frac{r}{4}$. The neutral axis of the uniformly varying stress would have cut across the circle as shown in Fig. 444, and the entire load of the foundation, derrick and all, would have been carried by the larger segment of the circle and would have given p_1 a maximum intensity larger than 3000 pounds per square foot.

462. The middle quarter. The discussion in **459** was based upon the assumption that the maximum intensity of pressure was limited to a known value, p_1, at the point B, and that the pressure varied uniformly.

These assumptions led to the confining of the center of pressure to the circumference of a circle whose radius was $\frac{r}{4}$.

This limitation of the area for the center of pressure is similar to the limitation of the center of pressure on a rectangular surface when the neutral axis is parallel to one of the sides. If the axis is one of the sides itself, the center of pressure (as was shown in the case of a rectangle on the side of a tank with its upper edge just at the surface of the liquid) is $\frac{2h}{3}$ from the top, or $\frac{2}{3}$ of the width of the surface from the neutral axis. In the case of masonry this has led to the familiar stone mason's rule, that the "center of pressure shall be within the *middle third*." The same rule should apply to the circle, that is, the center of pressure must not be without the circle whose radius is $\frac{r}{4}$.

If the center of pressure is outside, then the neutral axis is no longer at A, and, as there can be no assumed tension in the supporting material the segment cut off by the neutral axis adjacent to A gives no support whatever, for that position of the load on the derrick. However, so long as the center of pressure is between the center of the circle and the point Bm, and so long as the pressure at the point B, which we have called p_1, is not so great that the material sinks and allows the founda-

tion to lean sensibly, no harm is done. The danger is, however, that it *will lean* too much, since part of the foundation is, for the time being, useless. To assume that p_1 can be indefinitely large, and that the center of pressure may go anywhere within the foundation of the base, would seem to be poor practice.

463. A foundation which is to support a chimney when the latter is acted upon by a high wind. When a chimney rigidly attached to and supported by a mass of masonry is exposed to a strong wind, the base of the foundation is the surface of action of a uniformly varying pressure. The problem is to find *first* the economic, but safe dimensions of the foundation, and second, the proper thickness of the chimney wall or shell at its base. The chimney is assumed to be a hollow cylinder, and the foundation is a solid cylinder of masonry reinforced by steel.

The Foundation.

Let the height of the chimney above the foundation be H, its exterior diameter be $2R$, and its weight be W. Let $2r$ be the diameter, and h the height of the foundation, and its weight be $\pi r^2 hw$, as in the last problem. Let the average wind pressure upon a vertical plane perpendicular to its direction be p pounds per square foot. A cylindrical surface is found by experiment to offer approximately one-half as much resistance to wind as a square chimney of equal height and diameter. Hence the total wind pressure is HRp, and its "*center*" is approximately at a point $\frac{H}{2}$ above the base.

Chimneys usually taper towards the top, but the strength of the wind usually increases towards the top. It is therefore a fair assumption to make both R and p constant. Its overturning moment about a diameter in the *bottom* of the *foundation* is

$$HRp\left(\frac{H}{2}+h\right)=M.$$

The masonry of the foundation is supposed to be sheltered from the wind, yet no account is taken of the lateral support of the filling about it, it is so uncertain and unreliable.

464. The lateral action of the wind brings into play a horizontal force of equal magnitude, and acting in the opposite direction, thereby forming a couple. This is shown by the large arrows in Fig. 445. The horizontal force acting over the whole area of the base circle represents the frictional resistance which the substratum of the

clay, gravel, or sand, offers to the tendency of the wind to move the whole structure sideways. This tendency may be resisted more or less by the filling around the foundation, but as the trench may not be filled, we omit its possible action here.

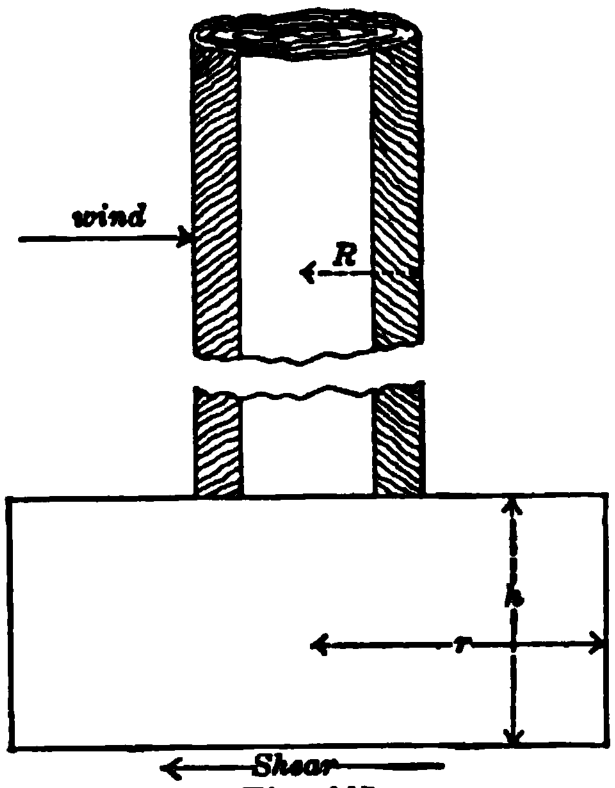


Fig. 445

The frictional force acts uniformly at right angles to the vertical supporting force and hence does not enter into our consideration of the vertical action which is a uniformly varying force when the wind blows.

465. The combination of the vertical force $W+\pi r^2hw$ and the couple M causes the resultant to act eccentrically, that is, it does not act thru the center of the base of the foundation. (See Fig. 446.) The equations for the resultant are

$$\left.\begin{aligned} P&=W+\pi r^2hw \\ CQ&=\frac{M}{P}=\frac{HRp\left(\frac{H}{2}+h\right)}{W+\pi r^2hw} \end{aligned}\right\}$$

In the *extreme case* which we are considering, the supporting vertical pressure varies from zero at A to the maximum allowable value p_1 at B. This is always the condition to be assumed for a minimum foundation. Calling the total supporting pressure P, we find as in the last problem

$$\left.\begin{aligned} W+\pi r^2hw=P&=\frac{1}{2}\pi r^2p_1 \\ CQ&=\frac{r}{4} \end{aligned}\right\} \quad (1)$$

These equations give

$$r^3=\frac{8HRp\left(\frac{H}{2}+h\right)}{\pi p_1} \quad (2)$$

At this point two courses are open:

(*a*) To give to h the arbitrary value $h=\frac{4}{5}r$, and solve the resulting equation by approximation:

$$r^3-\frac{32HRp}{5\pi p_1}\cdot r=\frac{4H^2Rp}{\pi p_1}. \quad (3)$$

Or (*b*) To substitute for h in (2) from Eq. (1)

$$h = \frac{p_1}{2w} - \frac{W}{\pi r^2 w},$$

and then solve by approximation, the equation:

$$\pi p_1 r^3 + \frac{8HRWp}{\pi r_2 w} = 4H^2Rp + \frac{4HRpp_1}{w}. \tag{4}$$

If h be omitted from equation (2) as relatively immaterial, that equation becomes

$$r^3 = \frac{4H^2Rp}{\pi p_1}. \tag{5}$$

466. A common method of procedure is to ignore p_1 altogether, and to equate the overturning moment (omitting h in the wind moment) to the moment of the weights about the axis at B. Thus:

$$\frac{H^2Rp}{2} = (W + \pi r^2 hw)r.$$

Then letting $h = \frac{4}{5}r$, and using a factor of safety, $f = \frac{5}{2}$, derive the equation

$$r^4 + \frac{5W}{4\pi w}r = \frac{25H^2Rp}{16\pi w}.$$

467. In the case of a **steel chimney** the weight of the chimney would hardly do more than compensate for the gas flue in the foundation, so that the term W would reasonably be omitted.

Making $W = 0$ in **465** and getting r in terms of p_1 we have for the foundation of a Steel Chimney,

$$r^3 = \frac{4H^2Rp}{\pi p_1} + \frac{4Rp}{\pi w}.$$

If in **466**, $W = 0$, $h = \frac{4}{5}r$, and $f = \frac{5}{2}$, we get for a **steel chimney:**

$$r^4 = \frac{25}{16}\frac{H^2Rp}{\pi w}.$$

With $w = 150$ lbs., and $p = 50$ lbs., this last equation gives

$$2r = 1.07(2RH)^{\frac{1}{2}}.$$

(See "Kent," page 927.)

Chimneys.

468. A brick or concrete chimney. Let us now consider the distributed force which supports a brick or concrete chimney, when it rests upon a *rigid foundation* and is acted upon by a high wind. Let the supporting surface be a horizontal ring formed by two concentric rings whose radii are r_1 and r_2. Fig. 446.

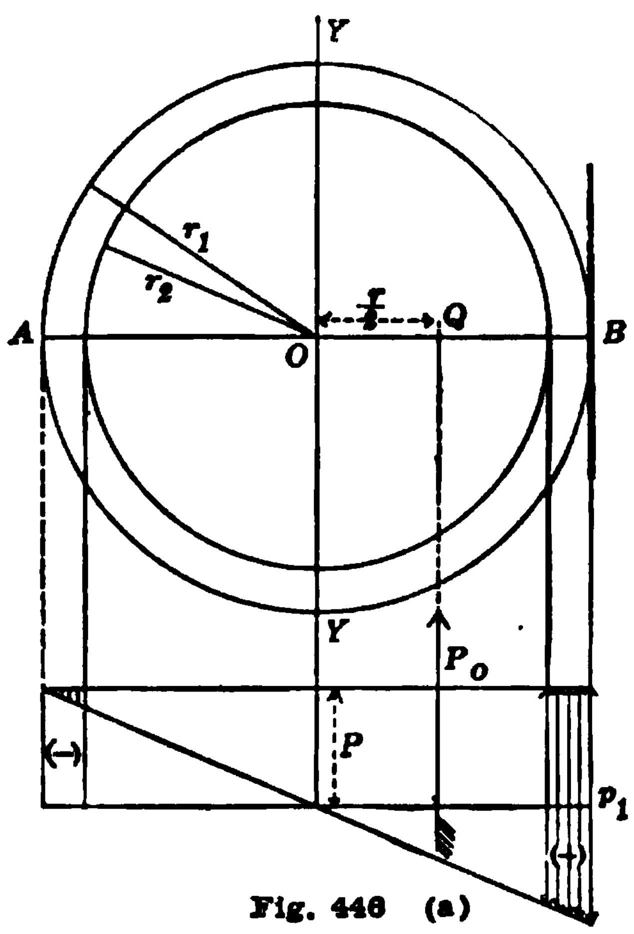


Fig. 446 (a)

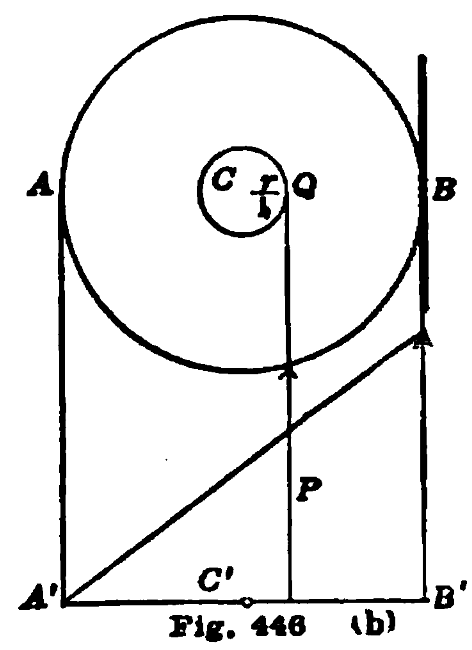


Fig. 446 (b)

Tall chimneys are usually made double with a taper in the exterior part, which must be stable and safe at every horizontal section. The interior chimney furnishes a flue of uniform size, and stands on a ring base of its own; it is not supposed to add either weight or stability to the outer chimney, and consequently is neglected in the following analysis. The exterior portion affords shelter and protection to the interior flue, and is not injuriously affected by changes of temperature.

No attempt is now made to design an ideally perfect chimney. Our object is to show how the principles of distributed and balanced forces are illustrated in a chimney so built as to withstand high wind with safety. No dimensions except the height and the thickness of the base will be assumed, and the total weight will be represented by the single letter W. It is sufficiently exact to assume the wind pressure as Hr_1p (See **463**).

The chimney is a cantilever beam fixed at the lower end and acted upon by a uniformly distributed force whose center is $\frac{H}{2}$ above the base, and whose magnitude is Hr_1p; hence the bending (or overturning) moment is

$$M = \frac{H^2r_1p}{2}.$$

As no tensile stress is allowed, the intensity of stress in the base ring cannot fall below zero. When it is zero at A and rises to p_1, the greatest allowable stress at B, the condition is fairly represented in Fig. 446. The magnitude of the support must be

$$W = p_0(\pi r_1^2 - \pi r_2^2) = \frac{p_1}{2}(r_1^2 - r_2^2)\pi.$$

If the actual stress be divided into:

First. A uniform stress of intensity p_o; and

Second. A uniformly *varying* stress whose magnitude is zero (being half positive and half negative) with an extreme fiber stress $\pm p_o$. Hence the moment of the whole about an axis thru the center of the circles is

$$M=\frac{p_o}{r_1}\cdot\left(\frac{\pi r_1^4}{4}-\frac{\pi r_2^4}{4}\right)=\frac{\pi p_o}{4r_1}(r_1^4-r_2^4)$$

The uniform stress has no moment about the axis thru O. Hence equating moments

$$\frac{H^2r_1p}{2}=\frac{\pi p_o}{4r_1}(r_1^4-r_2^4) \qquad (1)$$

We may properly assume that for chimneys of moderate height and for the upper half of the tall chimneys, $p_o=wH$. The enlarged, ornamental top and cap, justify this assumption. If t be the thickness of the cylindrical wall

$$\left.\begin{array}{l} t=r_1-r_2 \\ r_2=r_1-t. \end{array}\right\}$$

hence $$r_2^4=r_1^4-4r_1^3t+6r_1^2t^2+4r_1t^3+t^4.$$

Substituting for p_o and r_2 in (1) we have

$$r_1^3-\left(\frac{3t}{2}+\frac{Hp}{2\pi wt}\right)r_1^2+t^2r_1=\frac{t^3}{4}. \qquad (2)$$

469. A practical example. This equation will give the value of r_1 if t, H, p and w are known.

Suppose $t=2\frac{1}{4}$ feet$=27''$; $H=120$ feet; $p=50$ lbs. per square foot; $w=130$ lbs. per cubic foot. The above equation becomes

$$r_1^3-6.64r_1^2+5.06r_1=2.85$$
$$r_1=5.8 \text{ feet nearly.}$$
$$2r_1=D=11'.6.$$

If either r or t be made less, there would, *under the conditions assumed,* be tension in the masonry on the windward side.

It goes without saying that the radius can be calculated at different heights from the stone foundation.

470. Center of pressure at any bed joint. If we divide M in **468** by W, we have,

$$\frac{M}{W}=OQ=\frac{\frac{\pi p_o}{4r_1}\left(r_1^4-r_2^4\right)}{\pi p_o(r_1^2-r_2^2)}=\frac{r_1^2+r_2^2}{4r_1}$$

If now we substitute for $r_2 = r_1 - t$, we have

$$OQ = \frac{r_1}{2} - \frac{t}{2} + \frac{t^2}{4r_1}$$

As the thickness t *increases*, the value of OQ diminishes towards the limiting value $\frac{r_1}{4}$, as shown in Fig. 246 (b), and as anticipated in **118**, where the center of hydrostatic pressure is found. As the thickness t *decreases*, the value of OQ increases towards the limit $\frac{r_1}{2}$ as intimated in Fig. 246 (a).

This result follows from the assumption that the stress or pressure falls to zero on the windward side and that the pressure on the lea side is $p_1 = 2p_0 = 2Hw$.

In this connection, a paragraph quoted from Kent will be found interesting.

"Rankine, in a paper printed in the Transactions of the Institute of Engineers in Scotland, for 1867-8, says:—'It has been previously ascertained by observation of the success or failure of actual chimneys, and especially of those which respectively stood and fell during the violent storms of 1856, that, in order that a round chimney may be sufficiently stable, its weight should be such that a pressure of wind of 55 pounds per square foot of a plane surface directly facing the wind, or 27½ lbs. per foot of the plane projection of a cylindrical surface . . . shall not cause the resultant pressure at any bed-joint to deviate from the axis of the chimney by more than one-quarter of the outside diameter at the joint.' "

471. Steel chimneys. When the chimney is a steel cylinder the radius is practically constant. Such a stack may be bolted to a broad cast iron flange which rests upon and is securely anchored to a stone or concrete foundation. We may therefore consider the foundation as extending to the plane section ST, so that we have a steel cylinder resting on a steel ring. Fig. 447.

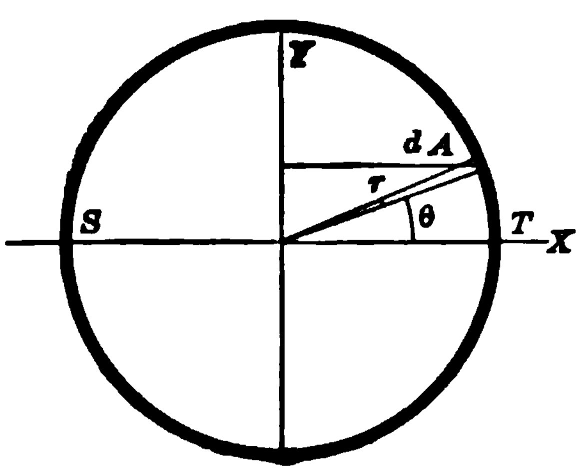


Fig. 447

This again, is a vertical cantilever beam, acted upon by a distributed horizontal force. As in the last section the equality of the bending moment and the moment of resistancegives

$$M = aI$$

$$M = \frac{H^2 rp}{2}, \text{ and } I = \frac{\pi}{4}(r_1^4 - r_2^4), \text{ and as } r_2 = r_1 - t$$

$$I = \pi t\left(r_1^3 - \frac{3}{2} r_1^2 t + r_1^2 - \frac{t^3}{4}\right)$$

But t is so small that the last three terms in the parenthesis may be neglected when compared with r, so that

$$I=\pi t r_1^3.$$

This value of I could have been obtained more directly as follows: Let dA be an element of the section of the stack, the thickness of the shell being t.

$$(r^2\cos^2\theta)dA=(trd\theta)\ (r^2\cos^2\theta)=dI$$

and
$$I=2r^3t\int_0^{\pi}\cos^2\theta d\theta=2tr^3\left(\frac{\theta+\sin\theta\cos\theta}{2}\right)\Bigg]_0^{\pi}=\pi tr^3$$

Proceeding with our analysis, as $a=\frac{p_1}{r}$, we have

$$\frac{H^2rp}{2}=\frac{p_1}{r}\cdot\pi tr^3$$

so that
$$tr=\frac{H^2p}{2\pi p_1}\tag{31}$$

The value of r is usually determined by the amount of gases (the products of combustion, hot air, etc.) the chimney is to discharge into the atmosphere, and the formula used to find the value of t is

$$t=\frac{H^2p}{2\pi rp_1}.$$

All dimensions are given in feet.

472. Steel endures tension as well as compression and in this formula, p_1 is tensile stress on the windward side and compressive on the lea side. When we combine with the stresses due to the wind, the uniform stress due to the weight, we increase p_1 on the lea side and diminish p_1 on the windward side. But the stress due to the weight, in the case of a steel chimney, is so small compared with p_1 that it is almost negligible. For a 150-foot steel chimney the vertical load is about 500 lbs. per square inch while p_1 per square inch is fully 10,000 lbs.

However, the effect of the weight is to partially compensate for the rivet holes which slightly weaken the steel on the windward side.

Practical Example.

Let $H=150$ feet; $p_1=(9000\times144)$ lbs. per square foot; $r=3$ feet; $p=50$ lbs. per square foot; $t=$ thickness of plate in feet.

Formula in **471** give

$$t=\frac{(150)^2 50}{2\pi.3.9000\times144}$$

$$t=0.046 \text{ feet}=0.552 \text{ inch}.$$

It would be beyond the scope of an elementary work on Mechanics to take up a full discussion of the foundations and sections of chimneys. At this point it is only desired to apply some of the principles developed with reference to the uniformly varying stress, and centers of stress. For further information the student would do well to consult the authorities named in Kent's Pocket Book, article on "Chimneys." The article is of itself exceedingly full and it refers the reader to a variety of discussions upon different points.

473. Varying pressures in semi-solids; Retainingwalls. Inter; mediate between fluids and solids are numerous materials or mixtures which under some conditions appear to be solids, and under other conditions appear to move or flow like liquids. Sand, loose earth, grain, powders of all kinds are familiar examples. Their fluidity is greatly dependent upon their wetness, but in every case, masses of such materials will maintain a state of equilibrium under their own weight, with upper surfaces more or less out of the horizontal. Piles of sand, grain, mixed clay, embankments of earth, broken stone, etc., are illustrations.

Every such substance has a limit to its slope. At the limiting slope, the upper layers are upon the point of sliding. The uppermost is held in place by the vertical action between it and the one below it, provided there is a certain adequate support at the bottom of the slope. Let Fig. 448 represent a vertical section of a retaining wall, and a bank of earth having a uniform slope, the line OY being the line of greatest declivity. The angle θ is not the maximum slope angle. A slope once established may be extended indefinitely, the conditions at a given depth below the surface being constant thruout the slope, but there must be some *lateral* support in sections parallel to ZY, which is constant and extends all the way up the slope.

Consider the condition of a prism represented by $ABCD$, with its length Δx extending horizontally in a direction perpendicular to AD. If BC is parallel to AD, the downward inclined action on the face DC must be *exactly balanced* by the upward inclined action on the face AB.

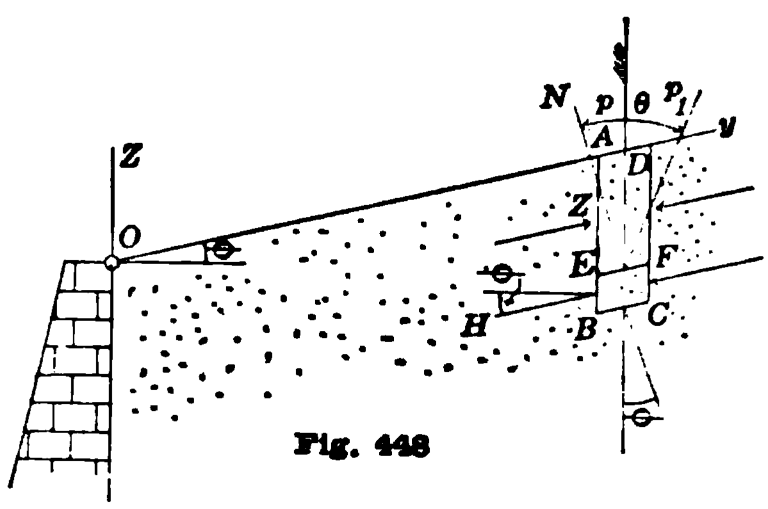


Fig. 448

If these actions did not balance, the conditions would not be the same at different heights on the slope which is contrary to observed facts.

Being thus supported laterally, the weight of the prism is carried by the supporting action of the base DC. If now we consider the situation

of an element $BCFE$, with upper and lower bases parallel to the sloping plane AD, with all lateral faces vertical, we shall find balancing forces in three directions: 1. Vertical, parallel to OZ; 2. Up the slope, parallel to OY; 3. Horizontal, perpendicular to the plane of ZY.

The pressure on the face $BEFC$ is normal and directly balanced by the pressure on the opposite face.

Let p be the intensity of the oblique, vertical stress on the face EF; its magnitude is $\Delta y.\Delta x.p$. The upward pressure on the face BC is $\Delta y\Delta x.(p+dp)$: the latter balances the former, *plus* the weight of the element; hence

$$\Delta y.\Delta x(p+dp)=\Delta y.\Delta x.(p)+\Delta z.\Delta x.\Delta y.\cos\theta.w$$

in which w is the specific weight of the material.

Hence
$$dp=dzw\cos\phi.$$
$$p=zw\cos\phi.$$

Hence p is known for a given depth, in a given material having a given slope.

The pressure on the face BE, $p'\Delta x.\Delta z$ is directly balanced by the pressure on FC, and an equation yields no solution; however, p' is closely related to p since they are "*conjugate stresses*"; they have by construction the same obliquity, and each is parallel to the faces upon which the other acts. Hence, their ratio is determined by formula in **205**, and

$$p'=zw\cos\phi\cdot\frac{\cos\phi-\sqrt{\sin^2\beta-\sin^2\phi}}{\cos\phi+\sqrt{\sin^2\beta-\sin^2\phi}}$$

in which β is the "angle of repose" of the material or the greatest possible declivity of the sloping surface plane.

Having found the intensity of the two conjugate pressures p and p', we can now, by **204** find the magnitude and direction of the *greatest pressure* in the plane YZ at the point E, vertically below the surface at the distance z.

If p_1 and p_2 are put for p_x and p_y in article **204** we have

$$\frac{p_1+p_2}{2}=\frac{p+p'}{2\cos\phi}=\frac{wz\cos\phi}{\cos\phi+\sqrt{\sin^2\beta-\sin^2\phi}}$$

$$\frac{p_1-p_2}{2}=\sqrt{\frac{(p+p')^2}{4\cos^2\phi}-pp'}=\frac{wz\cos\phi\sin\beta}{\cos\phi+\sqrt{\sin^2\beta-\sin^2\phi}}$$

$$p_1=\frac{wz\cos\phi(1+\sin\beta)}{\cos\phi+\sqrt{\sin^2\beta-\sin^2\phi}}$$

$$p_2=\frac{wz\cos\phi(1-\sin\beta)}{\cos\phi+\sqrt{\sin^2\beta-\sin^2\phi}}$$

These are the values of the greatest and least pressures at the point. They are principal stresses at right angles to each other. The direction of p_1 is given by the angle ϕ measured from the normal to the plane of slope towards and beyond the vertical, determined by the formula, derived directly from Fig. 218,

$$\cos 2\phi = \frac{2p\cos\phi - p_1 - p_2}{p_1 - p_2}$$

The formulas found above are of great importance in designing retaining walls, and foundations resting upon material of a granular character. They are based upon the assumption of complete uniformity of condition, and that condition the most unfavorable possible.

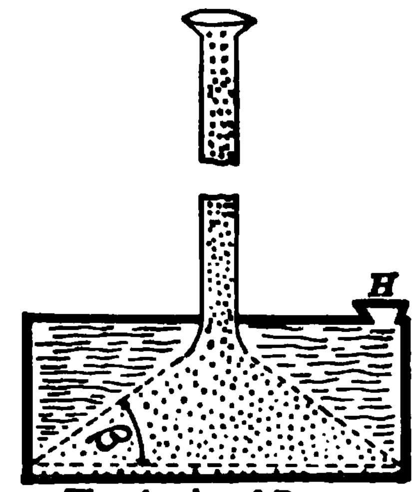
474. The insecurity of steep banks of earth. "A structure of earth, whether produced by excavating or by embankment, preserves its figure at first partly by means of the friction between its grains, and partly by means of their mutual cohesion or tenacity; which latter force is considerable in some kinds of earth, such as clay, especially when moist. It is by its tenacity that a bank of earth is enabled to stand with a vertical face, or even an overhanging face for a few feet below its upper edge; whereas friction alone would make it assume a uniform slope.

"But the tenacity of earth is gradually destroyed by the action of air and moisture, and of the changes of temperature; so that its friction is the only force which can be relied upon to produce permanent stability. In the present investigation, therefore, the stability of a mass of earth, or of shingle or gravel, or of any other material consisting of separate grains, will be treated as arising wholly from the mutual friction of those grains, and not from any adhesion amongst them."—*Rankine.*

475. A practical test under pressure. The maximum obliquity is, of course, the angle of repose, which must be found by experiments, made after many changes of weather, and with different degrees of moisture.

One of the pneumatic caissons of the St. Louis Bridge (known as the "Eads Bridge") was filled with sand, after having been solidly grounded on native rock covered with concrete. The ceiling of the air-chamber had been built with pyramidal recesses having for every pyramid a vertical tube rising from the vertex thru the masonry to the upper air. Thru these tubes sand was introduced. In order to determine the maximum obliquity, or angle of repose of wet sand under a water pressure of 100 feet, an experiment was made as shown in Fig. 449.

Sand with plenty of water was run down the pipe until the vertex of the cone of sand reached almost to the inlet. The water meanwhile escaping thru the manhole at H. The tank was then opened and the declivity was measured. Next the man hole was closed and the pipe was filled with water to a height of about 100 feet. It was found that the increased hydrostatic pressure caused *no perceptible change* in the "angle of repose" of the sloping sand.



The Angle of Repose
Fig. 449

The slope of the conical surface determined the slope of the *edges* of the pyramids in the ceiling of the air chambers.

When at last all the air chambers were filled with concrete and sand, the entry pipes were also filled solid to the very top.

The maximum declivity of loose material must not be estimated from embankments which have been exposed to rain and snow. The wash of running water, and the expansion, liquifaction and crumbling caused by freezing and thawing are matters not now under consideration.

476. The stability of retaining walls. Not only liquids but loose and granular materials of all kinds exert a lateral or overturning moment upon a wall, and the intensity of the action varies with the depth below the surface of no action at the top of the material. The centers of pressure in the case of liquids upon plane surfaces have already been studied, see Chapter VII.

The stability of a vertical wooden wall is secured by a strut or brace as shown. The end of the brace B should always be above the center of pressure, C, and the tenon of the post at A should be thick, not less than one third of the post itself, and the pin must be large and strong; in fact, it is well to spike a piece of plank as shown from a to b, since the upward thrust of the brace DB causes a tension in the post BA.

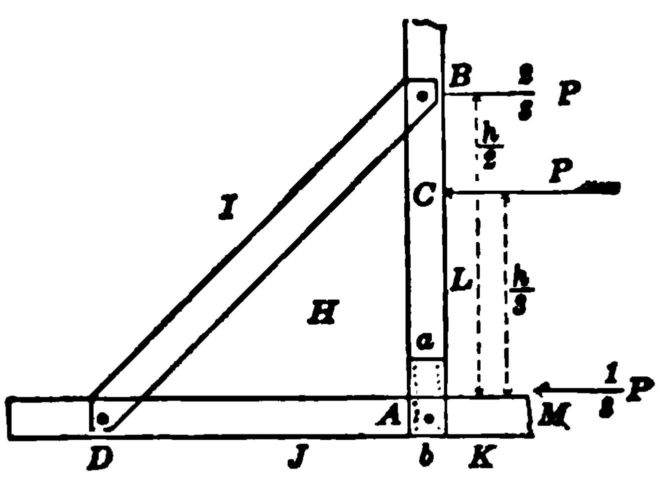


Fig. 450 (a)

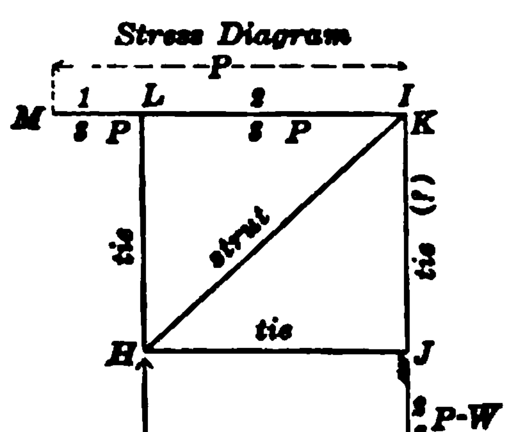


Fig. 450 (b)

Tension in BA = Thrust in DB tan θ − Weight on AB.

Assuming that B is $\frac{h}{2}$ above A, and that $ADB = 45°$, the stress diagram is drawn as shown.

The support at D may be furnished from below, or by the sill member AD as a beam. The tension in AB (if there is any) is more than balanced by the sill DM and the weight of the water it carries.

477. A retaining wall of masonry may present a vertical face to the material retained, or that face, and the other as well, may have a batter, with or without steps, and the material *retained* may have a level or a sloping surface. As a general case, Fig. 451, suppose the material has an inclination ϕ; that the angle of repose for the material is β; and that the masonry has a double batter.

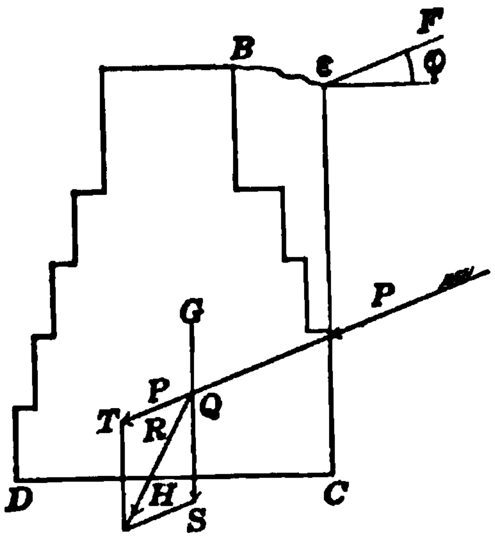


Fig. 451

Let the vertical plane EC be parallel to the axis of the wall and at right angles with the plane of greatest declivity EFC. The wedge of material BCE may be regarded as a part of the wall, but of less density. The weight of the wedge is supported by the foundation, but transmits the thrust of the material which acts upon the plane EC in the direction of EF, as already shown.

The center of the thrust is two-thirds of the distance down EC, and its magnitude is (if we take a typical section l feet long from the plane CEF to a parallel plane, making $l=1$):

$$P=\int_o^{z_1} pdz = \int_o^{z_1} wzdz\cos\phi \cdot \frac{\cos\phi-\sqrt{(\sin^2\beta-\sin^2\phi)}}{\cos\phi+\sqrt{(\sin^2\beta-\sin^2\phi)}}$$

$$P=\frac{wz_1^2}{2}\cos\phi\cdot\frac{\cos\phi-\sqrt{(\sin^2\beta-\sin^2\phi)}}{\cos\phi+\sqrt{(\sin_2\beta-\sin^2\phi)}}$$

in which z_1 = the depth EC.

Let G be the center of mass of the masonry and the wedge BEC. The resultant action of the thrust P and gravity is readily seen to be the force R which pierces the plane of the base at H. If H lies in the "middle third" of the foundation, and if the angle of repose for base of the wall and the material under it is less than the angle at H, the retaining wall is stable.

If $\phi=0$, the slope vanishes and the value of P is

$$P=\frac{wz_1^2}{2}\cdot\frac{1-\sin\beta}{1+\sin\beta}$$

If $\phi=\beta$, the slope is as great as it can be with that material, and P is

$$P=\frac{wz_1^2}{2}\cos\beta$$

and the conjugate pressures in the material are equal, as already stated.

478. Foundations resting upon clay or sand. Assuming as before that the sand or clay is loose, having lost the initial adhesion of a clay bank or packed sand, and is moist or homogenous, our formulas may be applied to explain a phenomenon much too common near the base of heavily loaded walls.

The intensity of pressure along the plane AC is readily found

$$p = \frac{W}{\text{Area } AC}.$$

The conjugate pressure, which is horizontal, is

$$p' = p \cdot \frac{1-\sin\beta}{1+\sin\beta}$$

This horizontal pressure at A, produces a *new conjugate pressure upwards* whose value is

$$p'' = p' \frac{1-\sin\beta}{1+\sin\beta} = p\left(\frac{1-\sin\beta}{1+\sin\beta}\right)^2$$

This is the vertical or lifting pressure along the plane AD at, or near, A. This pressure should be balanced by the weight of material above it, *i. e.*, wh.

Hence $$wh \text{ should} = p\left(\frac{1-\sin\beta}{1+\sin\beta}\right)^2 = \frac{W}{AC}\left(\frac{1-\sin\beta}{1-\sin\beta}\right)^2$$

That is to say: The weight of a column of the material, in which the foundation stands, whose height if equal to the depth of the foundation, and whose base is one foot square, should not be less than the pressure per foot sq. under the foundation multiplied by the fraction

$$\left(\frac{1-\sin\beta}{1+\sin\beta}\right)^2.$$

If the weight of material surrounding the foundation wall, *on both sides*, is insufficient, the wall settles, and the material on both sides rises. Plenty of illustrations of this action may be found where foundations are laid on granular or soft material. Sometimes a foundation will not settle so long as the material is not flooded with water, but settles as soon as moistened.

Perhaps the most familiar example of this double or secondary, conjugate action is seen when one puts his foot down upon a layer of loose moist clay, or sand wet or dry. The material rises all around the foot to a height proportioned to the depth reached by the foot. The reader will recall the reports of engineers on the Panama Canal at the Culebra Cut. The slopes on the sides were at first left too steep, or $\phi > \beta$. The consequence was frequent cases of sliding. It is hoped that the slope has finally been reduced to β. The double conjugate action has recently been shown by extensive upheavals in the bed of the canal in consequence of the weight of the banks, and the high degree of fluidity in the material of the cut.

479. If one inspects a steel armor plate which has been pierced by a projectile with a flat or pointed end, he will find that to a certain extent the steel has risen all around the hole on the entrance side, showing that under enormous pressures, even steel becomes a partial fluid so as to develop conjugate stresses.

Just as moisture diminishes the maximum declivity of some materials, an increase of temperature does the same. Most solids become semi-fluids when very hot, and thick liquid becomes thin when warm. On the other hand some granular substances harden with heat.

480. Stress in cables and chains. *The common Catenary.* A uniform wire or cable connecting two points not in the same vertical, loaded only with its own weight, takes the form of a curve known as the *Catenary.* The two points of suspension may or may not be in the same horizontal plane. Consider the hanging wire AB. The curve of equilibrium is the same as it would be if the wire were extended to C, since the portion AB hangs just the same whether stopping at B or C.

The equation of the curve is found as follows:— Let P be a point on the curve. Let the lowest point O where the tangent is horizontal be chosen as the origin of co-ordinates.

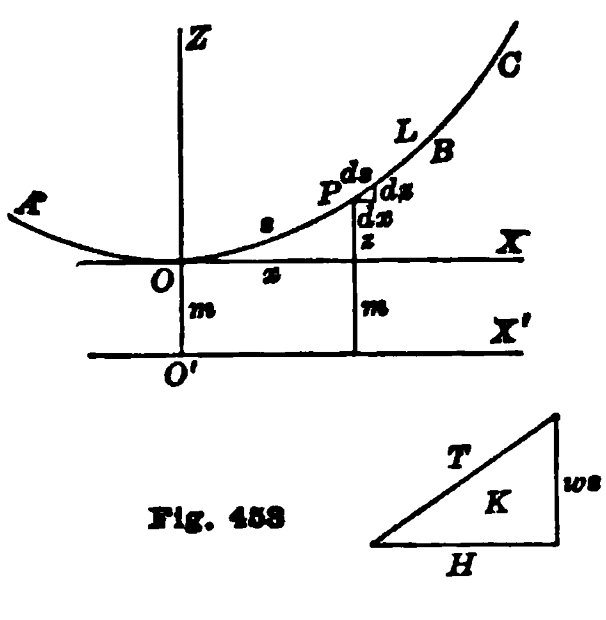


Fig. 458

The portion of the curve OP has a variable length s, and a weight ws, where w means the weight per unit of length. Let the horizontal tension at O be H. The tension, T, at P is in the direction of the tangent at that point. The static triangle K, for the portion OP, is similar to the elementary triangle ds,dz,dx; hence, if we put $H = wm$

$$\frac{dz}{dx} = \frac{ws}{H} = \frac{s}{m}.$$

It is evident that the value of H (and m) does not depend upon the position of the point P whose current co-ordinates are x and z. In other words, for *this* catenary m is a constant. Squaring the above equation, and adding 1 to each side, we readily derive

$$\text{(1)}\qquad \frac{dx^2+dz^2}{dx^2}=\frac{ds^2}{dx^2}=\frac{s^2+m^2}{m^2}$$

$$\text{(2)}\qquad \int_0^s \frac{ds}{(s^2+m^2)^{\frac{1}{2}}}=\int_0^x \frac{dx}{m}$$

$$\text{(3)}\qquad \log\frac{s+\sqrt{s^2+m^2}}{m}=\frac{x}{m}$$

$$\text{(4)}\qquad \frac{s+\sqrt{s^2+m^2}}{m}=\epsilon^{\frac{x}{m}}$$

$$\text{(5)}\qquad s=\frac{m}{2}\left(\epsilon^{\frac{x}{m}}-\epsilon^{-\frac{x}{m}}\right)$$

which is the equation of the Catenary in terms of x and s.

A second equation can be found by squaring $\frac{dx}{dz}=\frac{H}{ws}=\frac{m}{s}$ and proceeding as before

$$\frac{ds}{dz}=\frac{\sqrt{s^2+m^2}}{s}$$

$$\text{(6)}\qquad \sqrt{s^2+m^2}=z+m$$

$$\text{(7)}\qquad s=\sqrt{z^2+2zm}$$

A third equation is readily formed between x and z by eliminating s from (5) and (7) thereby giving

$$\text{(8)}\qquad z+m=\frac{m}{2}\left(\epsilon^{\frac{x}{m}}+\epsilon^{-\frac{x}{m}}\right)$$

The tension in the wire at the point P is given from the static triangle

$$\text{(9)}\qquad T=\sqrt{H^2+w^2s^2}=w\sqrt{m^2+s^2}=w(z+m)$$

Now m is evidently a length of the wire which weighs H, since $H=wm$; and T is the weight of a length of wire $(z+m)$, so that if m be laid down from O, it reaches to a horizontal line $O'X'$, whose distance from P is $z+m$. So that, if O' were taken as the origin instead of O, the equations (6), (8) and (9) would become

$$\text{(10)}\qquad \left\{\begin{array}{l} z'=\sqrt{s^2+m^2} \\ z'=\dfrac{m}{2}\left(\epsilon^{\frac{x}{m}}+\epsilon^{-\frac{x}{m}}\right) \\ T=wz' \end{array}\right\}$$

The quantity m is called the *parameter* of the curve, since it is the only constant in the equation, and the quantity which distinguishes one catenary from another. All catenaries are similar as are all circles and all parabolas. They differ only in scale or size.

It is evident that the tension in a hanging wire varies, being least at the lowest point where $z=0$ and where $T=H=wm$, and it is greatest at the highest point of suspension where

$$T=(z_1+m)w \tag{11}$$

Everything is a function of m, and yet m is never a given quantity. The given quantities are x_1 and $h_1=z_1$; or s_1 and x_1; or x_1 and T_1; or possibly $l=s_1+s_2$, $k=x_1+x_2$, and $h=z_1-z_2$. See Fig. 454. It is, therefore, necessary to find m from the data.

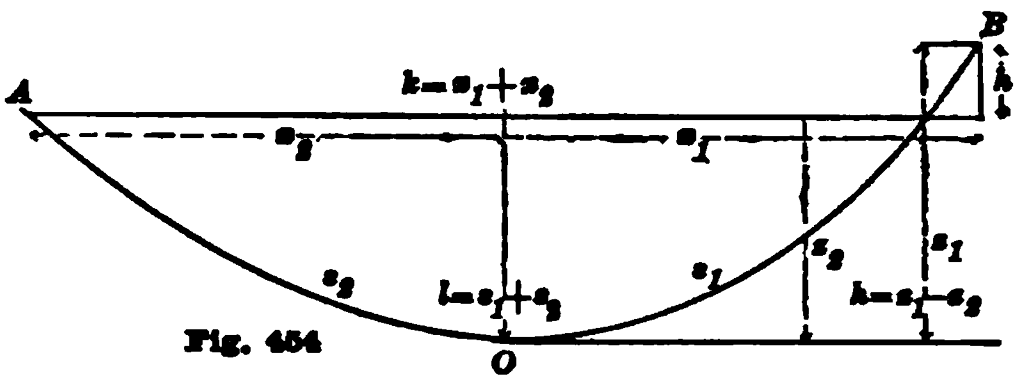


Fig. 454

Let us take the last case which is the most troublesome.

Let $l=$ the total length;
$k=$ the total span;
$h=$ the difference in the heights of the points of suspension.

The lowest point O is wholly unknown. It will be convenient to regard both x_1 and x_2 as positive, and to let both segments of the chain have the same origin and similar equations. We have then

$$x_1+x_2=k, \text{ a known quantity.}$$

Let

$$x_1-x_2=u, \text{ an unknown quantity,}$$

hence

$$x_1=\frac{k+u}{2}, \quad x_2=\frac{k-u}{2}$$

Putting these values of x_1 and x_2 in (5) and (8), we get

$$l=s_1+s_2=\frac{m}{2}\left[\left(\epsilon^{\frac{k+u}{2m}}-\epsilon^{-\frac{k+u}{2m}}\right)+\left(\epsilon^{\frac{k-u}{2m}}-\epsilon^{-\frac{(k-u)}{2m}}\right)\right]$$

$$=\frac{m}{2}\left[\epsilon^{\frac{k}{2m}}\left(\epsilon^{\frac{u}{2m}}+\epsilon^{-\frac{u}{2m}}\right)-\epsilon^{-\frac{k}{2m}}\left(\epsilon^{\frac{u}{2m}}+\epsilon^{-\frac{u}{2m}}\right)\right]$$

$$l=\frac{m}{2}\left[\left(\epsilon^{\frac{k}{2m}}-\epsilon^{-\frac{k}{2m}}\right)\left(\epsilon^{\frac{u}{2m}}+\epsilon^{-\frac{u}{2m}}\right)\right] \tag{12}$$

$$(13)\qquad z_1 - z_2 = h = \frac{m}{2}\left(\epsilon^{\frac{k}{2m}} - \epsilon^{-\frac{k}{2m}}\right)\left(\epsilon^{\frac{u}{2m}} - \epsilon^{-\frac{u}{2m}}\right)$$

Hence

$$l^2 - h^2 = \frac{m^2}{4}\left(\epsilon^{\frac{k}{2m}} - \epsilon^{-\frac{k}{2m}}\right)^2$$

$$(14)\qquad \sqrt{l^2 - h^2} = \frac{m}{2}\left(\epsilon^{\frac{k}{2m}} - \epsilon^{-\frac{k}{2m}}\right)$$

which contains only the *data* and m, so that m can be found by approximation.

Having found the numerical value of m, we next find from (12) and (13) the values of $l+h$ and $l-h$, and obtain the equations

$$\frac{l+h}{l-h} = \epsilon^{\frac{u}{m}}$$

$$(15)\qquad u = m \log_e \frac{l+h}{l-h}.$$

The values of x_1 and x_2 are now readily found, and the above equations suffice for finding whatever may be needed.

If the points of suspension are at the same level, and x_1 and s_1 are given, and equation (5) suffices for finding m.

The limiting value of T is a controlling datum in many cases. T is the total tension, and its intensity is $p_1 = \frac{T}{A}$, if A = area of a cross-section of the wire or cable. When $z_1 = z_2$, z_1 is the "sag" of the wire, and $T = w\,(z_1 + m) = p_1 A$, and

$$z_1 = \frac{p_1 A}{w} - m.$$

481. Dangerous stresses. The quantity w must include not only the weight of the wire per foot, but whatever the wire carries in the form of ice, snow, or covering of any kind if it be reasonably uniform. The working value of p_1 is given in this table.

Ultimate and Working Stress per sq. inch in pounds	Steel Wire		Copper	
	Ultimate	Working	Ultimate	Working
	150,000	25,000	60,000 to 64,000	10,000 to 16,000

One very important fact must always be kept in mind, in adjusting wires and ropes; that is, that the length depends upon the temperature or moisture or both. Everybody knows how a rope tightens when wetted, and how steel contracts as the temperature falls. A steel wire

hung with the working tension p_1 in a hot day in summer is in danger of breaking on a cold day in winter.

There is no simple formula showing the relation of ΔT (a change in the tension) to Δs (a change in the length of s due to a change in the temperature). A change in s produces a change in m, the *span* being constant, as shown in formula (5).

A change in both m and s produces a change in T as shown in formula (9). If s is made *less*, m is made *greater* (generally *much* greater).

An increase in w due to masses of snow and ice combined with low temperature often breaks wires hung in warm weather with long spans and insufficient sag. Manufacturers of steel and copper wire generally publish tables giving precautionary directions for hanging wires, allowing for both spans and temperatures.

"If a copper wire were to be erected in summer at a temperature of 70°F., with a span of 300 feet, it would be necessary to allow a sag of 5 feet 9 in.; then should the temperature drop in winter to −10°F., this sag would be decreased to 2 ft. 8½ in., with a tensile stress of 16000 lbs per sq. inch," no allowance being made for snow and ice.*

This statement should be checked by the use of formulas given above, and the additional formula from physics that the co-efficient of expansion for copper is 0.00000887 per 1°F.

482. The maximum span. Problem. To find the ratio of *sag* to *span* when $(z_1 = z_2)$, and when T is a *minimum.*

No amount of tension can eliminate the sag, tho every one knows that the greater the tension, the less the sag; and conversely, the greater the sag, the less the tension; but this last statement is not true indefinitely. There is a point where an increase of the sag would cease to diminish the tension. That point we are now to find, the span $2x_1$ being constant, and the length being indefinite.

We have the formula $T = w(z+m)$

hence
$$dT = w(dz+dm)$$

When T is a minimum, $dT = 0$; hence for the required point
$$dz+dm=0.$$

Differentiating (8), in **480** and remembering that we are considering the tension at A, so that $x = x_1 = \text{constant}$, we get

$$0 = dz+dm = \frac{dm}{2}\left(\epsilon^{\frac{x_1}{m}} + \epsilon^{-\frac{x_1}{m}}\right) + \frac{m}{2}\left(-\frac{x_1}{m^2}\epsilon^{\frac{x_1}{m}} + \frac{x_1}{m^2}\epsilon^{-\frac{x_1}{m}}\right)dm$$

* Mather & Platt, Manchester, England.

which reduces to
$$\epsilon^{\frac{x_1}{m}}+\epsilon^{-\frac{x_1}{m}}=\frac{x_1}{m}\left(\epsilon^{\frac{x_1}{m}}-\epsilon^{-\frac{x_1}{m}}\right)$$

Let $\frac{x_1}{m}=y$, for brevity. Then

$$\epsilon^{y}(y-1)=\epsilon^{-y}(y+1)$$

$$\frac{y+1}{y-1}=\epsilon^{2y}$$

By very easy approximation, we find the value of y which is evidently greater than 1 and less than 2. By trials, we get, with sufficient accuracy

$$y=\frac{x_1}{m}=1.2 \qquad \text{hence } m=\frac{x_1}{1.2}$$

These values in Eq. (8) **480,** give

$$\frac{z_1}{2x_1}=\frac{1}{4.8}\left(\epsilon^{1.2}+\epsilon^{-1.2}-2\right)=0.33+,$$

$\therefore$ *when the sag is one-third the span, T is a minimum.*

The reader may be curious to know the length of the longest possible span of a steel or copper wire, assuming that the supports are on the same level, and that they are high enough to admit of the *sag*. If the value of m is taken to be $\frac{\text{span}}{2.4}$, and the tensile strength of steel is taken at 100,000 lbs. per sq. inch, and the weight per cubic foot at 490 lbs., the span is about ($7\frac{1}{3}$) miles. The corresponding figures for copper are approximately:—

40,000 lbs. per sq. inch; 552 lbs. per cubic ft., and a span of $2\frac{2}{3}$ miles.

It will be a useful exercise on the part of the student to check or correct these figures.

483. Cords with loads so distributed that *w* is the same per foot horizontally. The load includes the cord itself. Referring to figure 453, and to the static triangle for the cable OP, we note that for the cable now under consideration, we have wx for the load and *not* ws. Consequently our proportion is

$$\frac{H}{wx}=\frac{dx}{dz}$$

$$H\int_0^z dz=w\int_0^x x\,dx, \text{ and } x^2=\frac{2H}{w}z$$

which shows that the cord takes the form of a parabola, and a tangent at P bisects the abcissa x.

If the span and the sag are given, H is found from the equation. H is the least tension.

The tension where it is greatest, viz., at the top of a pier, is found by the equation $H=\frac{wx_1^2}{2z_1}$.

$$T=\sqrt{H^2+w^2x_1^2}$$

The length of the entire cable from tower to tower is found from the proportion

$$\frac{T}{H}=\frac{ds}{dx}=\frac{\sqrt{H^2+w^2x^2}}{H}$$

If, as in the case of the catenary, we let $\frac{H}{w}=m$, this becomes

$$ds=\frac{1}{m}\cdot(m^2+x^2)^{\frac{1}{2}}\,dx.$$

$$\text{Hence }\ l=2s_1=\frac{2}{m}\int_o^{x_1}(m^2+x^2)^{\frac{1}{2}}\,dx$$

$$=\frac{2}{m}\left[\frac{x}{2}(m^2+x^2)^{\frac{1}{2}}+\frac{m^2}{2}\log_e\left(x+(m^2+x^2)^{\frac{1}{2}}\right)\right]_o^{x_1}$$

$$=\frac{x_1}{m}(m^2+x_1^2)^{\frac{1}{2}}+m\log_e\frac{x_1+(m^2+x_1^2)^{\frac{1}{2}}}{m}$$

$$l=x_1\left(1+\frac{w^2x_1^2}{H^2}\right)^{\frac{1}{2}}+\frac{H}{w}\log_e\frac{wx_1+(H^2+w^2x_1^2)^{\frac{1}{2}}}{H}.$$

484. An arch with hanging loads. If, in the place of a cable hanging upon lofty piers, a roadway is supported by an arched rib, the total distributed load being approximately uniform, the axis of the rib takes the form of a parabola, with the vertex high in the air. The ends of the rib rest upon pins mounted in skew backs which rest on firm foundations. Under changes of temperature the vertex of the parabola rises and falls, and as the radius of curvature at the vertex changes, excessive bending moments may be developed in the upper and lower members, if the *depth of the rib* be large.

485. Stress in a hanging wire or rope. The writer was once lowered into a copper mine, thru a vertical shaft one mile deep. The cage was suspended by a wire rope 1⅜ inches in diameter, which was unwound from a drum some 30 feet in diameter. The cage, when

loaded with a car full of copper-bearing rock, must have weighed about six tons. Exact figures show that the cross-section of the rope has an effective area of 0.88 square inch, and that the load at the end was 11,500 lbs. The steel rope weighed 3 lbs. per foot. The tension *per square inch of the rope* at the bottom, when the fully loaded cage was hanging ready to start from the bottom of the mine, was about 13,000 lbs. The rope 5280 feet long, at 3 lbs. per foot weighed 15,840 lbs. Hence the total tension at the top when the loaded bucket was ready to start from the bottom was 27,340 lbs.

If at the start, or while getting under full speed (a "mile per three minutes"), the acceleration was as high as $\frac{g}{10}$, the required tension was about $\frac{11}{10}$ of the weight of car, cable and mineral, making $T=30{,}074$ lbs., or $p=\frac{30074}{0.88}=34{,}000$ per square inch.

While a sound rope would probably hold under that tension, no engineer ought to approve of a working stress so high.

The ideal arrangement would be a cable with increasing strength from bottom to top.

Before the invention of high pressure steam engines, deep mines were pumped out by force pumps operated by engines (atmospheric and condensing) (see Newcomen's engine, Encyclopedia Britainica) situated near the top of the shaft. The connecting rod was built in sections, increasing in size and strength gradually from bottom to top, so that the stress per square inch was approximately the same thruout. The problem then was one of varying cross-section, rather than of varying stress. Nevertheless, the problem is appropriate here. Suppose a tapering wire supports a weight or load, at its lower end, and its own weight as well; what is the radius of its ccross-section at a distance x above the lower end if the unit stress is constant?

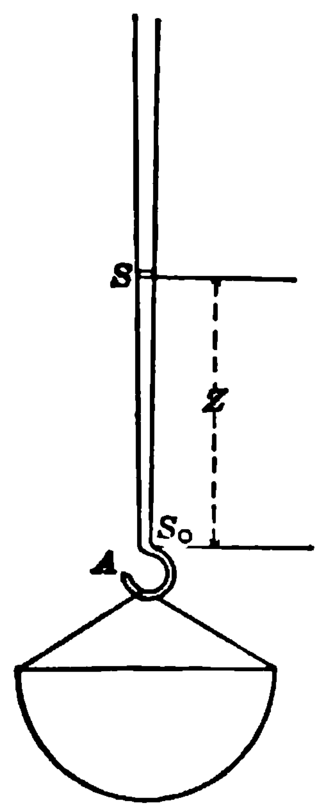


Fig. 455

If the load at the bottom is W_1 and if the uniform stress allowed is p_1, we find the law of form of the wire or rod as follows:

At the height z from A, take an element Sdz in which S is the cross-section at that point. The downward action upon the element is Sp_1+wSdz; the action up is $p_1\,(S+dS)$; as these are equal, we have

$$wSdz=p_1dS$$

or

$$wdz=p_1\frac{dS}{S}$$

Integrating from 0 to z, and from S_o to S

$$wz = p_1 \log_e \frac{S}{S_o} = p_1 \log_e \frac{p_1 S}{W}$$

whence
$$S = \frac{W}{p_1} \epsilon^{\frac{wz}{p_1}}, \text{ or } \log_{10} S = \log_{10} \frac{W}{p_1} + \log_{10} \epsilon^{\frac{wz}{p_1}}$$

$$\log_{10} S = \log_{10} \frac{W}{p_1} + \frac{wz}{p_1}(0.4343)$$

which gives the cross-section at any height, $S_o = \frac{W}{p_1}$ being the cross-section in sq. feet at A.

Example. Let W the load = 4000 lbs. Let $p_1 = 10{,}000 \times 144$ lbs. per square foot, and $w = 490$ lbs. per cubic ft.; find S_o, and S at a height of 5000 feet. Use the formula,

$$\log_{10} S = \log_{10} W - \log_{10} p_1 + \frac{wz}{p_1}(0.4343)$$

Ans. S at the height of 5000 ft., 2.19 sq. inch.
S_o at the bottom 0.4 sq. inch.

486. Variation of stress in liquids. It was shown in XII that if the stresses on two rectangular planes were normal, equal, and of the same kind, the stress on every plane was the same. That is the case in fluid pressure.

Given a mass of liquid, whose specific weight is w, at rest in a tank. Tho the density is not strictly constant under varying pressure, it is so nearly so that it is unnecessary for engineers to allow for change in density provided the temperature is constant. Consider a cubic element dx, dy, dz, at a distance z below its horizontal upper surface. Let p be the intensity of pressure on the upper surface of an element and $p+dp$ the intensity upon the lower surface. The downward pressure upon the upper surface then is $pdxdy$. The upward pressure on the bottom is $(p+dp)dxdy$. The pressures upon the lateral faces balance and need not be considered.

The upward action exceeds the downward action by $dxdy.dp$. This is balanced by the weight of the element

$$dxdydp = wdxdydz$$

where w means the weight of a cubic unit of the liquid and z measures the depth; hence

$$dp = wdx$$

$$\int_{p_o}^{p} dp = w \int_{o}^{z} dz = p - p_o = w \int_{o}^{z} dz = wz$$

$$p = p_o + wz$$

The pressure upon the upper surface of the liquid (if there be any) being p_0. The pressure p, is called the absolute pressure. In ordinary steam and gas gauges, the pressure $p-p_0$ is shown, which is the *excess over the atmospheric pressure.* It is seen that the *gauge pressure* is proportional to the depth.

The normal value of p_0 is 14.7 lbs. per sq. inch; this is very nearly the weight of 30 cubic inches of mercury at a temperature of 32°F. The normal pressure of air upon a square foot is 2116.4 lbs. At 4°F. water weighs about 1000 ounces, or 62.5 lbs.; hence at a temperature of 60°F, the absolute pressure at a depth in water of about 34 feet is two atmospheres, and the increase is an added atmosphere for every added 34 feet in depth. Hence when the lower edge of a diving bell, or of a pneumatic caisson is 102 feet below the surface of the water, the absolute pressure, at the lower edge of the bell or at the floor of an air chamber at that depth, is *four atmospheres.* The above figures are for pure water, and they are sufficiently exact for ordinary purposes.

487. How bars are built at the mouths of rivers. Ordinary sea water weighs 1.62 lbs. more per cubic foot than fresh water, so that 30 feet of sea-water would balance 30 feet and 9 inches of fresh water, if they stood in separate tanks with a small connecting pipe at the bottom. When a fresh water stream enters the sea, it is forced to rise and spread over the surface, after the manner of oil upon water. This is well illustrated by the waters of the Mississippi as they enter the Gulf of Mexico; they spread out like a fan, carrying a load of sedementary matter far out into the gulf where it gradually drops its load thus gradually extending the *Delta* of the river. Before the jetties of the "*South Pass*" were built, the far edge of the fan was about seven miles from the point where the fresh water met the salt. The jetties built by Capt. James B. Eads conduct the water out bodily beyond the bar which was at the far edge of the fan, and drops their load in the deep water. With some help from dredges which keep the heavy sedementary load from blocking the channel, the great river is now building another fan-like deposit with a future bar some seven miles away. Capt. Eads estimated from the known depth of the Gulf and from the character and quantity of the material carried by the river, that a new bar would require an extension of the jetties in about 120 years.

488. Buoyancy. When a body is partly or wholly immersed in a liquid, the resultant action of the liquid upon the body is a vertical force, which may partly or wholly balance the action of gravity and the atmospheric pressure above. The horizontal action of the water is perfectly balanced, no matter what the shape of the body.

Suppose a heavy prismatic body, with vertical bases ABC and a vertical lateral face AC is held in water as shown in Fig. 456. The horizontal pressure of the water upon the face AC is exactly balanced by the horizontal components of the pressure on all the elements of the surfaces AB and BC.

Take for example the surfaces between two horizontal planes dz apart. Letting p be the gauge pressure wz, the horizontal action on the strip at D is $p(ldz)$. The normal action on the correspondnig surface lds at E is $wzlds$. But $ds = dz \sec\theta$, and the horizontal component of the pressure is $(wzldz \sec\theta)\cos\theta = wlzdz$, the same as at D. The same is true for every horizontal element, whatever may be the inclination; there is therefore no horizontal resultant.

With the action upon a vertical element the vertical component of the pressure at each edge is $wlzdy$ times the depth. At H it is wlz_2dy upward; at F it is wlz_1dy downward; the resultant is $wldy(z_2 - z_1) = dB$, the differential of the Buoyancy; hence

$$\text{Buoyancy} = \int wl(z_2 - z_1)dy = wV \tag{14}$$

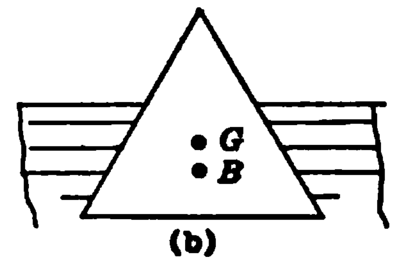
in which V is the entire volume of the body. Hence the Buoyancy of an immersed body is equal to the weight of an equal volume of the liquid.

489. When a body floats. If a body is only partially immersed, the buoyancy is equal to the weight of the liquid displaced. If the average density of the body is less than the density of the water it will float. When a body floats, the forces balance, the buoyancy exactly balances the weight. The center of gravity of the figure of the displaced water is called the Center of Buoyancy; it is of necessity in the same vertical line with the center of mass of the body. If the floating body is a homogeneous solid, its center of mass is above the center of buoyancy; consequently the equilibrium may be either *stable*, *unstable* or *indifferent*.

If the body be an equilateral triangular prism, it will be stable if one immersed face be horizontal, (b) Fig. 457.

If the prism be turned slightly by some disturbing couple, the C. G. of the body G will scarcely move, while the center of buoyancy will

move towards the side of the deeper vertex; the result will be a couple which will tend to restore the original position. The equilibrium is therefore *stable*. When, however the position is as in Fig. (*a*), a slight displacement converts the two forces, W acting at G, and B acting at B, into a couple which tends to increase the displacement; hence the position is *unstable*. In the case of a homogenous circular cylinder, solid or hollow, tho the center of buoyancy is below the center of gravity of the body, it will always be directly below it no matter how much it is turned; hence the equilibrium is *indifferent*.



(b)

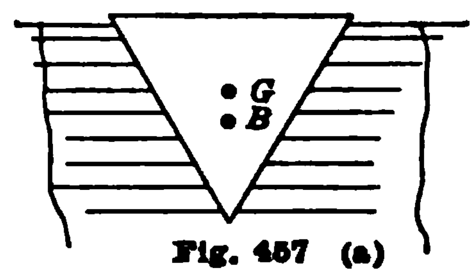


Fig. 457 (a)

The equilibrium of broad flat barges is stable, with ordinary loads. Deep and narrow vessels usually carry "ballast," or dense cargoes, near the keels, so as to bring the center of mass below the center of buoyancy, securing thereby great stability in spite of wind.

490. The line of immersion of a floating body. It is sometimes very necessary to find the *line of immersion* of a body before it is actually launched. Various methods, more or less empirical, are employed by ship builders for finding such lines, and engineers must at times launch very irregular bodies, like the pneumatic caisson of a bridge abutment, calculations for which are much complicated. Some simple examples will serve to point the way to a solution of a more complex problem.

Problem 1. A right circular cylinder whose average density is $\frac{3}{4}$ of that of water, floats upon still water. What is the unwetted arc 2θ? (Fig. 458.)

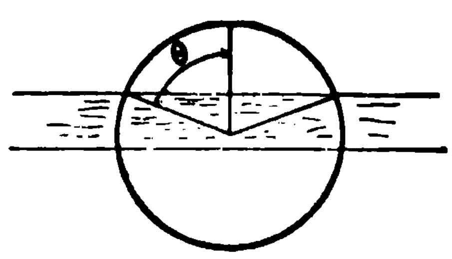


Fig. 458

As the novice is apt to be puzzled by a trigonometric equation, the solution of this problem is given with some detail.

If l be the length of the cylinder, and the radius be r, we have its weight $l\pi r^2 w'$; and the weight of the displaced water is $lr^2(\pi-\theta+\sin\theta\cos\theta)w$; hence

$$\pi\frac{w'}{w}=\pi-\theta+\sin\theta\cos\theta$$

Assuming (in general) that $C=\dfrac{w'}{w}$ we have

$$2\theta-\sin 2\theta=2\pi(1-C) \tag{1}$$

a general formula involving the ratio $C=\dfrac{w'}{w}$.

Now let $C=\frac{3}{4}$, and we have

$$2\theta-\sin 2\theta=1.5708 \tag{2}$$

Now knowing that if C were $\frac{1}{2}$, 2θ would be $180°=\pi$, we see that a fair estimate of θ is $66°\frac{1}{2}$, so that $2\theta=135°$. The radian value of 2θ is $\frac{135}{180}\pi=\frac{3}{4}\pi=2.3562$; and $\sin 135°=\sin 45°=0.707$; hence we have

$$2\theta-\sin 2\theta=2.3562-0.707=1.649.$$

This is too large as we see from (2). If we take a larger value for the angle, 2θ will be larger and $\sin 2\theta$ will be less, and our result will be more in error. So we try a smaller value. We let $2\theta=132°=132\cdot\frac{\pi}{180}$.

$$\begin{array}{llr} \text{We find} & \log\frac{\pi}{180} & =8.24188 \\ & \log 132 & =\underline{2.12057} \\ & \log 2.3038 & =0.36245 \\ \text{Hence} & 2\theta & =2.3038 \\ & \text{and}\sin 2\theta & =\underline{0.74314} \\ & & 1.56066 \end{array}$$

which is a little too small. The correct value of 2θ is between 135° and 132°.

The error for 135° was $1.649-1.5708=+0.0782$.

The error for 132° was $1.5607-1.5708=-0.0101$.

The change was 0.0883, for 3°.

To reduce the error to zero we must increase the last value of 2θ by $\frac{101}{883}$ of $3°=20'$ nearly. Hence the correct value should be very nearly $2\theta=132°\ 20'.=132°.33$.

If we try this value we get:

$$\begin{array}{rr} \log 132.33 & =2.12166 \\ \log\frac{\pi}{180} & =\underline{8.24188} \\ & 0.36354 \\ 2\theta & =2.310 \\ \sin 132°\ 20' & =\underline{0.7392} \\ 2\theta-\sin 2\theta & =1.5708 \end{array}$$

Hence the fairly correct answer is $2\theta=132°\ 20'$.

491. *Problem 2.* To find the water line (or line of immersion) on a solid homogeneous prism of length l, whose bases are scalene right triangles.

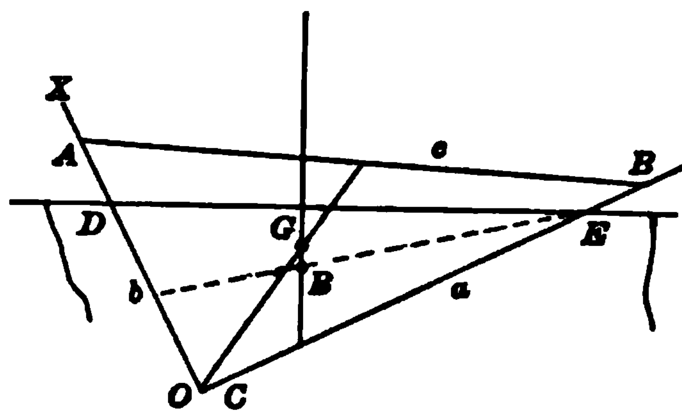


Fig. 459

Let the end (or front base) of the prism be the triangle ABC, Fig. 459. Assume a water line ED. We are to find the intercepts CD and CE. Let CA and CB be the rectangular axes X and Y respectively. Let $CD=x_1$, and $CE=y_1$. Let the sides of the triangle, ABC, be a, b, c. The co-ordinates:—of the center of gravity G are $\left(\frac{b}{3}, \frac{a}{3}\right)$; of the center of Buoyancy B are $\left(\frac{x_1}{3}, \frac{y_1}{3}\right)$.

We have $$\frac{x}{x_1}+\frac{y}{y_1}=1 \text{ the eq. of } ED. \tag{1}$$

We have $$\frac{y-\frac{a}{3}}{x-\frac{b}{3}}=\frac{\frac{y_1}{3}-\frac{a}{3}}{\frac{x_1}{3}-\frac{b}{3}}=\frac{y_1-a}{x_1-b} \text{ the eq. of line } GB. \tag{2}$$

from (1) $$y=-\frac{y_1}{x_1}x+y_1 \tag{3}$$

from (2) $$y=\frac{y_1-a}{x_1-b}\left(x-\frac{b}{3}\right)+\frac{a}{3} \tag{4}$$

But the points G and B must be in the same vertical line, while DE is horizontal; hence (3) and (4) are at right angles, and by Analytic Geometry, the co-efficient of x in one is the *negative* of the *reciprocal* of the co-efficient of x in the other; hence

$$\frac{y_1}{x_1}=\frac{x_1-b}{y_1-a} \tag{5}$$

A second equation is necessary; this is found by equating the weight of the prism and the weight of the displaced water.

$$\frac{labw_1}{2}=\frac{lx_1y_1w}{2} \text{ or } x_1y_1=ab\,\frac{w_1}{w} \tag{6}$$

Equations (5) and (6) suffice for finding the values of x_1 and y_1, when numerical values for a, b and the ratio $\frac{w_1}{w}$ are known. The solution may be effected by numerical approximation, or by graphics; the equations may be taken to represent rectangular hyperbolas which intersect in one real point whose co-ordinates are both positive.

It will be very instructive if the student (or the class) can prepare a prism (not too small) with great accuracy, paint or shellac the surface so that it will not absorb water, determine $\frac{abw_1l}{2} = W$ by careful weighing; calculate x_1 and y_1 and draw fine water lines with paint on the prism; let it dry, and then place the prism carefully in water. The degree of accuracy of the work will at once appear. Wood with knots, and mixtures of *sap* and *heart* are not homogeneous; pure heart from a large log is best. It is well to utilize opportunities for practice in solving numerical equations of 3d and 4th degrees.

In ordinary sea water a vessel floats higher, or the water line is lower in consequence of the greater density of salt water. In the case of a swimmer who displaces approximately 2.4 cubic feet of water (150 lbs.) the buoyancy in sea water is about 4 lbs. more than in fresh water. The extraordinary density of the water in Salt Lake, Utah, makes it so buoyant that a bather floats with one-sixth of his person out of water.

492. Air pressures. Unlike liquids, gases increase in density as the pressure increases and special formulas are often needed for the determination of pressures and densities, but the changes in tanks, reservoirs, etc., are so small that they are usually neglected. In deep mines and on high mountains, in balloons and aeroplanes, the differences are very important. Temperature is always a modifying factor which cannot be fully taken into account for two reasons: the state of the atmosphere at different heights cannot be known; and secondly, in consequence of air currents the conditions are constantly changing. The best we can do is to assume an *unvarying temperature*, or a temperature varying with the height from the *datum plane*, or the sea level, and derive a formula for pressure *on that condition.*

Consider a cubic element of air, as in the case of liquids, whose volume is dx, dy, dz, and whose specific weight is w; and let it be at the height z *above* the datum plane of the earth's surface. Let p be the intensity of pressure on the lower base $dxdy$, and $p+dp$ the pressure on the upper base. The lateral pressures balance, and the resultant vertical pressure balances the weight.

Hence $dxdy.p - dxdy.(p+dp) = dxdydz.w.$

$$-dp = wdz \qquad (1)$$

But w is a function of p. Assuming that the temperature is constant, we know that $pv = R$ in which v is the *volume* of a unit weight of the air under a pressure p, and R is a physical constant depending upon the temperature. See XXVII. Let v_0 and w_0 be the values of v and w

when p is p_o. Now since $p_o v_o$ also equals R, we have $pv = p_o v_o$, and

$$v = \frac{p_o v_o}{p} = \frac{R}{p}.$$

Moreover, since w is the weight of a cubic unit and v is the number of cubic units in a lb., we have $vw = 1$; hence

$$w = \frac{1}{v} = \frac{p}{R}. \qquad (2)$$

Substituting this value of w in (1), we have

$$\frac{dp}{p} = -\frac{1}{R}dz$$

Integrating from $p = p_o$ and $z = 0$,

$$\log_e \frac{p}{p_o} = -\frac{z}{R} \quad \text{or} \quad p = p_o \epsilon^{-\frac{z}{R}} \qquad (3)$$

in which ϵ is the base of Napierian logarithms, 2.718.

From (3) we get

$$z = R \log_e \frac{p_o}{p}$$

giving the height when p is known, by means of a barometer. A type of aneroid barometers gives values of z for different values of p, with corrections for temperature.

For the pressure in deep mines, the integration is from 0 to $-h$, and from p_o to p.

$$\log \frac{p}{p_o} = +\frac{h}{R}$$

$$p = p_o \epsilon^{\frac{h}{R}}$$

Table giving Pressures at Various Elevations, at 32° F, and Depths at 85° F.

Normal pressures and boiling points at different altitudes.

Elevation	*Pressure per Sq. Inch*	*Temperature of Boiling Water*
At Sea-level	$p_o = 14.7$ lbs.	212°.F.
$\frac{1}{4}$ mile	$p = 14.02$ lbs.	209°.4F.
$\frac{1}{2}$ mile	13.3 lbs.	207°.F.
$\frac{3}{4}$ mile	12.66 lbs.	204°.F.
1 mile	12.02 lbs.	202°.F.
$1\frac{1}{4}$ mile	11.42 lbs.	199°.F.
$1\frac{1}{2}$ mile	10.88 lbs.	197°.F.
2 mile	9.80 lbs.	192°.F.
−261 feet	14.86 lbs.	212°.5F.
−511 feet	15.01	213°.F.

Since the boiling temperature of water is raised by increasing the atmospheric pressure (thereby making it more difficult for the liquid to spring into gas) so it is lowered by a lower pressure. Mountain climbers often gauge their altitude by noting the temperature of boiling water.

493. Air currents in chimneys. Just as the heaviness of salt water forces it under an element of fresh water, by *overbalancing* the pressure of the fresh water thus compelling it to rise, so heavier (cooler) air compels lighter (warmer) air to rise, thereby forming vertical currents of air. When a chimney whose height is h, is full of hot gases at a temperature t_2, and a heaviness w_2, while external air is at a lower temperature t_1, and a greater heaviness w_1, the pressure *within* the base of the chimney is very nearly $(w_1-w_2)h$ **less** than the pressure near, but *without*, the base of the chimney.

This is shown as follows:—Let w_o be the average weight of a unit volume of external air, near the chimney, then if p_o be the atmospheric pressure on a unit of surface on a level with the top of the chimney, the pressure on a unit of surface *outside* the base of the chimney is $p_1=p_o+w_oh$.

The pressure *within* the chimney at the base is $p_2=p_o+w_2h$: the difference is

$$p_1-p_2=h(w_1-w_2) \tag{1}$$

This *difference* suffices to support the weight of a column of external air of unit cross-section

$$\frac{h(w_1-w_2)}{w_1}=H=h\left(1-\frac{w_2}{w_1}\right). \tag{2}$$

H is called the "Dynamic Head" in the study of liquids and gases, and it is used in the calculation of the velocity of a steady current of external air thru an opening at the base of the chimney if a clear smooth opening be made. The relation of H to V is given very closely by the equation, if we represent velocity by V,

$$V=2gH=2gh\left(1-\frac{w_2}{w_o}\right).\quad \text{See XXVI.} \tag{3}$$

If we treat the air as a perfect gas, we have its equation $pv=RT=\dfrac{p}{w}$ in which T is the *absolute temperature.*

$$w=\frac{p}{RT}$$

so that

$$\frac{w_2}{w_1}=\frac{p_2T_1}{p_1T_2}$$

Now, though the *difference* $(p_1 - p_2)$ is small it is very important and can rarely be ignored; still we can generally put the *ratio* $\frac{p_2}{p_1}$, as equal to unity without sensible error. Hence we may write

$$\frac{w_2}{w_1} = \frac{T_1}{T_2}, \text{ and } V = \sqrt{2gh\left(1 - \frac{T_1}{T_2}\right)} \tag{4}$$

494. For an explanation of the "absolute" temperature T, see Chapter XXVII. It is sufficient now to bear in mind the following relations:—

For Fahrenheit thermometers

$$T = 459^\circ.6 + t^\circ$$

For Centigrade thermometers

$$T = 273^\circ.1 + t^\circ$$

Hence we get for ideal conditions

$$V = \sqrt{2gh \cdot \frac{t_2 - t_1^\circ}{459^\circ.6 + t_2^\circ}} \tag{5}$$

If air at the temperature of melting ice be heated to 523°.6 F, the *pressure remaining the same*, a cubic ft. of air will weigh only one-half as much as it did at 32°. Let us then assume that $\frac{w_2}{w_0} = \frac{1}{2}$, and find the approximate velocity with which cool air would force its way into the base of a chimney where the average temperature was 524°F. Formula (3) gives

$$V = \sqrt{gh}.$$

If $h = 200$ feet, $V = 80$ ft. per sec. It is of course assumed that the inflowing air is immediately heated to 524°, and that all gases in the chimney are approximately "perfect."

For a "steady flow," allowance must be made for friction within the chimney, which always raises the pressure required to maintain the current in the chimney above that due to mere difference in temperature.

495. Congestion of gases in a chimney is harmful. The formula derived above assumed that there was no congestion of gases in the chimney, but this assumption is quite contrary to observed facts. Chimneys are often injuriously contracted at or near the top thereby necessitating a higher velocity and a greater driving pressure. Again,

the products of combustion, particularly if the combustion be perfect, form a large volume, and the cross-section of the chimney is frequently not large enough to keep down the congestion.

496. Draft. The difference of pressure, $p_1 - p_2 = h(w_1 - w_2)$ which is popularly known as "*draft*" is readily measured by means of water in a U-tube. Fig. 460. When the arms are vertical, water stands at the same level if $p_1 = p_2$; but if the arm C connects with the chimney flue, we have $p_1 - p_2$ shown by the *difference* of level in the two arms. If the arms are graduated *downward* the "draft" is read as so many inches or fractional parts of an inch:

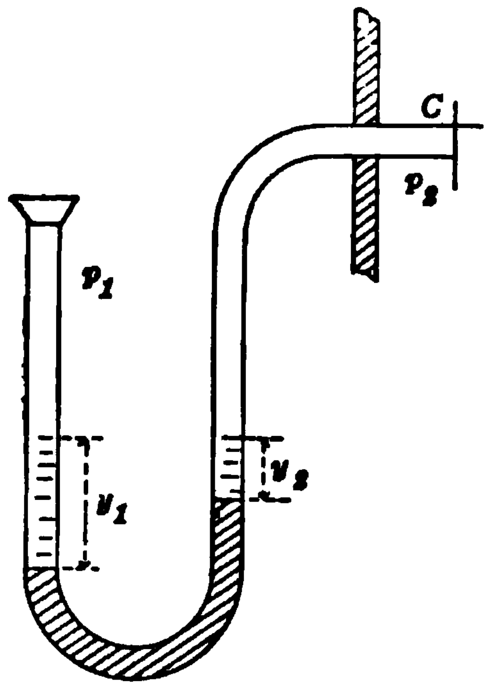


Fig. 460

$$y_1 - y_2 = \text{draft in inches.}$$

An inch of water corresponds to about 0.074 of an inch of mercury, or 0.036 lbs. per square inch.

The use of the word "*draft*" to explain the jet of inflowing air thru the furnace door or up a chimney, is responsible for many misconceptions on the part of even cultivated people. A writer on ventilation says that "an air current in a vertical flue is like a rope—you can *pull* it but you cannot *push* it." Hence, a contractor once proposed to ventilate the Hall of Representatives in the Capitol at Washington, by building a high flue, and maintaining **a fire at the top to draw the foul air up!**

What is called "*draft*" is really a "*blast*" created by the greater external pressure. If the external pressure is shut off by *air-tight* floors, windows and doors, the only current in the chimney flue will be that due to the creation of additional gases by combustion, which are apt to "*diffuse*" thru the room. A closed cold-air box is often to blame not only for an over-heated and ruined furnace, but for insufficient ventilation. Some knowledge of air pressures as related to air currents is a good investment.

CHAPTER XXV.

The Strength of Columns—Short and Long.

497. Short columns. A short column, either solid or hollow, with a uniformly distributed load and an equally uniform support, has the full strength p_1A, A being the cross-section and p_1 being the allowable working stress.

Eccentric loading of short columns. When a short column or pier is loaded eccentrically, its supporting ability is diminished and some-

times the diminution is very great. Only two simple cases will be here considered, both of which involve uniformly varying distributed forces almost identical with these already discussed.

When a highly concentrated load is brought upon a column or pier a very rigid plate or shoe is used to distribute the load over the top of the pier or column. If the load is properly placed upon a rigid plate or abacus, it will be evenly distributed, and the supporting pressure will be practically uniform. If, however, it is eccentrically placed, the pressure varies uniformly. Fig. 461 shows how an eccentric load may be applied.

Let Q be the point where the line of action of the load W intersects the top of the column. The case is exactly like the eccentric resultant of the action of weights and wind in the case of foundations for derricks and chimneys. This load must be balanced by a distributed supporting force whose "center" is Q.

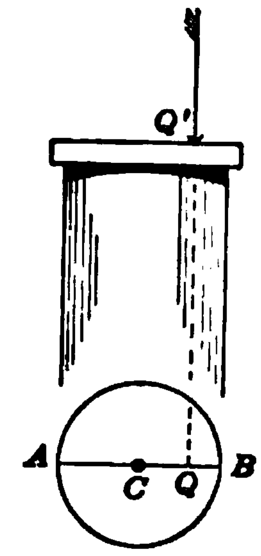


Fig. 461

All supporting material is limited in its capacity to give satisfactory support. It has been intimated more than once, that the limit for every material has been determined fairly by trial and experiment.

498. Let p_1 be the limiting intensity of pressure allowed on the column. Then the *normal load*, well centered and evenly distributed for a solid cylindrical column is

$$W_0 = \pi r^2 p_1$$

when an eccentric load, applied at Q, causes a pressure of intensity p_1 at B, no increase in W is allowed. The magnitude of the supporting ability depends upon the eccentricity CQ.

Let $CQ = nr$ in which $n < \frac{1}{4}$. Let AB be the diameter of the top of the column, and let the intensity of the supporting force at B be the limiting value p_1. As the "center" must be at Q, the intensity at A must be less, for example p_2. It was shown in **459** that in a case just like this the magnitude of the support and, of course, the magnitude of the load W was

$$W = \pi r^2 \left(\frac{p_1 + p_2}{2}\right) \tag{1}$$

but p_2 must be found in terms of n.

Since the "center" of the supporting force is known to be at Q, we have

$$CQ = nr = \frac{M}{W} = \frac{\frac{p_1 - p_2}{2r} \cdot \frac{\pi r^4}{4}}{\pi r^2 \cdot \frac{p_1 + p_2}{2}} = \frac{aI_0}{W}$$

The moment M is found as follows: The given pressure may be resolved into: a *uniform* pressure whose intensity is p_0, magnitude W, and moment zero; and a uniformly varying pressure, magnitude zero, maximum pressure $\frac{p_1-p_2}{2}$ and moment

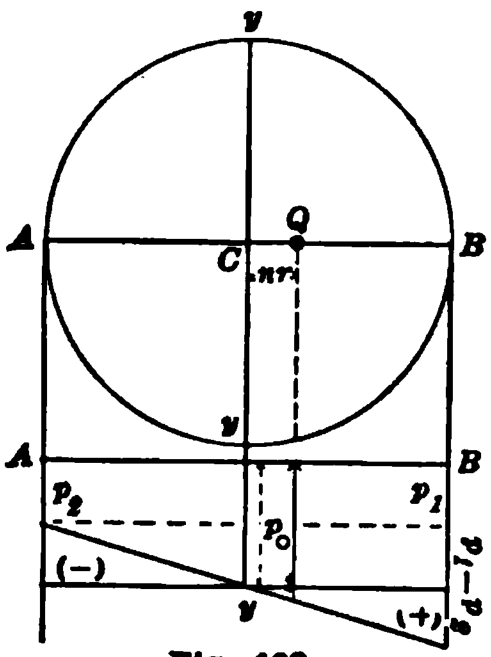


Fig. 462

$$=aI_0=\frac{p_1-p_2}{2r}\cdot\frac{\pi r^4}{4}$$

as above.
Hence

$$n=\frac{p_1-p_2}{4(p_1+p_2)}$$

and

$$p_2=\frac{1-4n}{1+4n}p_1.$$

Hence

$$W=\frac{\pi r^2}{2}(p_1+p_2)=\frac{\pi r^2 p_1}{1+4n}=\frac{W_0}{1+4n} \qquad (2)$$

From (2) it appears that when $n=\frac{1}{4}$, $p_2=0$; and when $n>\frac{1}{4}$, p_2 is negative. But as p_2 cannot be negative unless the material is capable of *tension*, the formula does not hold for earth and masonry for values greater than $\frac{1}{4}$, as has been before shown.

In all cases of eccentric loading and a limiting value for p_1, W, the working load, is less than the normal load, which is perfectly centered. Thus for solid wood, and steel columns with rigid but separate caps:

If $n=\frac{1}{16}$	$W=\frac{4}{5}W_0$
$n=\frac{1}{8}$	$W=\frac{2}{3}W_0$
$n=\frac{1}{4}$	$W=\frac{1}{2}W_0$

When the Cap or Plate Merely Rests on the Column.

499. For columns which do not admit of tension between the plate or caps, and the shaft below, an eccentricity greater than $n=\frac{1}{4}$ causes, in elastic or yielding material, a crack or opening at A; and the neutral axis becomes a chord of the circular top.

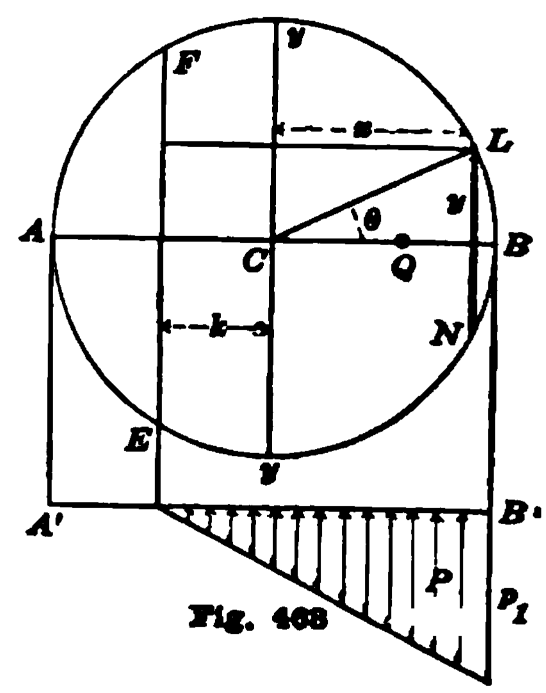


Fig. 463

In Fig. 463 the circle represents the top of the column. The resultant action of the imposed load is directly over Q. The intensity of the supporting pressure at B is p_1, the limiting

value. The intensity diminishes uniformly until it becomes zero at EF as shown by the arrows, which picture the reaction of the column. Knowing that the "center" of this support is at Q, and knowing also the intensity at B, we know that the position of the neutral axis EF is determined; that is, the value of k.

If W is the total support, we have

$$W.CQ = M = \int_{-k}^{r} (2y dx) px$$

$(2ydx)$ being an element of the surface (LN), p the intensity of pressure along the element, and x the distance from the element to the moment axis YY. The expression for P is

$$W = \int_{-k}^{r} 2ypdx$$

Hence

$$CQ = \frac{\int_{-k}^{r} yxpdx}{\int_{-k}^{r} ypdx}$$

The value of p is found by proportion from the figure.

$$\frac{p}{p_1} = \frac{k+x}{k+r}, \text{ or } p = \frac{k+x}{k+r} \cdot p_1$$

Substituting the values of p and y and integrating we get

$$M = \frac{2p_1}{k+r} \int_{-k}^{r} (r^2 - x^2)^{\frac{1}{2}} (k+x) x dx$$

$$= \frac{2p_1}{k+r} \left[k(r^2 - k^2)^{\frac{1}{2}} \cdot \frac{5r^2 - 2k^2}{24} + \frac{r^4}{8} \left(\frac{\pi}{2} + \sin^{-1} \frac{k}{r} \right) \right]$$

and

$$W = \frac{2p_1}{k+r} \int_{-k}^{+r} (r^2 - x^2)^{\frac{1}{2}} (k+x) dx$$

$$= \frac{2p_1}{k+r} \left[(r^2 - k^2)^{\frac{1}{2}} \cdot \frac{2r^2 + k^2}{6} + \frac{kr^2}{2} \left(\frac{\pi}{2} + \sin^{-1} \frac{k}{r} \right) \right]$$

Hence

$$CQ = \frac{k(r^2 - k^2)^{\frac{1}{2}} \cdot \frac{5r^2 - 2k^2}{24} + \frac{r^4}{8} \left(\frac{\pi}{2} + \sin^{-1} \frac{k}{r} \right)}{(r^2 - k^2)^{\frac{1}{2}} \cdot \frac{2r^2 + k^2}{6} + \frac{kr^2}{2} \left(\frac{\pi}{2} + \sin^{-1} \frac{k}{r} \right)}$$

If $k=r$, we have the case of the neutral axis being a tangent at A, and $CQ=\frac{r}{4}$, as already shown. If $k=0$ the neutral axis is a diameter and only half of the top of the column is in use. This means a very eccentric load for then we have

$$CQ=\frac{3}{16}\pi r=\frac{3}{5}r \text{ nearly,}$$

a value which the reader has met several times in the early chapters.

Substituting in the value of W the values of k we have used, we get these important results:

When $\quad k=r,\ CQ=\frac{r}{4}\cdot$ and $W=\frac{1}{2}W_o$

When $\quad k=o,\ CQ=\frac{3}{16}\pi r$, and $W=\frac{2}{3\pi}W_o$.

500. If the column be square with a square rigid plate, and the load W be placed on the point Q, and the intensity of the supporting pressure is p_1 along the edge at B, the pressure will diminish uniformly to, or towards, the edge at A, the rate of variation depending upon the eccentricity of the load which we will now call c.

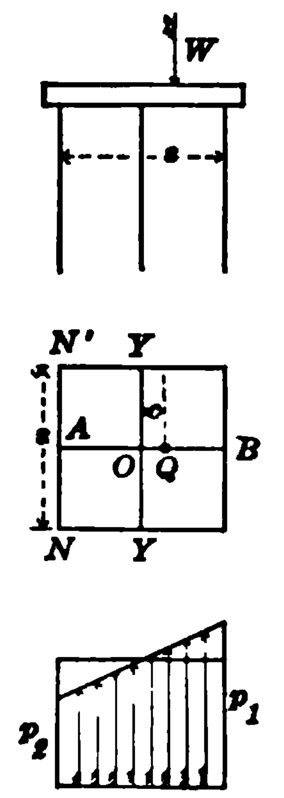


Fig. 464

Assuming that the pressure at A is p_2, the pressure is fairly represented in Fig. 464. By the laws of equilibrium, the *amount* of this pressure equals W; and the moment of that pressure about YY is equal to Wc. The magnitude of the pressure is evidently

$$\frac{(p_1+p_2)s^2}{2}=W \tag{1}$$

The moment is evidently the same as that of a uniformly varying pressure which is $\left(+\frac{p_1-p_2}{2}\right)$ on the extreme right, and $\left(-\frac{p_1-p_2}{2}\right)$ on the extreme left.

Hence
$$M=a.I=\frac{p_1-p_2}{s}\cdot\frac{s^4}{12}=\frac{p_1-p_2}{12}\cdot s^3=cW. \tag{2}$$

From (1) and (2) we get

$$p_2=\frac{s-6c}{s+6c}p_1. \tag{3}$$

If $c=0$, the load is perfectly centered, and $p_2=p_1$ and $W=s^2p_1=W_o$, the maximum load allowed.

Substituting for p_2 in (1) we get

$$W = \frac{W_o}{1 + \frac{6c}{s}} \tag{4}$$

Equations (1) to (4) hold so long as p_2 is not negative, *i. e.*, so long as c is not $> \frac{s}{6}$

If $\frac{c}{s} = \frac{1}{18}$	$p_2 = \frac{p_1}{2}$	$W = \frac{3}{4} W_o$
If $\frac{c}{s} = \frac{1}{12}$	$p_2 = \frac{p_1}{3}$	$W = \frac{2}{3} W_o$
If $\frac{c}{s} = \frac{1}{9}$	$p_2 = \frac{p_1}{5}$	$W = \frac{3}{5} W_o$
If $\frac{c}{s} = \frac{1}{8}$	$p_2 = \frac{p_1}{7}$	$W = \frac{4}{7} W_o$
If $\frac{c}{s} = \frac{1}{7}$	$p_2 = \frac{p_1}{13}$	$W = \frac{7}{13} W_o$
If $\frac{c}{s} = \frac{1}{6}$	$p_2 = 0$	$W = \frac{1}{2} W_o$

When this point is reached, the formulas fail, because an eccentricity greater than $\frac{s}{6}$ moves the neutral axis, which has been at NN', or off the plate, to a line on the surface. Let n equal the distance from A to the new axis. For such a neutral line the center of pressure is $\frac{2}{3}(s-n)$ from the neutral line, or

$$\frac{2}{3}(s-n) - \left(\frac{s}{2} - n\right) = c$$

to the right of the center, so that $n = \frac{6c - s}{2}$, and

$$W = \left(\frac{3}{4} - \frac{3}{2} \cdot \frac{c}{s}\right) W_o.$$

Now, as c is to be greater than $\frac{s}{6}$ we proceed, there being no p_2,

If $c=\frac{s}{5}$	$W=\frac{9}{20}W_0$	$n=\frac{s}{10}$
$c=\frac{s}{4}$	$W=\frac{3}{8}W_0$	$n=\frac{s}{4}$
$c=\frac{s}{3}$	$W=\frac{1}{4}W_0$	$n=\frac{s}{2}$
$c=\frac{s}{2}$	$W=0$	$n=s$

501. The reader should not fail to appreciate these figures. He must not be misled by an experiment with an eccentric load. No experiment is of value unless one has the means to measure the intensity of the pressure at B. In all of the above cases the intensity at B is p_1, and it is taken for granted that the pressure should not be allowed to exceed that limit.

502. Eccentric loads on the top of a short thin cylinder with a rigid cap.

Let the thickness of the shell be t, which is supposed to be small, and the mean radius r. Then if the limiting pressure be p_1 the maximum load when accurately placed over the axis of the column, is

$$W_0 = 2\pi r t p_1$$

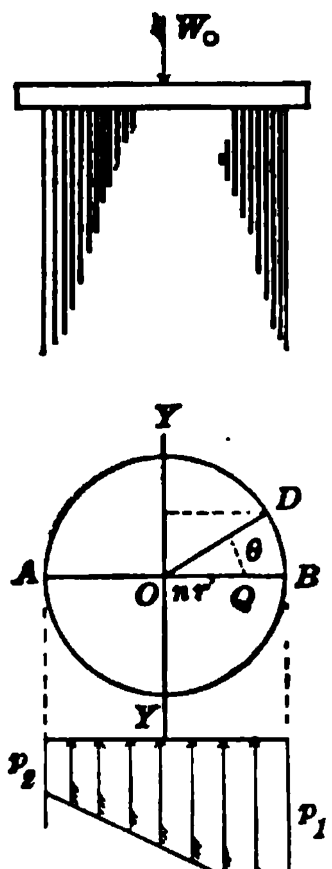


Fig. 466

Let the resultant of the eccentric load act at Q, a distance nr from the center. We shall find the possible load by finding the total of the distributed pressure on the thin ring of the column top, when its center is at Q. Fig. 466.

If D is an element of the ring $(rd\theta)t$, and if p be the intensity of pressure on that element its magnitude is

$$dP = ptrd\theta$$

and its moment about YY is

$$dM = ptr^2 \cos\theta d\theta$$

From a figure, we find by proportion that

$$p = p_2 + \frac{1+\cos\theta}{2}\cdot(p_1-p_2)$$

$$= \frac{p_1(1+\cos\theta)+p_2(1-\cos\theta)}{2}$$

Hence integrating between $\theta=0$ and $\theta=\pi$ and doubling.

$$P=rtp_1\int_0^\pi(1+\cos\theta)d\theta+rtp_2\int_0^\pi(1-\cos\theta)d\theta$$

$$P=\pi rt(p_1+p_2)=W \tag{1}$$

and

$$M_y=r^2tp_1\int_0^\pi(1+\cos\theta)\cos\theta d\theta+r^2tp_2\int_0^\pi(1-\cos\theta)\cos\theta d\theta$$

$$M_y=\pi r^2t\cdot\frac{p_1-p_2}{2}. \tag{2}$$

The folly of placing a load away from the center is clearly shown above. Eccentricity is even more unscientific when long columns are loaded, for the inequality of the stresses p_1 and p_2 causes a bending which increases the eccentricity at the middle section of the column.

The assumption is that the column is elastic and that the load deforms the top to a certain extent; that is, the longitudinal fibers on the side nearest to Q are shortened. The *amount* of the shortening depends upon the length of the column, but the length does not modify the varying pressure of the cap. So long as the *cap is rigid*, if $n>\frac{1}{2}$, the neutral axis will be across the top of the column, and there will be a crack over the unloaded portion, tho it may be exceedingly small. It is quite worth while to derive a formula giving the relation of the line of no pressure to the position of Q when n is greater than $\frac{1}{4}r$.

Now as the "center" of this supporting pressure must be at Q, we have

$$nr=\frac{M_y}{P}=\frac{r}{2}\cdot\frac{p_1-p_2}{p_1+p_2}$$

and

$$p_2=\frac{1-2n}{1+2n}\cdot p_1 \tag{3}$$

Substituting this value of p_2 in (1), we find the value of P and hence the magnitude of the load the column will carry if centered at Q.

$$P=W=\frac{2\pi rtp_1}{1+2n}=\frac{W_o}{1+2n}$$

By examining the value of p_2 we see that p_2 is greater than zero if n lies between zero and $\frac{1}{2}$. When $n=\frac{1}{2}$, $p_2=0$.

If $n>\frac{1}{2}$, p_2 would be negative if the formula held; but the formula

does not hold, for as a rule the column is not rigidly connected with the plate upon which the load is placed. The values of W as n varies from zero to $\frac{1}{2}$ are given in the table below.

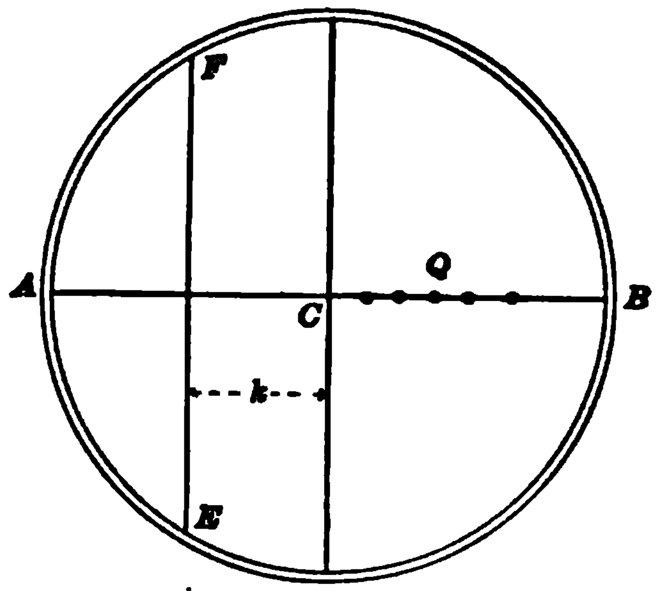


Fig. 467

When $n > \frac{1}{2}$, the pressure does not extend over the entire ring, it ceases at a "neutral axis" EF, as in the case of a solid column.

503. The practical weakening of a thin short column, not rigidly attached to the bearing plate, by an eccentric load is shown in this table. The eccentric load is applied at Q. The pressure at B is always p_1 the maximum allowed. When $nr > \frac{r}{2}$, there must be a crack or gap at A. See Fig. 467.

Under no circumstances should the eccentricity exceed $\frac{r}{2}$, no matter how light the load may be.

The Magnitudes of Eccentric Loads.

Eccentricity $CQ=nr$	Pressure at $A=p_2$	Pressure at B	Neutral axis	Maximum load allowed
0	p_1	p_1	none	$W_0=2\pi rtp_1$
$\frac{1}{8}r$	$\frac{3}{5}p_1$	p_1	outside	$W=\frac{4}{5}W_0$
$\frac{1}{4}r$	$\frac{1}{3}p_1$	p_1	outside	$W=\frac{2}{3}W_0$
$\frac{3}{8}r$	$\frac{1}{7}p_1$	p_1	outside	$W=\frac{4}{7}W_0$
$\frac{1}{2}r$	0	p_1	At A	$W=\frac{1}{2}W_0$
$\frac{2}{3}r$ nearly	Crack at A	p_1	$k=\frac{r}{2}=EF$	$W=\frac{2}{5}W_0$ nearly
0.785 r	Crack at A	p_1	$k=o$ EF is YY	$W=\frac{7}{22}W_0$

504. The "kernal" or limiting area for loads. It may happen that the thrust of an arch or the weight of a part of a building comes eccentrically upon the skewback or cap of a rectangular bed or pillar, the allowable intensity of pressure being limited as above to p_1, no

tensile stress being possible, and no cracks being permitted. In such a case the center of action is confined to a central rectangle whose area is but *one-ninth* of the area of the support; see figure 467½.

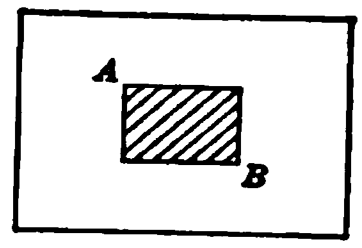


Fig. 467½

This area is often called the "kernel." If the center of the load is on the *perimeter* of the kernal, the strength of the pillar is reduced 50 per cent., and the *neutral axis* touches the large rectangle at one of the corners:—it coincides with an edge if Q is on an axis of symmetry.

505. The supporting strength of long columns. In the discussion of Eccentric Loads on the top of Pillars and Posts, in above sections no account was taken of the tendency of the pillar to bend or buckle; yet it is at once evident that an eccentric load which causes a varying pressure at the top must cause an unequal shortening of the fibers, and that that alone, if uniform thruout the length causes a straight column to become curved. The effect is the same as the effect of unequal shrinkage due to the drying or seasoning of wood.

In a long column, however, the curvature is increased by a bending moment which increases with the curvature. The analysis of the problem of strength is somewhat difficult, and practical formulas are empirical as well as theoretical; that is, they are made to fit the results of laboratory experiments. They are based, however, on a solution given long ago by Euler.

Euler's Formula. The column (or post) is supposed to be ideally perfect, prismatic or cylindrical except that the ends are rounded, so that the points of application and support are centered on the axis. It has been found by experiment* that under a light load the erect position is stable, that is, a slight horizontal action near the middle causing a minute deflection does not cause a permanent curvature, for when the horizontal action ceases, the column springs back to its erect position. If the load or pressure is increased to a certain magnitude, the column does not spring back when the horizontal action ceases. If then the load is increased, the deflection increases more and more until the column breaks or collapses. There is therefore a load which will just hold the bent column in a position of unstable equilibrium; a little more load, and the column gives way; a little less load, and it straightens itself. Euler's Formula gives the load at the critical point.

506. Let P, Fig. 468(a), be the load or pressure on the top, l the length, I the *least Moment of Inertia* of a cross-section; and c the lateral deflection at the middle of the column, which is taken as the origin.

* See T. C. Fidler's "Treatise on Bridge Construction." London.

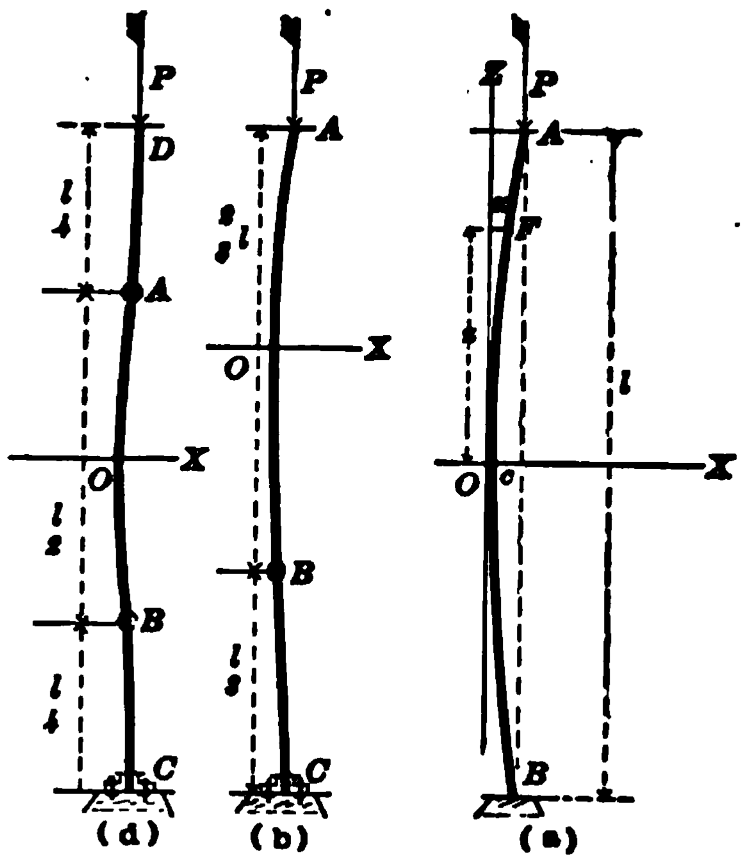


Fig. 408

At the point F the bending moment is

$$M = P(c - x)$$

As a column is *nearly* straight, we can use the formula

$$\frac{d^2x}{dz^2} = \frac{M}{EI}.$$

Both E and I are constant as z varies, and we have

$$EI\frac{d^2x}{dz^2} = M = P(c - x)$$

Multiplying by dx, and remembering that d^2x is the differential of dx, and that the formula for radius of curvature was derived on the supposition that dz was constant, we integrate into

$$\frac{EI}{(dz)^2}\cdot\frac{(dx)^2}{2} = P\left(cx - \frac{x^2}{2}\right) + C = \frac{1}{2}EI\left(\frac{dx}{dz}\right)^2.$$

When $\qquad x = 0, \dfrac{dx}{dz} = 0$, so that $C = 0$.

Changing the form of the equation to

$$\frac{dx}{(2cx - x^2)^{\frac{1}{2}}} = dz\sqrt{\frac{P}{EI}}$$

and integrating, we have the general form

$$\text{arc versin}\,\frac{x}{c} = z\sqrt{\frac{P}{EI}}, \tag{1}$$

the constant of integration being zero.

If $\qquad x = c, z = \dfrac{l}{2}$, so that

$$\text{arc versin}\,1 = \frac{l}{2}\sqrt{\frac{EI}{P}}.$$

The arc whose vers-sin is unity is $\dfrac{\pi}{2}$; hence

$$\pi = l\sqrt{\frac{P}{EI}}, \text{ and } P = \frac{\pi^2EI}{l^2}, \tag{2}$$

which is Euler's Formula.

If in (1) we make $\phi = \text{arc versin}\,\frac{x}{c} = z\sqrt{\frac{P}{EI}}$, we have

$$\left.\begin{aligned} 1-\cos\phi = \frac{x}{c}. \quad \therefore x = c(1-\cos\phi) \\ z = \phi\sqrt{\frac{EI}{P}} \end{aligned}\right\} \qquad (3)$$

which are the equations of the elastic curve of the axis of the column.

A column or strut with pin-joints is the same as a round-ended column provided the axis for the moment of inertia I is parallel to the axes of the pins. The strut may be regarded as having "fixed" ends in the plane of the axes of the pins.

507. If one end of the column as at C is fixed, Fig. 468(b), that is, if that end is so secured by bolting, riveting or merely squaring, that the tangent to the elastic curve of the axis at C is still vertical, the bent column designated by l in the formula is only $\frac{2}{3}$ of the l of the real column. The column being still very nearly straight it is evident that there is a point of inflection midway between O and C, and that the portion AB with no moment at either end is the l of the formula. Hence the limiting load for *such* a column is

$$P = \frac{\pi^2 EI}{\left(\frac{2}{3}l\right)^2} = \frac{9\pi^2 EI}{4l^2} \qquad (4)$$

If both ends are fixed, there are two points of inflection where $M=0$, and the column of Euler's Formula is only the part AB which has a length $\frac{l}{2}$. Hence the limiting load for *such* a column, Fig. d, is

$$P = \frac{4\pi^2 EI}{l^2}. \qquad (5)$$

508. Practical formulas for strength. The formulas given above do not deal with fiber stress, or elastic limits, or with factors of safety. In fact the P of Euler's formulas is the *critical*, and therefore the *dangerous* value, and hence they are of little use if unmodified.

The maximum fiber stress in a loaded column is made up in two parts.

First, the uniform stress due to the load P applied at the center

$$p' = \frac{P}{A}; \qquad (1)$$

Secondly, the extreme fiber stress due to the maximum bending moment (load into deflection)

$$M = P\Delta = \frac{2p''}{h} I$$

in which h is the diameter of the column in the plane of the elastic curve; hence

$$p'' = \frac{hP}{2I} \cdot \Delta \tag{2}$$

But Δ is not known and no formula gives it. It is known, however' that the deflection varies as the square of the length, and inversely as the least Moment of Inertia of the cross-section. See Table. **375.**

Hence
$$p'' = K\frac{Pl^2}{I} = K\frac{Pl^2}{Ak_o^2}; \tag{3}$$

So that the final fiber stress is $f = p' + p'' = \dfrac{P}{A}\left(1 + \dfrac{Kl^2}{k_o^2}\right)$

or
$$P = \frac{Af}{1 + \dfrac{Kl^2}{k_o^2}}. \tag{4}$$

The quantity K is determined by experiment; as is the allowable working stress f. There are various modifications of the above for the purpose of fitting it to the results of experiments. Mr. Theodore Cooper, in a recent pamphlet on "Standard Bridge Specifications," gives the following for *safe* loads of steel columns and struts:

For horizontal struts or chords
$$\begin{cases} \dfrac{P}{A} = 8{,}000 - 30\dfrac{l}{k_o} \text{ for live loads} & (5) \\ \dfrac{P}{A} = 16{,}000 - 60\dfrac{l}{k_o} \text{ for dead loads} & (6) \end{cases}$$

For vertical members
$$\begin{cases} \dfrac{P}{A} = 7{,}000 - 40\dfrac{l}{k_o} \text{ for live loads} & (7) \\ \dfrac{P}{A} = 14{,}000 - 80\dfrac{l}{k_o} \text{ for dead loads} & (8) \\ \dfrac{P}{A} = 10{,}000 - 60\dfrac{l}{k_o} \text{ for wind stresses} & (9) \end{cases}$$

509. The stresses within a bent cylindrical column. The activity of a long column in straightening itself under a light load, noted

by Fidler (See Footnote to **505**) is readily explained. If the unloaded column is bent by a symmetrical horizontal force H, either distributed or concentrated, the limiting stresses, p_1 and p_2 on the concave and convex sides of the section at the center, have equal magnitudes and opposite signs. Fig. 469(a). These stresses are due to the shape of the column and those magnitudes are quite independent of the forces which cause the bending. The algebraic sum of the stress at the middle section of the column is zero, and there is no "center of stress," or the center is at infinity. See Fig. 469(a). Now suppose a rigid and unyielding block is placed above and *just in contact* with the rounded end of the column, but not pressing upon it. Next, suppose the horizontal force H is diminished. At once the column endeavors to straighten, and presses up against the block. This pressure produced an internal change at the middle section. The deflection $\Delta = c$ is unchanged, but the algebraic sum of the normal stress is no longer zero; the compression at B has been increased, and the tension at A has been diminished, and there is now a "center of stress," far beyond the line of action of the vertical forces at top and bottom.

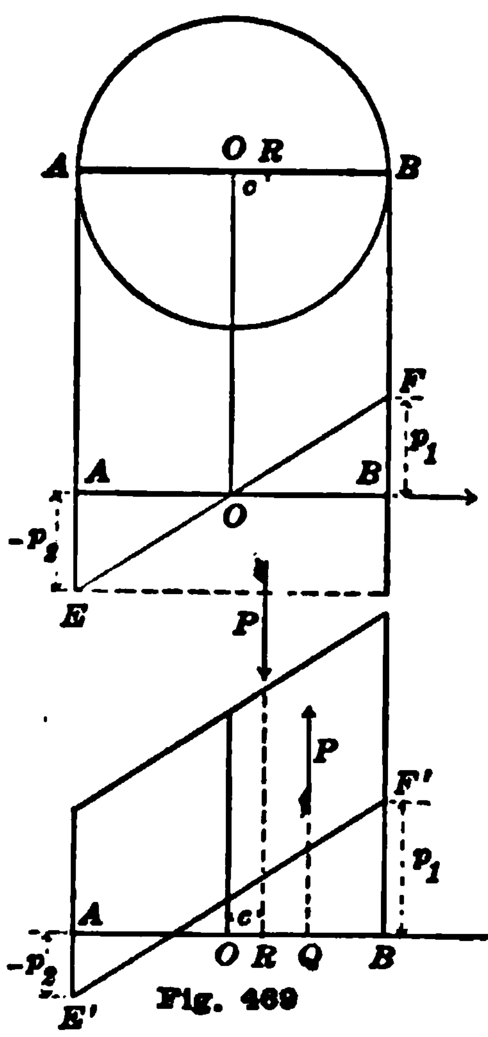


Fig. 469

The distance to that "center" Q is exactly measured by the quotient

$$OQ = \frac{M}{P}$$

M being the Moment of Resistance in the column, and P the magnitude of the pressure at the top.

It is evident that the resultant stress acting at Q, and the pressure P form a couple which has a tendency to straighten the column. The moment of the couple is

$$L = P(OQ - c)$$

which may be called *the straightening moment*. As H is diminished P must be increased to preserve the moment and the deflection, and the straightening moment becomes less. When finally H is zero, OQ becomes c, and $Pc = M_0'$, and the pressure at the top (or the load on the column) has reached a *maximum* for the given deflection.

If P were now increased in some way, the equilibrium would be destroyed. OQ would become *less* than c, and the couple $P(c - OQ)$ would tend to bend the column still more. Hence P is the maximum

pressure at the top, or the *maximum* load, *consistent with the given deflection* at the middle section.

If H were at first uniformly distributed, see Table, **375**,

$$M_0 = \frac{Hl}{8}$$

$$c = \Delta = \frac{5Hl^3}{384EI},$$

and as $Pc = M_0$, we must have

$$P = \frac{48}{5}\frac{EI}{l^2} \tag{1}$$

If H was a concentrated force acting at the middle of the column,

$$M_0 = \frac{Hl}{4}$$

$$c = \Delta = \frac{Hl^3}{48EI}$$

and

$$P = 12\frac{EI}{l^2} \tag{2}$$

If H were so distributed that the elastic curve of the axis of the column was represented by the equations

$$\left.\begin{aligned} x &= c\,(1 - \cos\phi) \\ z &= \phi\sqrt{\frac{EI}{P}} \end{aligned}\right\}$$

we should have

$$P = \pi^2\frac{EI}{l^2}. \tag{3}$$

510. It is seen that in every case the quantity c is not found in the value of P, which leads to the conclusion that c may vary for the same value of P, provided the deformation is so small that our equations for moment and deflection still hold; that is, so long as the column is very nearly straight.

A second conclusion is that P is always a function of the fraction $\frac{EI}{l^2}$. Hence the form of the empirical equations given in **508.**

Note.—The assumption that a column bent by a single concentrated load or by a uniformly distributed load will retain its *exact* shape under the load P when H is reduced to zero, is untenable, inasmuch as the *shape* would automatically change to the cosinoidal form

represented by the equations (S) **506.** In fact, the change of form is progressive from the instant P has value to the instant when

$$P=\pi^2\cdot\frac{EI}{l^2}.$$

Nevertheless the straightening phenomenon is fully explained.

CHAPTER XXVI.

THE ENERGY OF STREAMS AND IMPINGING SOLIDS.

511. The energy of a stream of water. A stream of water acting upon *moving* guides or surfaces, does work and gives up energy. When, however, it acts upon stationary guides or surfaces, it does no work, and (omitting all friction) it loses none of its energy. The two cases will be discussed separately.

1. A current of water acts upon stationary surfaces.

(*a*) In the elbow of a pipe.

When passing the bend or elbow, the stream is deflected by a deviating pressure but it loses no energy and does no work. We are to find the resultant action between the water and the pipe. (When we think of the water as acting against the wall of the pipe, we call it the "centrifugal force"; when we think of the pipe as acting laterally upon the water, we call it the "deviating force.")

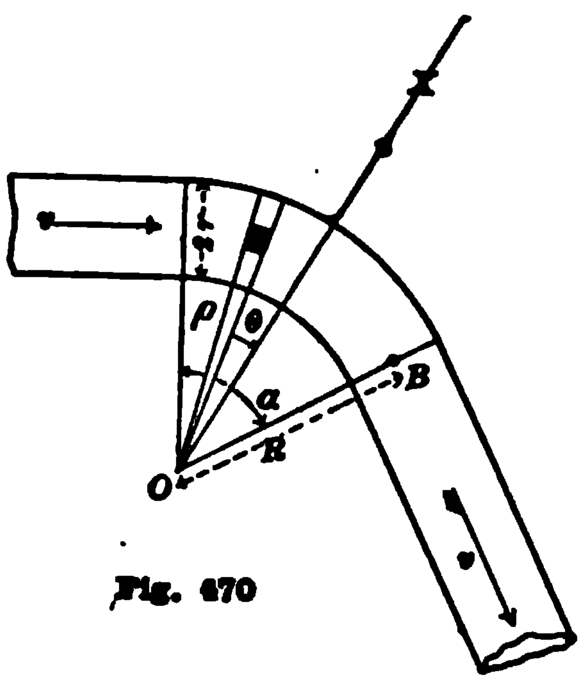


Fig. 470

Consider, Fig. 470, an element dV of the water. Its mass is $\delta dV=\frac{w}{g}(\rho d\theta)dA=dm$, in which dA is an element in the cross-section of the stream. Its centrifugal force is

$$d^2Q=\frac{w}{g}\frac{v^2}{\rho}\rho d\theta dA=\frac{(d^2m)v^2}{\rho}.$$

Its component along the bisector OX is $d^2Q\cos\theta=\frac{wv^2}{g}\cos\theta d\theta dA.$

Since v^2 is assumed to be constant for all points of the stream, we have the resultant along OX, the bisector

$$Q_x=\frac{wv^2}{g}\int\cos\theta d\theta\int dA=\frac{wv^2A}{g}\int\cos\theta d\theta$$

$$Q_x = \frac{wv^2\pi r^2}{g}\int_{-\frac{a}{2}}^{\frac{a}{2}}\cos\theta d\theta = \frac{2wv^2\pi r^2 \sin\frac{a}{2}}{g}.$$

The integration of dA takes in the cross-section of the pipe, and then the integration for θ takes in all the water there is in the elbow. It is to be noticed that Q_x is independent of R.

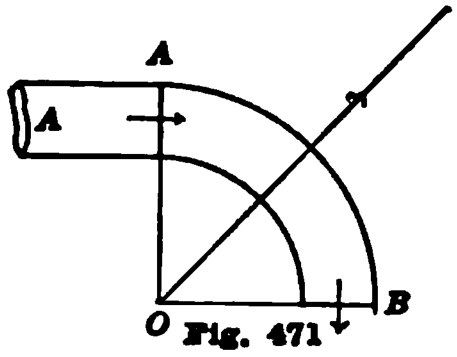


Fig. 471

If a is a right angle

$$Q_x = \frac{wv^2\pi r^2\sqrt{2}}{g} \tag{1}$$

and the component of Q_x parallel to the pipe at A and at B is

$$Q_A = \frac{wv^2\pi r^2}{g} = Q_B \tag{2}$$

If this be compared with the hydrostatic pressure P which may be supposed to cause the velocity v, we have

$$p = wh, \qquad P = pA = wh\pi r^2,$$

and $$v^2 = 2gh \therefore h = \frac{v^2}{2g}, \qquad P = \frac{w\pi r^2 v^2}{2g}$$

which shows that

$$Q_A = 2P \tag{3}$$

It is evident that the elbow may have many sub-divisions without affecting Q_A; in fact, the entire stream may strike a wall and be deflected 90° in all directions, with Q_A constant. The centrifugal force is still equal to *twice* the *hydrostatic pressure*, $P = wh\pi r^2$. This was formerly called the **Hydrostatic paradox** because a jet of water from a tank pressed against a vertical shield with a force twice as great as the pressure upon a surface equal to the nozzle on the wall of the tank at the same depth.*

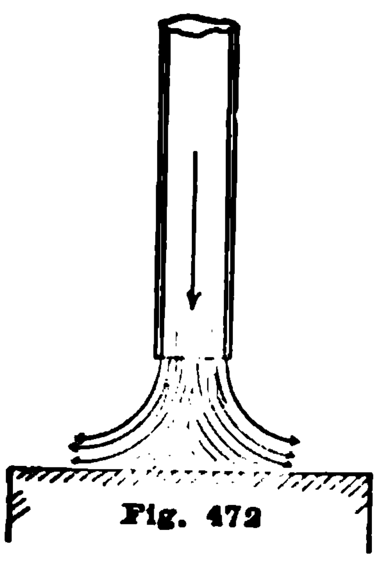


Fig. 472

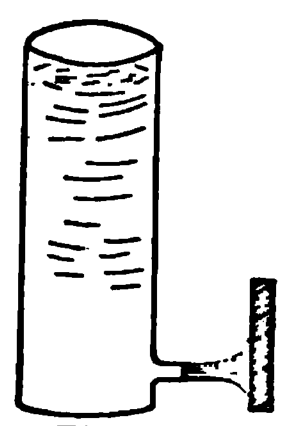


Fig. 473

* Writers have almost invariably explained the result shown in the text, which is only a problem in centrifugal force, by assuming that there is a change of momentum (that is a change in velocity) and they bring in the element of time. Omitting friction, there is no change in velocity, momentum or energy, and no "work" is done.

Had the deflection been thru the angle 180°, the integration of $\cos\theta d\theta$ above would have been from

$-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, giving. Fig. 474,

$$Q_x = Q_A = \frac{2wv^2\pi r^2}{g}. \qquad (4)$$

Fig. 474

In properly securing a pipe (or a hose) which makes a turn of much magnitude, and conducts water at high velocity, the above formulas should not be neglected.

Example. Suppose a steel pipe, internal diameter 6", has water, with a dynamic head of 160 feet,passing thru it at a velocity of 100 feet per second, and making a turn of 90°; what is the tension caused in the pipe on both sides of the elbow? or what should be the single external support to prevent the tension?

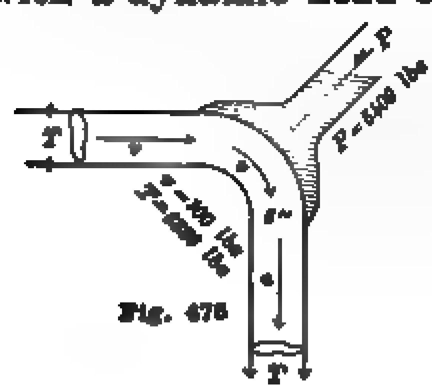


Fig. 475

The Formula gives, there being no external support at P_1

$$T = \frac{w}{g} \cdot \pi r^2 v^2$$

$$= \frac{1000}{16 \times 32} \times \frac{22}{7} \times \frac{1}{16} \times (100)^2 = 3836 \text{ lbs.}$$

512. The action of a jet of liquid or gas upon a receding vane or cup.

Let the current be v_1 and the velocity of the cup *in the same direction* be v_2. The relative velocity, that is, the velocity with which water enters and leaves the cup, is $v_1 - v_2$. Hence the thrust given the cup is by formula (4) of **511**, $\frac{2Aw}{g}(v_1 - v_2)^2$, in which A is the sectional area of the jet at the nozzle. The rate at which work is done on the cup is the thrust times v_2 or

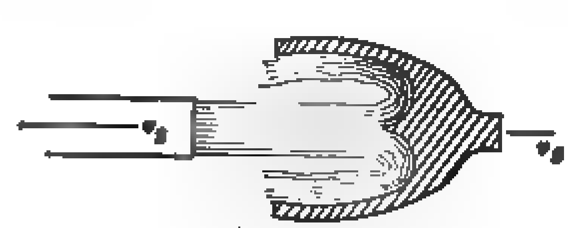


Fig. 476

$$U' = \frac{2Aw}{g}(v_1 - v_2)^2 v_2 \text{ per sec.}$$

This work is zero when $v_2 = 0$ and also when $v_2 = v_1$. It is a maximum when $v_2 = \frac{1}{3}v_1$. So that the *maximum work upon a single receding cup* is

$$\text{Maximum } U' = \frac{8}{27}\frac{Aw}{g} \cdot v_1^3 \text{ per sec.}$$

This is a low velocity of the cup, and a small amount of work. The energy of the jet per sec is $\left(\frac{Awv_1}{g}\right)\frac{v_1^2}{2}\cdot = \frac{mv_1^2}{2}$, a large portion of which is lost, partly because one-third of what issues in a second fails to reach the cup, during the second; and what does reach it, leaves it with a (backward) velocity of $\frac{1}{3}v_1$. Assuming, as we have done all the while, that the energy lost by external and internal friction is negligible, the energy lost (or not employed) in driving the cup is:—

1. The Kinetic energy of one-third of the jet which does not reach the cup,

$$\left(\frac{Aw}{g}\frac{v_1}{3}\right)\frac{v_1^2}{2} = \frac{Awv_1^3}{6g} \text{ per second.}$$

2. The Kinetic energy still in the two-thirds when it leaves the cup.

$$\left(\frac{Aw}{g}\frac{2v_1}{3}\right)\frac{\left(\frac{1}{3}v_1\right)^2}{2} = \frac{Awv_1^3}{27g}$$

Adding these losses to the work done, we have

$$\frac{Awv_1^3}{g}\left(\frac{8}{27}+\frac{1}{6}+\frac{1}{27}\right) = \frac{Awv_1^3}{2g}.$$

In order to utilize all the energy, we must have a *continuous row* of cups so closely arranged that the jet enters cups as fast as it issues. When a jet v_1 feet long enters cups every second instead of a jet (v_1-v_2) feet long, the work done will be $\frac{v_1}{v_1-v_2}$ times as much. Multiplying the work shown in (U') by this fraction, we have

$$U = \frac{v_1}{v_1-v_2}\left(\frac{2Aw}{g}(v_1-v_2)^2v_2\right) = \frac{2Aw}{g}(v_1-v_2)v_1v_2 \text{ per second.}$$

Now, the velocity of the cups must be so regulated that U will be a maximum. Putting $\frac{dU}{dv_2}=0$, we get $v_2 = \frac{1}{2}v_1$.

When $v_2 = \frac{1}{2}v_1$, the absolute velocity of the water on leaving the cups is zero, which is the ideal condition, and no energy is lost, since

$$U = \frac{2Awv_1}{g}\left(\frac{v_1}{2}\times\frac{v_1}{2}\right) = \frac{Awv_1^3}{2g} \text{ per unit of time,}$$

which was the energy of the jet.

THE "PELTON WATER WHEEL." Fig. 476. In the water jet wheel as now used, the cups are so constructed that the current is deflected 180° with the least possible loss from eddies and friction, and so close together that all the current is caught, and the linear velocity of the cups, ωr, is kept as near to $\frac{1}{2}v_1$ as possible, and the efficiency approaches closely to 100 per cent.

513. The velocity of a fluid jet. The two fundamental formulas of Hydraulics are

$$p = wh$$
$$v^2 = 2gh$$

in which h is the "Dynamic Head." The first was derived during our study of hydrostatics. The second will now be derived from the doctrine of the conservation of Energy.

Assume a tank, with a closed orifice at the bottom, to be filled with water. The atmospheric pressures above the tank and at the orifice balance each other. If the orifice is opened, the volume of water escaping at the velocity v in the time dt is $dV = avdt$, in which a is the cross-section of the jet at its *minimum point*. The mass of the volume dV is

$$dm = \frac{w}{g} avdt,$$

and its Kinetic Energy is

$$d(K.E.) = \frac{v^2}{2} \cdot \frac{w}{g} avdt.$$

Now this discharge is caused by a lowering of the entire mass within the tank above the orifice. If the average cross-section of the tank be A, and the height be h, the *weight* of the water moved downward is $W=wAh$; the distance moved is dh, and the *work done* by gravity is

$$dU=wAhdh;$$

hence, by our law, $d(\text{K. E.})=d(U)$

$$\frac{v^2}{2}\cdot\frac{w}{g}avdt=wAhdh$$

But of necessity $A(dh)=av(dt)=dV$ the volume of the discharge. Hence

$$\frac{v^2}{2g}=h$$

$$v^2=2gh. \qquad Q.E.I.$$

2. If a gas is escaping thru an orifice from one region to another where the pressure is less, the *dynamic head* is $h=\frac{p_1-p_2}{w}$ and $v^2=2gh$, v and w being measured before expansion takes place.

514. The energy of air jets. If, instead of water, an air or gas jet strikes cups or vanes, the same principles apply for moderate pressures (and densities), since the changes in density due to different pressures are so small compared with the uncertain co-efficients of friction, that they may be neglected. A gas jet may change in temperature and volume without any change is mass, and it is *mass alone* which is involved in problems of centifugal (or deviating) forces.

It would be out of place in an elementary book to go into a discussion of steam and water turbines, tho they are based on the principles already given. The student is referred to Special Treatises on these subjects.

515. The action between air currents and screw propellers.

1. Preliminary. Air under ordinary conditions is perfectly elastic. As we saw in Chapter XXIV, an increase in the pressure in general, increases the temperature and decreases the volume of a given mass, and as the *mass* of an air current, and not its temperature, is the matter to be considered here, and as changes in pressure are small during the revolution of a propeller, we omit all thermodynamic matters, as well as friction and irregular eddies.

The normal pressure, due to impact, of an air current upon a smooth

plane surface of a stationary object is found, as is the pressure due to a jet of water, as follows:

Let A be the cross-section of a stream of air, or of the air current which has a velocity v_1. S is the plane surface of the *stationary* vane BC. The angle which the current makes with S is θ. Let dm be an element of the curving current; its volume being $dA\rho da$, ρ being the radius of curvature of itspath, its mass is $\frac{w}{g}\rho dAda$; and

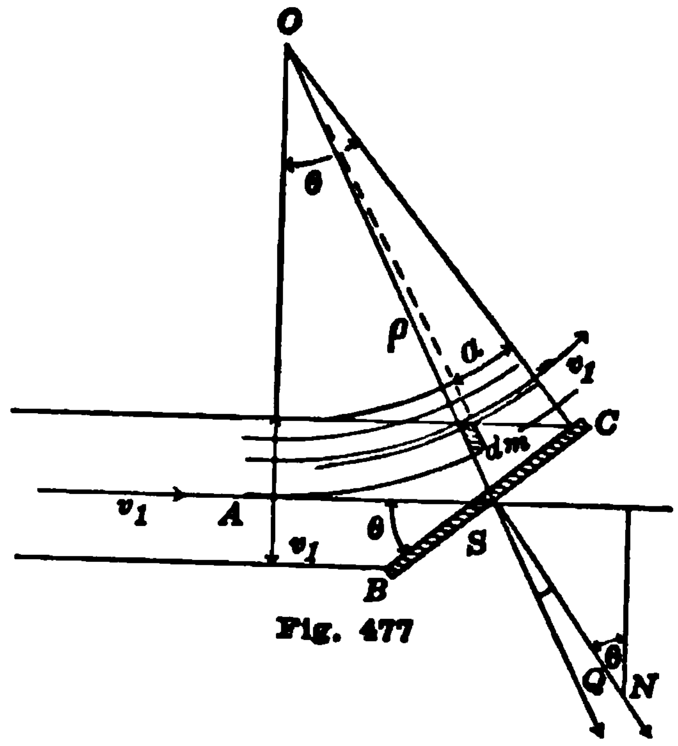


Fig. 477

its centrifugal force $\frac{dmv_1^2}{\rho}=\frac{w}{g}v_1^2dAda$. The *component of this force along the normal* to the plane is $\frac{w}{g}v_1^2\cos adAda$. Hence

$$d^2Q=\frac{w}{g}v_1^2dA\cos ada.$$

Now, every element in the curved stream which has dA for its cross-section, whatever may be its radius of curvature, has a *component part* in Q according to a.

Integrating for a from 0 to θ, we get

$$dQ=\frac{w}{g}v_1^2dA\int_0^\theta\cos ada=\frac{w}{g}v_1^2\sin\theta dA.$$

Next integrating for A, so as to include all such curved jets or elements, we get

$$Q=\frac{w}{g}v_1^2A\sin\theta \tag{1}$$

which is the total normal pressure on the surface S which is in the path of the current A. But $A=S\sin\theta$, substituting this value of A and dividing by S, we have

$$\frac{Q}{S}=p_n=\frac{w}{g}v_1^2\sin^2\theta \tag{2}$$

The pressure p_n is the intensity of pressure on the exposed surface. This quantity p_n ought rather to be called the *average* pressure, since it is well known that on a finite plane surface exposed to the wind, the pressure is far from uniform.

However, since the entire current A is deflected, perhaps some parts

more than θ and others less, the resultant pressure cannot vary much from Q.*

516. An air current acting upon a receding plane or vane, having a velocity v_2 in the same direction as v_1.

The *relative* velocity is the *difference* $v_1 - v_2$, which must take the place of v_1 in the former case. Hence Q in this case is

$$Q = \frac{w}{g}(v_1 - v_2)^2 A \sin\theta \tag{1}$$

The component of this pressure, in the direction of motion is

$$Q \sin\theta = \frac{w}{g}(v_1 - v_2)^2 A \sin^2\theta \tag{2}$$

so that the *work done* by the air current per unit of time is

$$U = \frac{(v_1 - v_2)^2 v_2 w}{g} \cdot A \sin^2\theta \tag{3}$$

Given v_1 constant, this amount of work done per unit of time is a maximum when $v_2 = \frac{v_1}{3}$. But this gives the amonut of work done as only $\frac{4}{27} \cdot \frac{w}{g} A v_1^3 \sin^2\theta$, while the energy of the air current per second is

$$E = \left(\frac{w}{g} A v_1\right)\frac{v_1^2}{2} = \frac{w}{2g} A v_1^3. \tag{4}$$

Therefore, is $\qquad \frac{U}{E} = \frac{8}{27} \sin^2\theta.$

The loss is partly due to the fact (as in the case of a water jet) one-third of the current flowing per second fails to reach the retreating vane. If additional vanes appear with sufficient frequency to catch the *entire current*, the mass in action continually is not $(v_1 - v_2)\frac{Aw}{g}$, but $\frac{v_1 A w}{g}$, so that the work done is

$$U_2 = \frac{v_1 A w}{g}(v_1 - v_2) v_2 \sin^2\theta$$

* For a full discussion of wind pressures upon stationary surfaces by Prof. F. E. Nipher, see *Proceedings* of the *St. Louis Academy of Science*. Vol. 8, No. 1, pp. 1–24.

This is a maximum when $v_2 = \frac{v_1}{2}$, and

$$\text{Maximum } U_2 = \frac{Awv_1}{g} \cdot \frac{v_1^2}{4} \sin^2\theta$$

so that the efficiency is $\frac{\sin^2\theta}{2}$.

If $$\theta = 90^\circ$$

$$\frac{U_2}{E} = \frac{1}{2}$$

This is for the ideal case in which there is no loss from eddies and friction.

517. The action of an air current upon a screw propeller, or the reverse. Fig. 478.

Let BC (with the width dr) be an element of the helicoid bounded by two consecutive helices, and two horizontal differentials dr. If the propeller were stationary, the action of the air current striking it would be the case of **515** over again.

Since the propeller is revolving with an angular velocity ω, the entire "vane," BC, is *receding* with a velocity depending upon ω, and the "pitch" of its screw surface. In the time dt the element moves $\omega r dt$ in its circle of rotation to the position $B'C'$. Fig. (b).

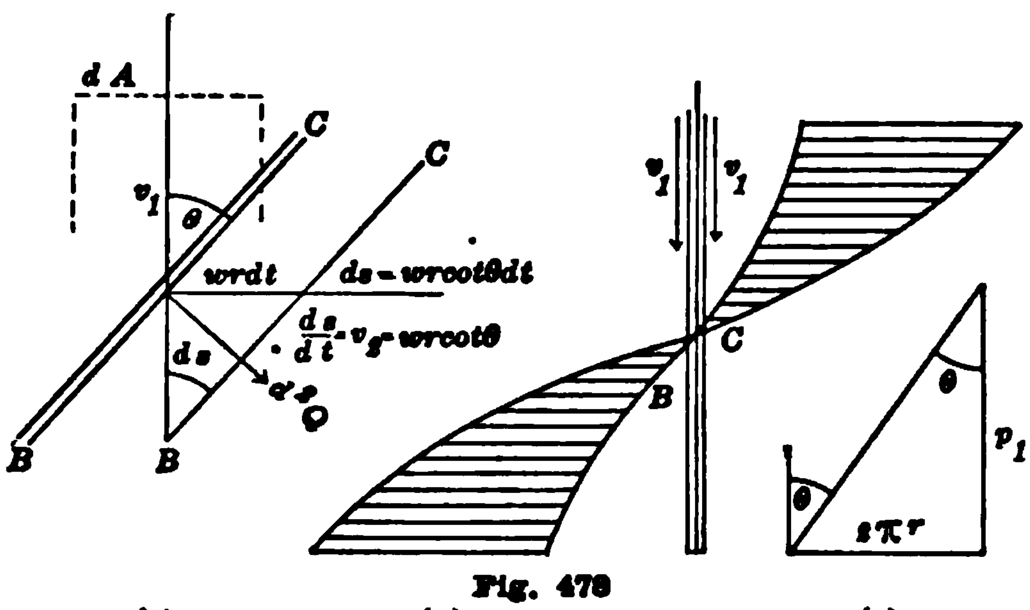


Fig. 478

(b) (a) (c)

This movement causes it to *retreat* or *recede* in the direction of the air current $ds = \omega r dt \cot\theta$ so that $v_2 = \frac{ds}{dt} = \omega r \cot\theta$. By the law of the helix, Fig. 478(c), $\cot\theta = \frac{p_1}{2\pi r}$, hence $v_2 = \frac{p_1\omega}{2\pi}$ (in which p_1 is now the pitch of the screw).

This "velocity of receding," v_2, is independent of r, and hence is the same for all points of the helicoidal vane, a fundamental property of a *screw surface*.

The normal pressure of the air upon the element, $BC \times dr = dS$, is by (1) in the last section

$$d^2Q' = \frac{w}{g}\left(v_1 - \frac{p_1\omega}{2\pi}\right)^2 dA \sin\theta \tag{1}$$

Since the propeller has no motion of translation the component of d^2Q' parallel to v_1 does no work. The component perpendicular to v_1 does work by *turning* the wheel. This component is

$$d^2Q' \cos\theta = \frac{w}{g}\left(v_1 - \frac{p_1\omega}{2\pi}\right)^2 \sin\theta\cos\theta dA \tag{2}$$

and its moment is

$$d^2M = d^2Q' r\cos\theta = \frac{w}{g}\left(v_1 - \frac{p_1\omega}{2\pi}\right)^2 r\sin\theta\cos\theta dA \tag{3}$$

But $dS\sin\theta = dA$ the cross-section of the current which acts upon dS. If dS be projected upon a circle of rotation, we have a clear idea of dA as an element of a circular sector. See Fig. 479. The arc HK depends upon the width of the blade. Evidently

$$dA = (dr)(rd\phi)$$

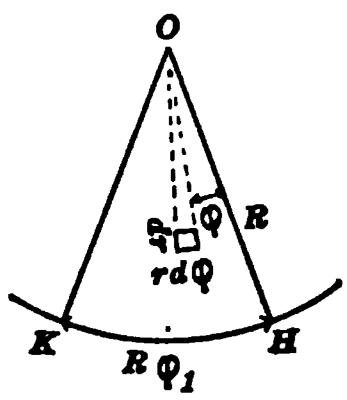


Fig. 479

Moreover, from Fig. 478(c)

$$\sin\theta = \frac{2\pi r}{\sqrt{4\pi^2r^2+(p_1)^2}} \qquad \cos\theta = \frac{p_1}{\sqrt{4\pi^2r^2+(p_1)^2}}$$

and $$\sin\theta\cos\theta = \frac{2\pi p_1 r}{4\pi^2r^2+p_1^2},$$ so that we have

$$M = \frac{w}{g}\left(v_1 - \frac{p_1\omega}{2\pi}\right)^2 .2\pi p_1 \int_0^R \frac{r^3dr}{4\pi^2r^2+p_1^2}\int_0^{\varphi_1} d\phi$$

Decomposing the rational fraction and integrating

$$M = \frac{w\phi_1 p_1}{2\pi g}\left[\frac{R^2}{2} + \frac{p_1^2}{4\pi^2}\log_e\cos\theta_1\right]\left((v_1 - \frac{p\omega}{2\pi}\right)^2 \tag{4}$$

in which θ_1 is the angle between the axis of the propeller and a tangent to the outermost helix. Taking the factor $\frac{R^2}{2}$ from the parenthesis, we have

$$M = \frac{wp_1}{2\pi g}\cdot\frac{\phi_1 R^2}{2}\left(1 + \frac{p_1^2}{2\pi^2R^2}\log\cos\theta_1\right)\left(v_1 - \frac{p_1\omega}{2\pi}\right)^2. \tag{5}$$

$\frac{\phi_1 R^2}{2}$ is the area of the sector A. If the number of blades is $\frac{2\pi}{\phi_1}$, their combined area of their projections will be πR^2. In such a case the continuous return of the blades would utilize *all the current* within

the circle πR^2, so that instead of being acted upon by the volume $\pi R^2\left(v_1 - \frac{p_1\omega}{2\pi}\right)$ every second, that volume would be $\pi R^2 v_1$.

The turning moment produced on such a propeller would be

$$M = \frac{wp_1}{2\pi g}\cdot \pi R^2\left(v_1 - \frac{p_1\omega}{2\pi}\right)v_1\left[1 + \frac{p_1^2}{2\pi^2R^2}\log\cos\theta_1.\right] \qquad (6)$$

Assuming that ω, v_1 and R are constant, it may be found by plotting the curve for M_1 for various values for p_1 it is found that M is a maximum for a definite value of p_1.

Now to find the **Thrust of the Propeller** parallel to the air current, which thrust must be balanced by the bearings.

Referring to the value of d^2Q

$$d^2Q = \frac{w}{g}\left(v_1 - \frac{p_1\omega}{2\pi}\right)^2 \sin\theta dA$$

we note that the component parallel with the air current is

$$d^2Q\sin\theta = \frac{w}{g}\left(v_1 - \frac{p_1\omega}{2\pi}\right)^2\sin^2\theta dA = d^2T$$

As before, substituting $dA = rdrd\phi$ and $\sin^2\theta = \dfrac{4\pi^2r^2}{4\pi^2r^2+p_1^2}$ we have

$$\text{Thrust } T = \frac{4\pi^2w}{g}\left(v_1 - \frac{p_1\omega}{2\pi}\right)^2\int_o^R\frac{r^3dr}{4\pi^2r^2+p_1^2}\int_o^{\varphi_1}d\phi$$

$$T = \frac{\phi_1 w}{g}\left(v_1 - \frac{p_1\omega}{2\pi}\right)^2\left[\frac{R^2}{2} + \frac{p_1^2}{2\pi^2R^2}\log\cos\theta_1\right]$$

and

$$T_1 = \frac{w}{g}\pi R^2 v_1\left(v_1 - \frac{p_1\omega}{2\pi}\right)\left[1 + \frac{p_1^2}{\pi^2R^4}\log\cos\theta_1\right]$$

when $\frac{\phi_1R^2}{2}$ becomes πR^2, and *all* the air current is utilized.

518. *The Propeller moves in still air, with a velocity v_1.*

The actions between the air and the propeller are exactly the same as when the conditions were reversed: the air moving, and the propeller supports stationary; the *relative* velocity is just the same. What was necessary in the last two sections, to *resist* the rotation of the propeller,

and to resist the thrust of the propeller upon its bearings, is now necessary to turn the propeller, and to keep the machine in forward motion. In other words, M now measures the constant moment which the motor must apply to the propeller shaft; and T is the thrust which the propeller must give to the machine in order to maintain the velocity v_1. This thrust is due to the *reaction* of the air which is set in motion and sent to the rear with a component velocity $v_2=\frac{p_1\omega}{2\pi}$ parallel to the axis of the propeller.

IMPACT.

519. Only simple ideal problems will be here discussed. The
CASE I. **Direct central impact** of homogeneous spheres or prisms.
weights are supposed to be balanced, as tho each moved on a smooth horizontal plane. Both bodies are supposed to be somewhat elastic; their centers move along the same straight line.

The masses are m_1 and m_2; their velocities, both positive, are u_1 and u_2; m_1 overtakes and impinges against m_2. There are three epochs: before, during, and after the impact. The respective velocities after impact are v_1 and v_2.

There is all the while a "center of mass," which moves as the bodies move, with a constant velocity u_o, which is found as follows:—The resultant momentum before impact is $m_1u_1+m_2u_2$. During impact their mutual action changes the momentum of each, but one gains exactly as much as the other loses, since

$$Fdt=m_2du_2=d(m_2u_2)=-m_1du_1=-d(m_1u_1)$$

hence
$$d(m_1u_1+m_2u_2)=0$$

$$m_1u_1+m_2u_2=a \text{ constant.}$$

F is the magnitude of their mutual action at any instant.

At the instant at the end of their deformation while in contact, m_1 and m_2 have the same velocity, viz.: u_o. Hence the "constant," is $(m_1+m_2)u_o=m_1u_1+m_2u_2$

$$u_o=\frac{m_1u_1+m_2u_2}{m_1+m_2}. \tag{1}$$

As soon as the deformation of the bodies has reached a maximum, restoration begins, since the material is more or less elastic. Let k^2 be the "co-efficient of restitution." Before impact the kinetic energy of m_1 relative to the C. G. of the system was

$$E_1=\frac{m_1}{2}(u_1-u_o)^2.$$

After the separation the energy is

$$E_2 = k^2 E_1 = \frac{m_1}{2} \cdot k^2(u_1 - u_o)^2.$$

Hence the velocity of m_1 *relative to the C. G.* is (being back words and therefore negative)
$$-k(u_1 - u_o) = k(u_o - u_1).$$

Hence the absolute velocity (relative to the earth) is

$$v_1 = u_o + k(u_o - u_1) = (1+k)u_o - ku_1 \tag{2}$$

In like manner with reference to m_2, except that the rebound is forward and therefore positive,

$$v_2 = u_o + k(u_o - u_2) = (1+k)u_o - ku_2 \tag{3}$$

Cor. 1. Since $u_1 > u_o$, it is evident from (2) that v_1 may be zero, or even negative. If $u_1 = \dfrac{1+k}{k} u_o$, $v_1 = 0$; and $v_2 = k(u_1 - u_2)$.

Cor. 2. If $k = 1$ (for perfectly elastic balls)

$$\begin{aligned} v_1 &= 2u_o - u_1 \\ v_2 &= 2u_o - u_2 \end{aligned} \tag{4}$$

Cor. 3. If m_2 is initially stationary, $k = 1$, and $m_1 = m_2$, we have from (1) $u_o = \dfrac{u_1}{2}$, and from (2) $v_1 = 0$, and $v_2 = 2u_o = u_1$ which means that m_1 stands still, while m_2 moves of *after* the contact is over, with the velocity which m_1 had before the impact.

Cor. 4. If the balls meet, the initial *velocity* of the one having the smaller momentum should be taken as negative, so that u_o may be positive. All the formulas involving the negative velocity hold.

Cor. 5. If m_2 is infinite and stationary (that is, an immovable rigid wall) then $u_o = 0$, and $u_2 = 0$ and $v_1 = -u_1 k$.

Cor. 6. The value of k in the case of a billiard ball, or golf ball, dropped upon a solid marble or metallic floor, may be found by measuring the heights of fall and rebound. The reader may prove the formula

$$k = \sqrt{\frac{h_2}{h_1}},$$

if h_1 is the height of fall, and h_2 that of rebound.

520. Impact oblique of sphere and wall.

CASE II. Suppose an elastic sphere strikes a rigid wall obliquely making the angle θ with the normal. The angle of rebound rarely approximates the angle of incidence, for two or more reasons:—

1. The ball is not perfectly elastic.
2. Friction causes retardation and rotation.
3. An initial angular velocity.
4. The attraction of the earth.

The degree of elasticity may sometimes be found as explained above, by experiment.

The friction may be reduced to a minimum by polishing and oiling.

The action of gravity is neutralized if the plane of θ and ϕ is horizontal.

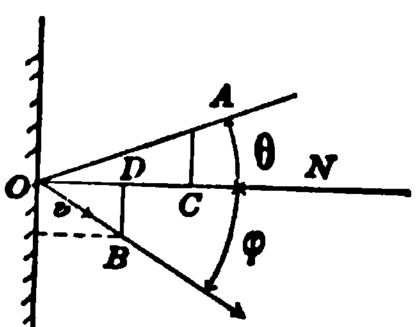


Fig. 480

Let us first suppose that the supporting plane of θ and ϕ is horizontal, that there is *no friction*, *no rotation*, and that k is known. Fig. 480.

Let AO be the velocity of approach, and OB, of the rebound. The tangential components of AO and OB are equal as there is nothing to modify motion parallel to the plane; that is $DB = v \sin\phi = AC = u \sin\theta$.

The normal components follow the law given in Cor. 5 of the last section; that is

$$OD = k(u\cos\theta) = v\cos\phi = k.OC \tag{1}$$

and hence

$$\left.\begin{aligned} \tan\phi &= \frac{\tan\theta}{k}. \\ v = \frac{u\sin\theta}{\sin\phi} &= u\ (\cos\theta\sqrt{k^2 + \tan^2\theta}) \\ &= u\sqrt{\sin^2\theta + k^2\cos^2\theta} \end{aligned}\right\} \tag{2}$$

If $k = 1$ (perfect elasticity)

$$v = u, \text{ and } \phi = \theta.$$

If $k = 0$ (inelastic)

$$v = u\sin\theta, \text{ and } \phi = \frac{\pi}{2}.$$

and the body glides along the plane.

Example 1. If $k = \frac{1}{2}$ and $\tan\theta = \sqrt{\frac{1}{2}}$ prove that the incident and the reflected paths are at right angles.

Example* 2. If an ivory ball lying within a rigid metallic circular hoop, both resting on a smooth plane, be driven along a chord making such an angle θ with the two radii OA and OB that after two rebounds it returns to A, prove that the relation between θ and k is given by the equation

$$\tan\theta = \sqrt{\frac{k^3}{1+k+k^2}}.$$

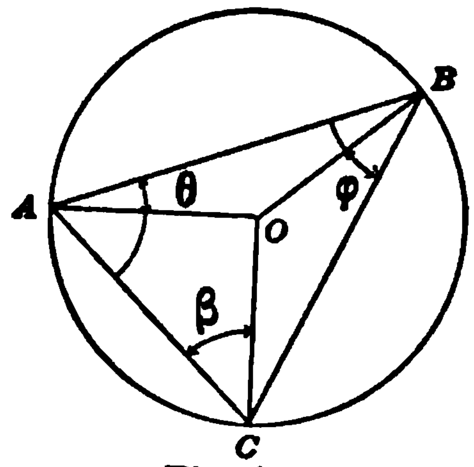


Fig. 481

Example 3. Prove that if the ball rebounds three times and returns to A, the relation is

$$\tan\theta = \sqrt{k^3}.$$

521. Eccentric impact. An ideal problem. Both balls are *smooth and* k is 1. The mass m_2 is at rest when struck by m_1. The velocity of m_1 along the common normal is $u_1\cos\theta$; the tangential velocity is $OT = u\sin\theta$, which is unchanged by the collision. The consequent normal velocities of m_1 and m_2, are found from formula (4) (**519**) by changing u_1 to $u_1\cos\theta$, and making $k=1$, $u_2=0$, and

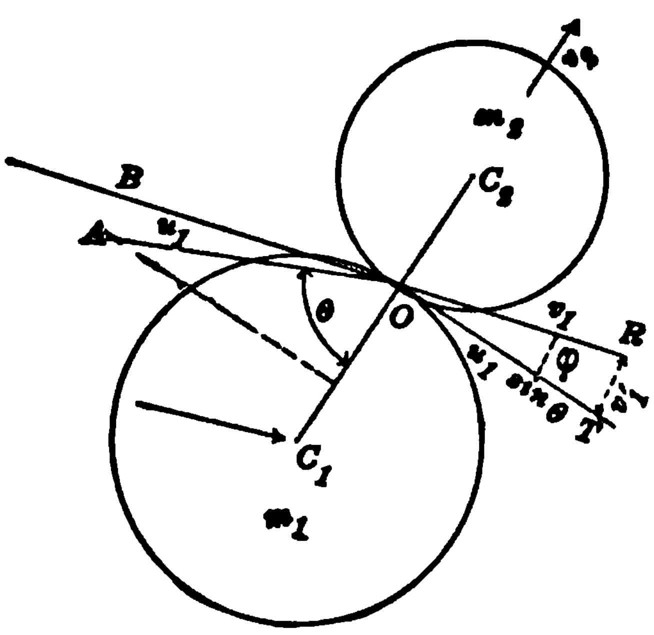


Fig. 482

$$u_o = \frac{m_1 u_1 \cos\theta}{m_1+m_2}.$$

The consequent velocities *along the line of centers are*

$$\left.\begin{aligned} v_1' &= \frac{2m_1u_1\cos\theta}{m_1+m_2} - u_1\cos\theta = \frac{m_1-m_2}{m_1+m_2}\cdot u_1\cos\theta \\ v_2 &= \frac{2m_1}{m_1+m_2}\cdot u_1\cos\theta \end{aligned}\right\} \qquad (1)$$

The resultant velocity, v_1, and direction, BR, of m_1 are found by combining v_1' with $u_1 \sin\theta$, *i. e.*: TR with OT, as seen in the figure. The path of m_1 has been *deflected* thru the angle $BOA = 90-(\theta+\phi)$, and its velocity has been *reduced* from

$$u_1 \text{ to } u_1\left(\sin^2\theta + \frac{(m_1-m_2)^2}{(m_1+m_2)^2}\cdot\cos^2\theta\right)^{\frac{1}{2}}.$$

---

* In these examples no allowance is made for sliding along the hoop during the period of contact.

Corollorary. If $m_1 = m_2$, we have

$$v_1' = 0$$
$$v_2 = u_1 \cos\theta$$
$$v_1 = u_1 \sin\theta$$

and the final directions of m_1 and m_2 are at right angles.

CHAPTER XXVII.

The Efficiency of Compressed Air.*

522. In this chapter the aim has been to present the principles in as simple and intelligible a form as possible. Every engineer or architect has occasion to deal with compressed air, and it is well worth while to master the Thermodynamic laws involved even if he omits the study of Imperfect gases. No attempt is made to present anything new, but an effort is made to show the extreme simplicity of the fundamental laws.

1. If v† is the volume of a unit weight of a PERFECT GAS which is thruout under a uniform pressure p, and at a uniform absolute temperature T, then the value of the expression $\frac{vp}{T}$ is constant. If this constant be called R we have

$$\frac{vp}{T} = R \tag{1}$$

The pressure p is given in pounds, kilograms, or atmospheres as may be desired, upon a unit of surface, and v gives the volume in cubes of the unit of length. The temperature may be given in degrees Fahrenheit or Centigrade; if in the former, T is 459.58 degrees more than shown by a Fahrenheit thermometer; if the degrees are Centigrade, T is 273.10 degrees more than shown by a Centigrade thermometer; that is

$$\left.\begin{aligned} T^\circ(\mathrm{F}) &= t^\circ + 459^\circ.58 \\ T^\circ(\mathrm{C}) &= t^\circ + 283^\circ.10 \end{aligned}\right\}$$

2. Equation (1) shows that if the pressure is constant, the volume increases and decreases with the temperature, or, more exactly, that

* This chapter is substantially as presented in a paper read before the Engineers' Club of St. Louis, Feb. 13, 1884, and printed in the Journal of the Association of Engineering Societies, in 1884.

† In this chapter v is *volume*, not velocity.

the volume is proportional to the temperature. Hence, if the temperature increases by equal increments, or degrees, the volume will increase by equal amounts. Again, if p is constant, the equal increments of volume show that equal amounts of external work are done by the expanding gas as the temperature increases by regular degrees.

3. If we consider next the amount of heat required to raise our unit weight of gas through one degree of temperature, and then another, and so on, the pressure being all the while constant, we shall see that a certain part of it is used to do the external work, which is the same in each case, because the increments of volume are equal, and that the remainder is used to increase the temperature of the gas itself, that is, in making the gas hotter. Now it has been found by experiment:

First. That for all ordinary attainable temperatures *air is a perfect gas.*

Second. That that portion of the heat required to merely heat the lb. of air one degree (not including that spent in doing the external work) is constant.

Third. That this last constant amount of heat is precisely the same as that required to heat from any one degree to the next a unit weight of air which is so confined that it cannot expand, and consequently can do no external work.

4. This last amount of heat, required to heat a unit weight of air through one degree at constant volume is therefore *constant,* whatever be the temperature and pressure; this amount of heat will be designated by C_v, and will be read: "Specific heat at constant volume." If in addition to being heated, the air is allowed to expand, the pressure being constant, more heat is required on account of the external work to be done; but as this amount of work is also constant, the sum is constant, and the total is denoted by C_p, and is read: "Specific heat at constant pressure." It is evident that C_p is greater than C_v by the heat required to do the external work involved in the expansion while the air is heated thru one degree.

5. It is now easy to show that this difference between C_p and C_v is just the quantity R given in equation (I). For let us suppose that, p being all the while constant, the temperature is reduced to absolute zero, where T becomes zero. In this ideal state of things all heat, and hence all energy, has gone out of the air; and it is therefore without the power of physical manifestation, shown by the logical necessity of writing the volume equal to zero also.

Next let us suppose that the air receives heat and expands at constant pressure till the volume is v and the temperature T. It is evident that the whole work done by the air as it expands from no volume till

its volume is v, is vp, and since this is the work done while the temperature is increasing from zero to T, we find the work done during the raising of the temperature one degree, by dividing the whole work by T, the number of degrees. Hence it is $\frac{vp}{T}$; but this is R; so that the difference between the two specific heats is R, and we have the equation

$$C_p - C_v = R \tag{II}$$

523. The measure of work done. It may be well to show that if a pound of air expand from v_1 to v_2 against a constant pressure p, the work done by the air is $(v_2-v_1)p$. Suppose the air to be in a cylinder with unyielding sides and one fixed end, and that as the air is heated and expands it slowly drives along a piston. Let A be the area of the piston. Then the length of the cylinder of air before expansion must be $\frac{v_1}{A}$ and after expansion $\frac{v_2}{A}$. Hence, the piston has been forced along a distance $\frac{v_2-v_1}{A}$. The pressure on the piston is pA, so that the work done, or the energy exerted, during expansion is $\frac{v_2-v_1}{A} \times pA = (v_2-v_1)p$, which was to be shown.

If the pressure during an expansion is not constant the work must be computed for small parts of the expansion separately and the results added. In other words, make the difference between v_2 and v_1 very small (denoted by dv) so that an element of work will be represented by pdv, and then sum the series (or integrate the differential). Hence, the general expression for work against a varying pressure is $\int pdv$. If the pressure is constant, and if v_1 is zero, we have

$$U = \int_0^v pdv = p\int_0^v dv = pv$$

where U stands for the work done, as was assumed in **522.**

In equation II. we have the difference between two amounts of heat placed equal to a certain number of foot-pounds or to a number of kilogram-meters. This shows that we are measuring heat by the *work* it can do, or in *dynamic* units. Heat is often measured in *thermal* units, as, for example, by the amount of water it can heat one degree or in the weight of ice it can melt. These units are convertible, but the simplest unit for our use is the dynamic.

If our units are the foot, the pound, and the degree Fahrenheit, the numerical value of R is thus found: Let p be the atmospheric pressure, or 2,116.3 pounds per square foot; T the absolute temperature of

melting ice under normal pressure, or 491.58 degrees; then, by measure, v for air is known to be 12.387 cubic feet. Hence the value of R, which is always equal to

$$\frac{vp}{T}, \text{ is } \frac{12.387 \times 2116.3}{491.58} = 53.33 \text{ foot-pounds.}$$

Had the units been the kilogram, the meter and the degree Centigrade, the value of R would have been numerically 29.27.

The values of C_p and C_v are 183.33 and 130.00 foot-pounds respectively.

The ratio of Cp to Cv is approximately 1.41.

524. Let us now suppose that one pound of air receives a small amount of heat, which we will denote by dQ; and that in consequence the air becomes slightly warmer, indicated by dT; and, at the same time, that it increases a little in volume, denoted by dv; in other words, that a part of the heat is used to increase temperature, and a part to increase the volume. To merely increase the temperature one degree would require C_v units of heat; to increase the temperature dT requires $C_v\,dT$ units; to do the external work requires pdv; the general equation for a *perfect gas* is therefore

$$dQ = C_v dT + pdv \tag{III}$$

It should be understood that no waste heat is included in dQ of this formula. Either we must suppose that the heat escaping into or through the material of the cylinder is left out of the account, or we must assume that the cylinder is non-conducting, so that there is no waste heat. Whichever way we regard it, the assumption is not to be forgotten.

Equations (I), (II) and (III) lead to all the formulae we shall use. If (III) be integrated from the condition v_1, T_1, p_1, to v_2, T_2, p_2, we shall have

$$Q = C_v(T_2 - T_1) + \int_{v_1}^{v_2} pdv;$$

we cannot integrate $\int pdv$ unless we know the relation of p to v for every condition intermediate between v_1, p_1 and v_2, p_2.

If there is *no change of temperature*, then $T_2 - T_1 = 0$, and since $p = \dfrac{TR}{v}$ from (I), we have

$$Q = TR\int_{v_1}^{v_2} \frac{dv}{v} = TR\,(\log. v_2 - \log. v_1) = TR \log. \frac{v_2}{v_1}. \tag{IV}$$

This is the quantity of heat used in "Isothermal Expansion." As there is no change in the temperature of the expanding air, all the heat Q is used in doing work. Another way of saying the same thing is: Q is the quantity of heat used to *keep the air from cooling* while it is doing work.

525. In the ordinary operations of compressing or using compressed air there is little or no direct transfer of heat; we shall therefore derive the equations for such operations by putting $dQ=0$, and by integrating equation (III) by means of equation (I).

To find the relation between p and T when there is no transfer of heat, we let $dQ=0$ in (III): whence

$$C_v dT + pdv = 0. \tag{V}$$

Differentiating (I) we have $pdv+vdp=RdT$,

or
$$pdv = RdT - vdp = RdT - \frac{RT}{p}dp,$$

Substituting in (V) we have

$$(C_v+R)dT = \frac{RT}{p}dp,$$

or
$$\frac{C_v+R}{R}\times\frac{dT}{T}=\frac{dp}{p},$$

or by (II)
$$\frac{C_p}{C_p-C_v}\times\frac{dT}{T}=\frac{dp}{p}=\frac{k}{k-1}\times\frac{dT}{T}$$

in which we denote by k the ratio of the two specific heats, or $\frac{C_p}{C_v}=k$.

Integrating our last equation from p_1, T_1 to p_2, T_2 we have

$$\frac{k}{k-1}\log.\frac{T_2}{T_1}=\log.\frac{p_2}{p_1};$$

whence
$$\log.\left(\frac{T_2}{T_1}\right)^k=\log.\left(\frac{p_2}{p_1}\right)^{k-1}$$

or
$$\frac{T_2}{T_1}=\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}};\ \text{or}\ \frac{p_2}{p_1}=\left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}} \tag{VI}$$

This equation gives the ratio between the absolute temperatures when the pressures are known, and the value of T_2 when T_1, p_1, p_2 are known.

526. To find the relation between p and v when $dQ=0$, we have, eliminating dT from (V) by substituting the value of dT obtained by differentiating (I),

$$C_v\frac{pdv+vdp}{R}+pdv=0.$$

Whence $$(C_v+R)\times pdv=-C_v vdp$$

or $$\frac{C_p}{C_v}\times\frac{dv}{v}=k\frac{dv}{v}=-\frac{dp}{p}.$$

Integrating from v_1, p_1 to v_2, p_2 we get

$$k\log.\frac{v_2}{v_1}=-\log.\frac{p_2}{p_1}=\log.\frac{p_1}{p_2};$$

whence $$\left(\frac{v_2}{v_1}\right)^k=\frac{p_1}{p_2},\text{ and }\frac{v_2}{v_1}=\left(\frac{p_1}{p_2}\right)^{\frac{1}{k}}. \tag{VII}$$

Combining (VI) and (VII) we readily get

$$\left(\frac{v_2}{v_1}\right)^k=\left(\frac{T_1}{T_2}\right)^{\frac{k}{k-1}};\text{ or }\frac{v_2}{v_1}=\left(\frac{T_1}{T_2}\right)^{\frac{1}{k-1}};$$

or $$\frac{T_2}{T_1}=\left(\frac{v_1}{v_2}\right)^{k-1} \tag{VIII}$$

This could also be found from (V) by eliminating p and integrating.

In section **522** we spoke of the work done by the expanding air as a positive quantity. To be consistent we should then prefix the negative sign to the work done (by an engine, for instance) in compressing the air. It will be more convenient, however, to denote the work done by whatever motor as positive; so that for $\int pdv$ in (V) we shall write $-U$ and (V) leads to

$$U=\int_{v_1}^{v_2}pdv=\int_{T_1}^{T_2}C_v dT=C_v(T_2-T_1) \tag{IX}$$

This equation gives the total work done in compressing one pound of air from the condition v_1, p_1, T_1 to the condition v_2, p_2, T_2.

527. Now let us find the work done by the engine while compressing a pound of air from p_1 to p_2, and then forcing it out of the cylinder into a reservoir where the pressure is constantly p_2. As has just been seen, the work of compression is $C_v\ (T_2-T_1)$, the volume of the air is now v_2, so that the work of forcing the air out of the cylinder is $v_2\,p_2$; hence the total work is

$$C_v(T_2-T_1)+v_2p_2.$$

But this is not wholly due to the engine; the pressure on the *outside* of the piston is uniformly p_1 throughout the *whole stroke*, so that the work v_1p_1 is actually done by the atmosphere. Substracting this from the total work and we have as the work of the compressing engine

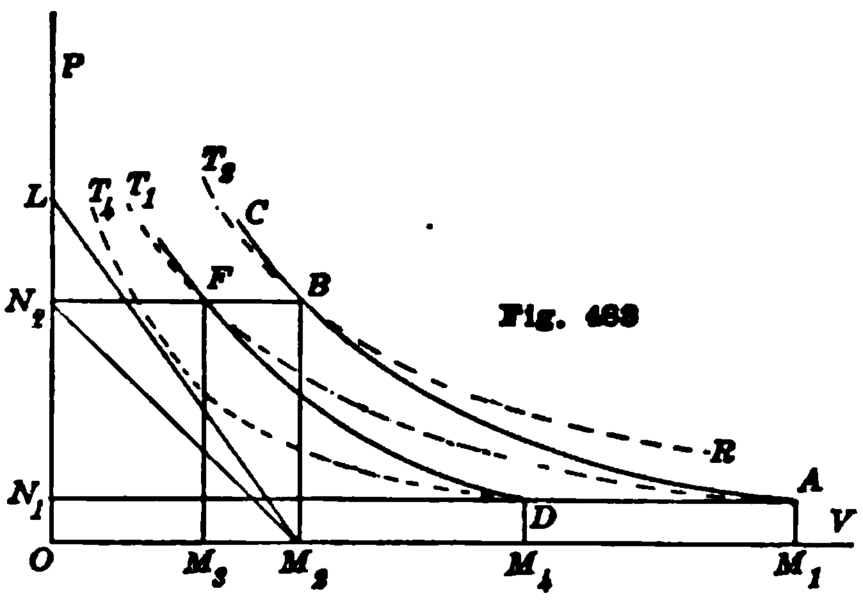
$$U_c = C_v(T_2 - T_1) + v_2p_2 - v_1p_1.$$

But when the volume was v_2 and the pressure p_2, the temperature was T_2, so that by (I) $v_2p_2 = RT_2$. Similarly $v_1p_1 = RT_1$, so that we have

$$U_c = C_v(T_2 - T_1) + R(T_2 - T_1) = (C_v + R)\ (T_2 - T_1) = C_p(T_2 - T_1). \quad \text{(X)}$$

This shows that the work of the compressor is directly proportional to the *change of temperature of the air while in the compressor.*

528. The work of compression. *Graphics.*—Our results may be illustrated very fully by diagrams. Equation (I), being an equation between three variables, may be regarded as the equation of a surface referred to rectangular co-ordinates, the axes being OP, OV, and OT. It is readily seen that the surface is a hyperbolic-paraboloid. If during a change in volume and pressure, T is constant, we get a curve of the surface which is called an "isothermal" curve; its projection upon the plane of volume and pressure axes is represented by the equation $pv = RT =$ a constant. This is the equation of an equilateral hyperbola. The three dotted lines, Fig. 483, are "isothermal lines, the temperatures being respectively T_1, T_2, and T_4.



ISENTROPIC AND ISOTHERMAL LINES.

Every point in the surface represents a separate condition of the pound of air, as regards v, p and T. Any change in the condition of the air, whether arising from expansion, compression, heating or cooling, will be accompanied by a change in the position of its representative point on the surface; so that a connected series of changes of condition is always represented by a line, straight or curved, on the surface. An isothermal curve represents a series of conditions which our pound of air can be made to pass thru only by the direct transfer of heat to or from it as it expands or is compressed. If, however, it expands or is compressed, *without any transfer of heat*, the change is said to be *isentropic*, and the curve of the surface representing the

successive conditions is called an *isentropic curve.* [Professor Rankine called it an *adiabatic* curve.] Equation (VII) gives the projection of an isentropic curve on the plane of volume and pressure, if in the place of v_2 and p_2 we write the general values v and p; thus:

$$v=v_1\left(\frac{p_1}{p}\right)^{\frac{1}{k}} \quad \text{or} \quad p=\frac{p_1v_1{}^k}{v^k} \tag{XI}$$

In Fig. 483, ABC is a portion of an isentropic curve, and RBT_2 an isothermal.* Let A denote the condition of the pound of air in the cylinder of a single-acting air-pump, when v, p and T are v_1, p_1, T_1. In like manner let B denote the condition v_2, p_2, T_2 after compression $OM_1.=v_1$, $OM_2=v_2$; $AM_1=p_1$, $BM_2=p_2$. Then the total work during the compression is

$$C_v(T_2-T_1)=\text{the } \textit{area } AM_1M_2B.$$

Since pdv in the figure represents an element of the *area*; and in the formula it represents an element of *work.* The work of forcing the air out of the cylinder into the reservoir is $p_2v_2=$ the *area* BM_2ON_2. The work done by the external air during the whole stroke is $v_1p_1=$ the *area* AM_1ON_1. Hence

$$U_c=C_p(T_2-T_1)=\text{the } \textit{area } A\ N_1\ N_2\ B.$$

529. The temperature with which the air leaves the compressor is not maintained. In the reservoir tank and in the pipe leading to the air-engine, where the air is to be used, the temperature generally falls nearly if not quite to the original temperature T_1, while the pressure is not measurably diminished. This loss of heat involves a loss of volume, so that the condition of our pound of air as it enters the cylinder of an air-engine is v_3, p_2, T_3, represented by the point F. These are simultaneous values and hence satisfy equation (I), as does every other set of simultaneous values. T_3 may not differ measurably from T_1, but in general it will differ several degrees. In the figure $T_3=T_1$.

In the use of the air in the engine it will be assumed that it enters the cylinder under full pressure, and then expands without direct transfer of heat till the pressure falls to p_1. The work done will be by the air, so that no change of sign is needed in this case.

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* The curves in the figures are drawn to scale from their equations. They were made by a student of the class to whom this discussion was first presented, Mr. Wm. S. Love. Their proportions are very nearly correct. The range of pressures is from *one* to *five* atmospheres. The graphical results, therefore, appeal to the eye with their proper force.

On entering the cylinder under full pressure the work done by our pound is $v_3\ p_2$, represented by $F.M_3ON_2$. It then expands till the pressure falls to p_1 and the temperature to T_4. The work done by the air in expanding is $C_v(T_3-T_4)$ in accordance with (IX). But a part of this work is done upon the external air which resists the motion of the piston by the constant pressure p_1. To get the *available work* of the air engine we must subtract from the workd one, the product v_4p_1, v_4 being the final volume of the air which leaves the cylinder at a temperature T_4. The work of the air-engine is then

$$U_e=C_v(T_3-T_4)+v_3p_2-v_4p_1$$

But $v_3p_2=RT_3$, and $v_4p_1=RT_4$, so that

$$U_e=C_v(T_3-T_4)+R(T_3-T_4)=C_p(T_3-T_4) \qquad \text{(XII)}$$

This result might have been inferred from equation (X).

530. We are now able to compare the work done by the air-engine per pound of air with the work done by the compressor. Dividing (XII) by (X), we have for the efficiency E.

$$E=\frac{U_e}{U_c}=\frac{T_3-T_4}{T_2-T_1}=\frac{T_3}{T_2}\times\frac{1-\dfrac{T_4}{T_3}}{1-\dfrac{T_1}{T_2}}=\frac{T_3}{T_2} \qquad \text{(XIII)}$$

for $\dfrac{T_4}{T_3}=\dfrac{T_1}{T_2}$ since each is equal to $\left(\dfrac{p_1}{p_2}\right)^{\frac{k-1}{k}}$ by (VI.).

This shows the utility of keeping the temperature T_3 up as near as possible to T_2, or at least of raising the temperature in the *service pipe* to T_2, or above.

If $T^3=T_1$,
$$E=\frac{T_1}{T_2}=\left(\frac{p_1}{p_2}\right)^{\frac{k-1}{k}} \qquad \text{(XIV)}$$

Since by (VI.) $T_2=T_1\left(\dfrac{p_2}{p_1}\right)^{\frac{k-1}{k}}$, we find that E of (XIII.) becomes

$$E=\frac{T_3}{T_1}\times\left(\frac{p_1}{p_2}\right)^{\frac{k-1}{k}} \qquad \text{(XV)}$$

which shows that for a given range of pressure the *efficiency will be increased by increasing T_3 or by decreasing T_1.*

531. *Graphical Representation of Work Lost.*—In Fig. 484 as in Fig. 483, let A represent the condition of the air at p_1, v_1, T_1; and B at p_2, v_2, T_2. The area AN_1N_2B was shown to represent the work of the compressor. While in the reservoir, or in the engine supply pipes, the pressure being constantly p_2, the temperature falls to T_3, and the volume is reduced to v_3; this condition is represented by the position of F. In the figure, T_3 is assumed to be the same as T_1. In the condition F, the air enters the cylinder of the air engine. The work it does at *full pressure* is represented by the area $FM_3ON_2 = v_3p_2$. The air then expands till the pressure is reduced to p_1, the volume is v_4, the temperature is T_4, and the condition of the exhaust is represented by the point D. Hence the work done during expansion is represented by the area FDM_4M_3. After subtracting the work v_4p_1 done in overcoming the atmospheric pressure, the *useful* work remaining is represented by the area FDN_1N_2; and as the work by the compressor was ABN_2N_1, the *work lost* thru the cooling of the air between the compressor and the air-engine is fully represented by the shaded area $ABFD$. The object of all modifications of the processes we have followed, should be to *diminish this loss*, or graphically, to reduce the area $ABFD$.

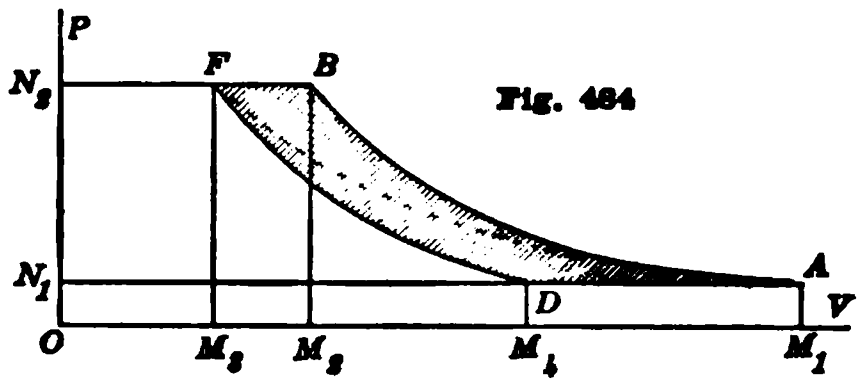


Fig. 484

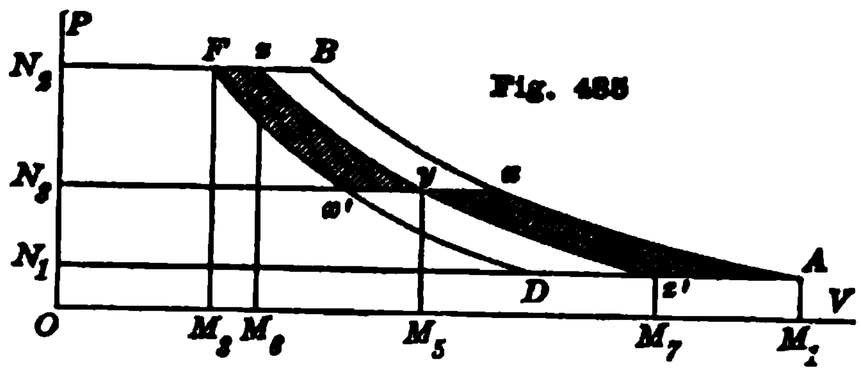
LOSS OF ENERGY WITH NON-CONDUCTING CYLINDERS.

532. This loss may be diminished in several ways:

(a.) Reduce the temperature T_1 of the air entering the compressor as much as possible. This will bring the point A nearer D (and consequently B nearer F), thus reducing the area on the outside. This reduction can be made by drawing the air through cold vaults, caverns, or cellars, where the volume of a pound of air is less than that of air at the average temperature.

(b.) Raise the temperature T_3 of the air entering the air-engine as much as possible. This will shorten the line BF, and also the line DA, and so save much that was lost. If the compressed air supply-pipe can be heated by what would otherwise be waste heat, such as solar heat, chimney gases, or hot refuse-water, it should be done. Sometimes the **service pipe** can pass through the combustion chamber of a gas or gasoline burner, in a coil, whereby the temperature can be materially raised, and the efficiency be increased.

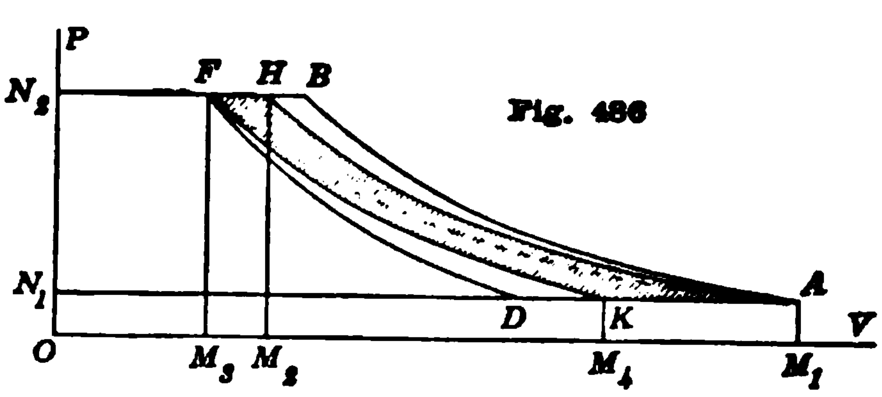
(c.) Divide the work of compression into two parts, allowing the temperature to fall to T_1, after the work of compression is partly done in one cylinder, and then completing the work in a second cylinder. The saving thus effected is shown in Fig 485 by the quadrilateral $xyzB$, which is saved on the work of compression. This will involve a second tank, and, generally, arrangements for rapid cooling of the air, and the saving will be a maximum when the intermediate pressure is a *mean proportional between the extreme pressures.* Thus, if p_2 the highest pressure is $9p_1$, the intermediate pressure should be $3p_1$; and it will be as much work to compress the pound of air from *one* atmosphere to *three,* as from *three* to *nine,* the temperatures at the two compressors being the same. The volume of the second cylinder should be to the volume of the first in the ratio of OM_5 to OM_1. Thus, if the air is to be compressed from one to nine atmospheres by a compound compressor, the intermediate cylinder should have 0.46 of the volume of the larger.*



SAVING OF ENERGY BY A COMPOUND SYSTEM.

(d.) In a similar manner the air could be used in a compound engine, the air being heated up to T_2 after escaping from the first cylinder. This would increase the useful work of the engine and diminish the area of loss in the diagram by a quadrilateral $x'yz'D$. The intermediate pressure should correspond to the intermediate pressure during compression.

(e.) Withdraw the heat from the air as rapidly as possible during compression by a *cooling jacket* or otherwise. The curve which represents the condition of the air under these circumstances is intermediate between an isentropic and an isothermal curve, as is shown in Fig. 486 by the curve AH. This effects a great saving.



SAVING EFFECTED BY HEATING AND COOLING JACKETS.

(f.) Supply heat to the expanding air in the air-engine cylinder by a *hot jacket.* This result is shown by the modified curve FK in Fig. 486, which exhibits a clear gain.

Two or more of these methods of saving may be adopted at the same time.

* The proof of these several statements is left to the student.

533. Numerical results. *When $T_3 = T_1$ and none of the methods given above for diminishing the loss are used.* Equation (XIV) gives

$$E = \left(\frac{p_1}{p_2}\right)^{\frac{k-1}{k}} \text{ and from}$$

$$\text{equation (VI.) we have } T_2 = T_1\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} = \frac{T_1}{E},$$

$$\text{and } T_4 = T_1\left(\frac{p_1}{p_2}\right)^{\frac{k-1}{k}} = T_1E;$$

$$\text{and } \frac{k-1}{k} = \frac{0.41}{1.41} = 0.29.$$

If the temperature of the air at first is 60 degrees Fahr.,

$$T_1 = 60 + 459 + 519.^*$$

The following table gives the Efficiency and the Temperatures, both of the high-pressure air, and at the exhaust, for various ranges of pressure:

$\frac{p_1}{p_2}$	$\frac{p_2}{p_1}$	E	T_2	T_4	t_2 deg. Fahr. Degrees	t_4 deg. Fahr. Degrees
1-2	2	0.815	634	423	175	— 36
1-3	3	0.727	714	377	255	— 82
1-4	4	0.669	776	347	317	—112
1-5	5	0.627	828	325	369	—134
1-6	6	0.595	873	309	414	—150
1-7	7	0.569	912	295	453	—164
1-8	8	0.547	949	284	490	—175
1-9	9	0.529	981	275	522	—184

Numerical Results for Compound Cylinders.—Since the results of the table just given do not depend on the absolute values of p_1 and p_2, but only on their ratio, it is evident that the efficiency of a compressor and engine working between three atmospheres and nine atmospheres is the same as between one and three atmospheres. It will therefore be seen on a moment's reflection that the efficiency of a compound

* The table was prepared, omitting the fraction of 459°.58, since at the time the calculations were made the value of T_0 was given as 459°.4. The omission does not affect the results materially.

compressor and compound engine as described above working between the extreme pressures, one and nine atmospheres will be the same as for a single compressor and engine between one and three atmospheres; and that, in general, the efficiency of a compound system (using the proper intermediate pressure and always bringing the pumped or the exhaust air back to T_1) working between two given extreme pressures, is equal to the *square-root* of the efficiency with a single system. Thus:

Ratio of extreme pressures	2	3	4	5	6	7	8	9
Efficiency of single system	.815	.727	.669	.627	.595	.569	.547	.529
Efficiency of compound system.	.90	.85	.82	.79	.77	.76	.74	.73

An examination of the table will also show that the efficiency in the column headed 6 is the product of the efficiencies in columns 2 and 3; so the efficiency under 8 is the product of the efficiencies under 2 and 4.

In a large compressing station a compound compressor is well worth while.

APPENDIX.

LOGARITHMS OF NUMBERS.

N	0	1	2	3	4	5	6	7	8	9	D
10	0000	043	086	128	170	212	253	294	334	374	42
11	414	453	492	531	569	607	645	682	719	755	38
12	792	828	864	899	934	969	1004	1038	1072	1106	35
13	1139	173	206	239	271	303	335	367	399	430	32
14	461	492	523	553	584	614	644	673	703	732	30
15	1761	790	818	847	875	903	931	959	987	2014	28
16	2041	068	095	122	148	175	201	227	253	279	26
17	304	330	355	380	405	430	455	480	504	529	25
18	553	577	601	625	648	672	695	718	742	765	24
19	788	810	833	856	878	900	923	945	967	989	22
20	3010	032	054	075	096	118	139	160	181	201	21
21	222	243	263	284	304	324	345	365	385	404	20
22	424	444	464	483	502	522	541	560	579	598	19
23	617	636	655	674	692	711	729	747	766	784	19
24	802	820	838	856	874	892	909	927	945	962	18
25	3979	997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	166	183	200	216	232	249	265	281	298	16
27	314	330	346	362	378	393	409	425	440	456	16
28	472	487	502	518	533	548	564	579	594	609	15
29	624	639	654	669	683	698	713	728	742	757	15
30	4771	786	800	814	829	843	857	871	886	900	14
31	914	928	942	955	969	983	997	5011	5024	5038	14
32	5051	065	079	092	105	119	132	145	159	172	13
33	185	198	211	224	237	250	263	276	289	302	13
34	315	328	340	353	366	378	391	403	416	428	13
35	5441	453	465	478	490	502	514	527	539	551	12
36	563	575	587	599	611	623	635	647	658	670	12
37	682	694	705	717	729	740	752	763	775	786	12
38	798	809	821	832	843	855	866	877	888	899	11
39	911	922	933	944	955	966	977	988	999	6010	11
40	6021	031	042	053	064	075	085	096	107	117	11
41	128	138	149	160	170	180	191	201	212	222	10
42	232	243	253	263	274	284	294	304	314	325	10
43	335	345	355	365	375	385	395	405	415	425	10
44	435	444	454	464	474	484	493	503	513	522	10
45	6532	542	551	561	571	580	590	599	609	618	10
46	628	637	646	656	665	675	684	693	702	712	9
47	721	730	739	749	758	767	776	785	794	803	9
48	812	821	830	839	848	857	866	875	884	893	9
49	902	911	920	928	937	946	955	964	972	981	9
50	6990	998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	084	093	101	110	118	126	135	143	152	8
52	160	168	177	185	193	202	210	218	226	235	8
53	243	251	259	267	275	284	292	300	308	316	8
54	324	332	340	348	356	364	372	380	388	396	8

LOGARITHMS OF NUMBERS.

N	0	1	2	3	4	5	6	7	8	9	D
55	7404	412	419	427	435	443	451	459	466	474	8
56	482	490	497	505	513	520	528	536	543	551	8
57	559	566	574	582	589	597	604	612	619	627	8
58	634	642	649	657	664	672	679	686	694	701	7
59	709	716	723	731	738	745	752	760	767	774	7
60	7782	789	796	803	810	818	825	832	839	846	7
61	853	860	868	875	882	889	896	903	910	917	7
62	924	931	938	945	952	959	966	973	980	987	7
63	993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	069	075	082	089	096	102	109	116	122	7
65	8129	136	142	149	156	162	169	176	182	189	7
66	195	202	209	215	222	228	235	241	248	254	7
67	261	267	274	280	287	293	299	306	312	319	6
68	325	331	338	344	351	357	363	370	376	382	6
69	388	395	401	407	414	420	426	432	439	445	6
70	8451	457	463	470	476	482	488	494	500	506	6
71	513	519	525	531	537	543	549	555	561	567	6
72	573	579	585	591	597	603	609	615	621	627	6
73	633	639	645	651	657	663	669	675	681	686	6
74	692	698	704	710	716	722	727	733	739	745	6
75	8751	756	762	768	774	779	785	791	797	802	6
76	808	814	820	825	831	837	842	848	854	859	6
77	865	871	876	882	887	893	899	904	910	915	6
78	921	927	932	938	943	949	954	960	965	971	6
79	976	982	987	993	998	9004	9009	9015	9020	9025	5
80	9031	036	042	047	053	058	063	069	074	079	5
81	085	090	096	101	106	112	117	122	128	133	5
82	138	143	149	154	159	165	170	175	180	186	5
83	191	196	201	206	212	217	222	227	232	238	5
84	243	248	253	258	263	269	274	279	284	289	5
85	9294	299	304	309	315	320	325	330	335	340	5
86	345	350	355	360	365	370	375	380	385	390	5
87	395	400	405	410	415	420	425	430	435	440	5
88	445	450	455	460	465	469	474	479	484	489	5
89	494	499	504	509	513	518	523	528	533	538	5
90	9542	547	552	557	562	566	571	576	581	586	5
91	590	595	600	605	609	614	619	624	628	633	5
92	638	643	647	652	657	661	666	671	675	680	5
93	685	689	694	699	703	708	713	717	722	727	5
94	731	736	741	745	750	754	759	763	768	773	5
95	9777	782	786	791	795	800	805	809	814	818	5
96	823	827	832	836	841	845	850	854	859	863	5
97	868	872	877	881	886	890	894	899	903	908	4
98	912	917	921	926	930	934	939	943	948	952	4
99	956	961	965	969	974	978	983	987	991	996	4

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